(*November 11, 2014*)

Waveforms, I

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/mfms/notes_2013-14/11_waveforms_I.pdf]

Hecke had observed examples where the zeta functions of rings of algebraic integers in complex quadratic extensions of $\mathbb{Q}$ are expressible as Mellin transforms of *binary theta series*. For example, with $\mathfrak{o}=\mathbb{Z}[i]$ the ring of Gaussian integers,

$$\zeta_{\mathbb{Q}(i)}(s) \;=\; \sum_{0\neq\alpha\in\mathfrak{o}/\mathfrak{o}^\times} \frac{1}{(N\alpha)^s} \qquad\qquad \text{(with } N \text{ the Galois norm } \mathbb{Q}(i)\to\mathbb{Q})$$

the simplest binary theta series

$$\theta_2(z) \;=\; \sum_{c,d\in\mathbb{Z}} e^{\pi i(c^2+d^2)z}$$

gives

$$\int_0^\infty y^s\,\theta_2(iy)\,\frac{dy}{y} \;=\; \pi^{-s}\,\Gamma(s)\,\zeta_{\mathbb{Q}(i)}(s)$$

Similarly, for the Eisenstein integers $\mathfrak{o}=\frac{-1+\sqrt{-3}}{2}$,

$$\zeta_{\mathbb{Q}(\sqrt{-3})}(s) \;=\; \sum_{0\neq\alpha\in\mathfrak{o}/\mathfrak{o}^\times} \frac{1}{(N\alpha)^s} \qquad\qquad \text{(with } N \text{ the Galois norm } \mathbb{Q}(\sqrt{-3})\to\mathbb{Q})$$

and binary theta series

$$\theta(z) \;=\; \sum_{m,n\in\mathbb{Z}} e^{\pi i(m^2+mn+n^2)z}$$

we have

$$\int_0^\infty y^s\,\theta(iy)\,\frac{dy}{y} \;=\; \pi^{-s}\,\Gamma(s)\,\zeta_{\mathbb{Q}(\sqrt{-3})}(s)$$

As with Riemann's treatment of $\zeta(s)$ of the rational integers $\mathbb{Z}$, these identities give *analytic continuations and functional equations* of the zeta functions. The theta functions are holomorphic weight-one *modular forms* for various congruence subgroups of $SL_2(\mathbb{Z})$. Similarly *grossencharacter L-functions* such as

$$L_{\mathbb{Q}(i)}(s,\chi_{4m}) \;=\; \sum_{0\neq\alpha\in\mathfrak{o}/\mathfrak{o}^\times} \frac{(\alpha/|\alpha|)^{4m}}{(N\alpha)^s}$$

useful in proving equidistribution of Gaussian primes in angular sectors, appeared as Mellin transforms of *harmonic binary theta series*

$$\theta_{2,4m}(z) \;=\; \begin{cases} \sum_{c,d\in\mathbb{Z}} (ci+d)^{4m}\,e^{\pi i(c^2+d^2)z} & \text{(for } m\geq 0) \\[2ex] \sum_{c,d\in\mathbb{Z}} (ci-d)^{4|m|}\,e^{\pi i(c^2+d^2)z} & \text{(for } m\leq 0) \end{cases}$$

by

$$\int_0^\infty y^s\,\theta_{2,4m}(iy)\,\frac{dy}{y} \;=\; \pi^{-s}\,\Gamma(s)\,L_{\mathbb{Q}(i)}(s-2|m|,\chi_{4m})$$

Again, $\theta_{2,4m}$ is a holomorphic modular form of weight $1+4m$, and is a *cuspform* for $m\neq 0$.

At the time, such devices seemed only to work for *complex* quadratic extensions of $\mathbb{Q}$, not *real* quadratic extensions such as $\mathbb{Q}(\sqrt{2})$. Although [Hecke 1918/20] had already exhibited other means to produce such zeta functions by Mellin transforms, Hecke proposed to Maaß to find some kind of *automorphic form* for real quadratic fields analogous to the holomorphic binary theta series for complex quadratic fields, and for grossencharacter L-functions.

Maaß succeeded in finding the *special waveforms* to function as hoped. They are a sort of binary theta series with an *indefinite* form, such as x^2-2y^2, rather than the *definite* forms such as x^2+y^2 in Hecke's treatment.

Rather than being *holomorphic* functions of $z\in\mathfrak{H}$ with an automorphy relation for suitable $\Gamma\subset SL_2(\mathbb{Z})$, Maaß special waveforms satisfy the analytic condition

$$\Delta^{\mathfrak{H}}u \;=\; \lambda\cdot u \qquad\qquad (\text{for some } \lambda\in\mathbb{C}, \text{ with } \Delta^{\mathfrak{H}} = y^2\big(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}\big))$$

and are *invariant* under suitable $\Gamma\subset SL_2(\mathbb{Z})$. Here $\Delta^{\mathfrak{H}}$ is the $SL_2(\mathbb{R})$-invariant Laplacian on $\mathfrak{H}$. [1] In fact, the special waveforms exhibited by Maaß to address Hecke's question indeed prove to be special among waveforms. One special aspect is that none of the cuspidal waveforms among Maass' special waveforms is of *level one*. They are only of higher level, depending on the real quadratic field extension. [Selberg 1956] used a *trace formula* to prove that there are infinitely-many cuspidal waveforms of level one. [2]

This part treats the basic features of waveforms, postponing several more sophisticated discussions. The first emphasis is on a family of *Eisenstein series*, each an eigenfunction for the invariant Laplacian. These are the most explicit waveforms, and have several important roles: in integral representations of L-functions as *periods* of Eisenstein seriess, in giving the most conceptual proof of non-vanishing of $\zeta(s)$ and other L-functions on the edges of the critical strip, and in Rankin-Selberg integral representations of L-functions attached to cuspforms.

Later, we will see the fundamental role of Eisenstein series in the spectral decomposition of $L^2(\Gamma\backslash\mathfrak{H})$.

---

1. *Waveform Eisenstein series* E_s

Let $G=SL_2(\mathbb{R})$ act on the upper half-plane $\mathfrak{H}$ by linear fractional transformations, as usual. Let P be upper-triangular matrices in G. Use coordinates $z=x+iy$ on $\mathfrak{H}$.

[1.1] Invariant Laplacian The differential operator

$$\Delta \;=\; \Delta^{\mathfrak{H}} \;=\; y^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)$$

[1] The $SL_2(\mathbb{R})$-invariance of $\Delta^{\mathfrak{H}} = y^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)$ can be proven by direct computation, but this is silly. Indeed, we want to *determine* such an invariant operator in this and other circumstances, rather than merely *verify* it after someone gives it to us. A dignified, broadly applicable approach to expressing an invariant Laplacian in coordinates is given in a supplement.

[2] A modern form of Selberg's argument will be given later. Elementary heuristic versions of the argument unfortunately fail to suggest how to legitimize many details, producing the hazard that similar-sounding arguments produce nonsense.

is G-invariant: the P-invariance is visible, but, again, the G-invariant should *not* be verified by direct computation, although such verification is possible in principle. The supplement shows how to obtain invariant operators. The simplest Δ eigenfunctions are the functions $z = x + iy \longrightarrow y^s$ for complex s, because

$$\Delta y^s \;=\; y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)(y^s) \;=\; y^2 \cdot s(s-1)y^{s-2} \;=\; s(s-1)\cdot y^s$$

With $\Gamma = SL_2(\mathbb{Z})$, since $\mathbb{Z}$ is a principal ideal domain, and $\mathbb{Z}^\times = \{\pm 1\}$ has cardinality 2,

$$(P\cap\Gamma)\backslash\Gamma \;\longleftrightarrow\; \{\pm1\}\setminus\{(c\quad d) : \text{coprime } c,d\} \qquad \text{by} \qquad \begin{pmatrix} * & * \\ c & d\end{pmatrix} \;\longleftrightarrow\; (\,c \quad d\,)$$

The $SL_2(\mathbb{R})$ invariant measure on $\mathfrak{H}$ is

$$\frac{dx\,dy}{y^2}$$

As with Δ, the P-invariance is easy to see, but the G-invariance is not, *and* should not be verified after-the-fact by brute force computation, but *derived* from generally-applicable considerations as in the supplement.

By coincidence, the *symmetry* property

$$\int_{\mathfrak{H}} (\Delta f)(z)\cdot\overline{g(z)}\;\frac{dx\,dy}{y^2} \;=\; \int_{\mathfrak{H}} f(z)\cdot\overline{(\Delta g)(z)}\;\frac{dx\,dy}{y^2}$$

is visible from Euclidean integration by parts for the Euclidean Laplacian and usual measure on $\mathbb{R}^2$: for $f,g\in C_c^\infty(\mathfrak{H})$,

$$\int_{\mathfrak{H}} (\Delta f)(z)\cdot\overline{g(z)}\;\frac{dx\,dy}{y^2} \;=\; \int_{\mathfrak{H}} \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) f(z)\cdot\overline{g(z)}\;dx\,dy$$

$$=\; \int_{\mathfrak{H}} f(z)\cdot\overline{\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)g(z)}\;dx\,dy \;=\; \int_{\mathfrak{H}} f(z)\cdot\overline{(\Delta g)(z)}\;\frac{dx\,dy}{y^2}$$

The convenient coincidence of cancellation of the y^2 factors is *not* essential for symmetry. [3] Also, $\Delta = \Delta^{\mathfrak{H}}$ has the *negative-semi-definite* property that

$$\int_{\mathfrak{H}} (\Delta f)(z)\cdot\overline{f}(z)\;\frac{dx\,dy}{y^2} \;\le\; 0$$

since, integrating *once* by parts,

$$\int_{\mathfrak{H}} (\Delta f)(z)\cdot\overline{f}(z)\;\frac{dx\,dy}{y^2} \;=\; -\int_{\mathfrak{H}} \Big|\frac{\partial f}{\partial x}\Big|^2 + \Big|\frac{\partial f}{\partial y}\Big|^2\;dx\,dy \;\le\; 0$$

Integrals of Γ-invariant functions on $\mathfrak{H}$ on the *quotient* $\Gamma\backslash\mathfrak{H}$ can be understood in an elementary way as integrals over the standard fundamental domain

$$\{z = x+iy\in\mathfrak{H} \;:\; |z|\ge 1,\; -\tfrac{1}{2}\le x\le \tfrac{1}{2}\}$$

[3] The same coincidence does *not* occur in other, analogous situations, such as n-dimensional hyperbolic space for $n\ge 3$: with coordinates (x,y) with $x\in\mathbb{R}^{n-1}$ and $y>0$ on hyperbolic n-space, the invariant Laplacian and invariant measure are found to be

$$y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) - (n-2)\,y\,\frac{\partial}{\partial y} \qquad\text{and}\qquad \frac{dx\,dy}{y^{n+1}}$$

Nevertheless, invariant Laplacians are *always* symmetric with respect to the corresponding invariant measures, for very general reasons. See the supplement.

Apart from other, subtler problems, such a viewpoint complicates understanding of why integration by parts for $\Delta = \Delta^{\mathfrak{H}}$ is correct for $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$. Thus, instead of integrating over a fundamental domain, one might eventually discern an *intrinsic* integral on the *quotient*, as is done in a supplement. In summary, first prove *surjectivity* of the averaging maps

$$\alpha \;:\; C_c^o(\mathfrak{H}) \longrightarrow C_c^o(\Gamma\backslash\mathfrak{H}) \qquad \text{by} \qquad (\alpha f)(z) \;=\; \sum_{\gamma\in\Gamma} (f\circ\gamma)(z)$$

and

$$\alpha \;:\; C_c^\infty(\mathfrak{H}) \longrightarrow C_c^\infty(\Gamma\backslash\mathfrak{H}) \qquad \text{by} \qquad (\alpha f)(z) \;=\; \sum_{\gamma\in\Gamma} (f\circ\gamma)(z)$$

where $C_o^\infty(\Gamma\backslash\mathfrak{H})$ is construed as right K-invariant compactly-supported smooth functions on the *smooth manifold* $\Gamma\backslash G$, using $\mathfrak{H} \approx G/K$. Second, show that there is a unique measure on $\Gamma\backslash\mathfrak{H}$ such that, for every $f \in C_c^o(\mathfrak{H})$,

$$\int_{\mathfrak{H}} f \;=\; \int_{\Gamma\backslash\mathfrak{H}} \Big(\sum_{\gamma\in\Gamma} f\circ\gamma\Big)$$

Indeed, similar iterated-integral identities are essential to *unwinding* arguments.

[1.2] Eisenstein series

The Eisenstein series E_s for Γ is

$$E_s(z) \;=\; \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \mathrm{Im}\,(\gamma z)^s \;=\; \frac{1}{2}\cdot \sum_{\text{coprime } c,d} \frac{y^s}{|cz+d|^{2s}}$$

As with the holomorphic Eisenstein series $\sum_{c,d} 1/(cz+d)^{2k}$, the coprimality condition and quotient by $\{\pm 1\}$ can be avoided by inserting all possible common factors and multiplying by 2:

$$2\,\zeta(2s)\cdot E_s(z) \;=\; \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2} \frac{y^s}{|cz+d|^{2s}}$$

For $\mathrm{Re}\,(s) > 1$ the series defining E_s converges absolutely, uniformly for z in compacts, much as for the holomorphic Eisenstein series $\sum_{c,d} 1/(cz+d)^{2k}$, as follows. Recall

$$\mathrm{Im}\begin{pmatrix} a & b \\ c & d\end{pmatrix}(z) \;=\; \frac{\mathrm{Im}\,(z)}{|cz+d|^2} \qquad \Big(\text{for } \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{R})\Big)$$

Since E_s is Γ-invariant, it suffices to consider z in a fixed compact C inside the usual fundamental domain

$$\{z = x+iy \in \mathfrak{H} \;:\; |z|\geq 1,\; -\tfrac{1}{2}\leq x\leq \tfrac{1}{2}\}$$

For such z,

$$(cx+d)^2 + (cy)^2 \;=\; (x^2+y^2)c^2 + 2x\cdot cd + d^2 \;\geq\; c^2 - |cd| + d^2 \;\geq\; \tfrac{1}{2}(c^2+d^2)$$

The sum over coprime (c,d) is certainly dominated by the sum over *all* (c,d), not both 0. Thus, the Eisenstein series is uniformly (for z in compacts inside the fundamental domain) dominated by

$$\sum_{c,d \text{ not both } 0} \frac{1}{(c^2+d^2)^{\mathrm{Re}\,(s)}}$$

An adaptation of an integral test proves that this converges for $\mathrm{Re}\,(s) > 1$.

Since Δ is G-invariant, it is certainly Γ-invariant. Thus, at least in the region of convergence $\mathrm{Re}(s) > 1$, the Eisenstein series E_s is a Δ-eigenfunction: [4]

$$\begin{aligned}\Delta E_s(z) \;=\; \Delta \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \mathrm{Im}(\gamma z)^s \;&=\; \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \Delta\,\mathrm{Im}(\gamma z)^s \;=\; \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \Big(\Delta\,\mathrm{Im}(z)^s\Big)\circ\gamma \\ &=\; \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \Big(s(s-1)\cdot\mathrm{Im}(z)^s\Big)\circ\gamma \;=\; s(s-1)\cdot\sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \Big(\mathrm{Im}(z)^s\Big)\circ\gamma \\ &=\; s(s-1)\cdot\sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \mathrm{Im}(\gamma z)^s \;=\; s(s-1)\cdot E_s(z)\end{aligned}$$

This eigenfunction property persists for the meromorphically-continued Eisenstein series (below), by the identity principle from complex analysis.

Note that the eigenvalue $\lambda_s = s(s-1)$ is *real* and non-positive, as would be expected of a negative-semi-definite symmetric/self-adjoint operator, only for either $\mathrm{Re}(s) = \frac{1}{2}$ or $s \in [0,1]$, but that the series characterization of the Eisenstein series does not converge there.

[1.3] Meromorphic continuation and functional equation

We follow [Godement 1966a]'s Poisson summation argument, descended from [Rankin 1939], attributed by Rankin to his advisor Ingham. This argument is elementary, but less informative than arguments that engage with the spectral theory. [Epstein 1903] had already meromorphically continued $E_s(z)$ by expressing it as a Mellin transform.

The usual $\zeta(s)$ with its gamma factor is $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, with functional equation $\xi(1-s) = \xi(s)$.

[1.3.1] Theorem: For each fixed $z \in \mathfrak{H}$, $s(1-s)\xi(2s)\cdot E_s(z)$ has an analytic continuation to an *entire* function [5] of s. The *functional equation* is

$$\xi(2s)\,E_s \;=\; \xi(2-2s)\,E_{1-s}$$

$E_s(z)$ has no pole in $\mathrm{Re}(s) > \frac{1}{2}$ other than at $s = 1$. The pole of E_s at $s = 1$ is simple with residue the constant function $3/\pi$. For fixed $\delta > 0$, via Phragmén-Lindelöf, letting $s = \sigma + it$,

$$\Big|y^{-s}\,\zeta(2s)\,E_s(z)\Big| \;\ll_\delta\; y^{1+2\delta}\cdot t^{1+2\delta} \qquad\qquad (\text{for } -\delta \le \mathrm{Re}(s) \le 1+\delta)$$

For any fixed s, the function $E_s(z)$ is *of moderate growth*, in the sense that there is a sufficiently large exponent A depending on s such that

$$E_s(z) \;\ll_s\; y^A \qquad\qquad (\text{as } y \to +\infty)$$

[1.3.2] Remark: Further, with z fixed, away from poles, if we are willing to invoke some *convexity* results such as Hadamard's three-circle theorem, there is the *convexity bound*

$$E_s(z) \;\ll\; t^{1-\mathrm{Re}(s)+\varepsilon} \qquad\qquad (\text{for } \tfrac{1}{2} \le \mathrm{Re}(s) \le 1, \text{ for every } \varepsilon > 0)$$

The argument for this is essentially as follows. For a holomorphic function $f(s)$ with polynomial vertical growth, let

$$\psi(\sigma) \;=\; \inf\{A \;:\; |f(\sigma+it)| \ll t^{A+\varepsilon},\ \text{for all } \varepsilon > 0\}$$

[4] Term-wise application of Δ is justified by checking that the series defining E_s converges in the C^k topology on smooth functions on $\mathfrak{H}$, for all $k \ge 0$.

[5] The Eisenstein series $s \to E_s$ is a *function-valued function* of s. Nevertheless, we mostly consider the scalar-valued functions $s \to E_s(z)$ with fixed $z \in \mathfrak{H}$.

Hadamard's three-circle and other *convexity* results show that ψ is *convex*. Since $\psi(1+\delta) = 0$ for every $\delta > 0$, and $\psi(-\delta) = 1 + 2\delta$, necessarily $\psi(\frac{1}{2}) = \frac{1}{2}$, and $\psi(\sigma) = 1 - \sigma$ for $\frac{1}{2} \le \sigma \le 1$.

Proof: For $v = (c\ d) \in \mathbb{R}^2$, consider the Gaussian

$$\varphi(v) \;=\; e^{-\pi|v|^2} \;=\; e^{-\pi(c^2+d^2)}$$

with $v \to |v|$ the usual length function on $\mathbb{R}^2$. For $g \in SL_2(\mathbb{R})$, define

$$\Theta(g) \;=\; \sum_{v\in\mathbb{Z}^2} \varphi(v\cdot g) \;=\; \sum_{(c,d)\in\mathbb{Z}^2} e^{-\pi|(c,d)g|^2}$$

where $v \in \mathbb{R}^2$ is a row vector. Consider the integral (a Mellin transform)

$$\int_0^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t}$$

where the t in the argument of Θ simply acts by scalar multiplication on $g \in SL_2(\mathbb{R})$. On one hand, integrating term-by-term gives

$$\int_0^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} \;=\; \sum_{v\neq(0,0)} \int_0^\infty t^{2s}\,e^{-\pi|tvg|^2}\,\frac{dt}{t}$$

Since

$$\pi|tvg|^2 \;=\; (t\cdot\sqrt{\pi}|vg|)^2$$

we can change variables by replacing t by $t/(\sqrt{\pi}|vg|)$ to obtain

$$\sum_{v\neq(0,0)} (\sqrt{\pi}|vg|)^{-2s} \int_0^\infty t^{2s}\,e^{-t^2}\,\frac{dt}{t} \;=\; \frac{1}{2}\pi^{-s} \sum_{v\neq(0,0)} |vg|^{-2s} \int_0^\infty t^s\,e^{-t}\,\frac{dt}{t}$$

$$=\; \frac{1}{2}\pi^{-s}\,\Gamma(s) \sum_{v\neq(0,0)} |vg|^{-2s}$$

We want $g \in SL(2,\mathbb{R})$ to map $i \to x+iy$. One reasonable choice is

$$g \;=\; g_z \;=\; \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}$$

Using this choice of g and writing out $v = (c,d)$ gives

$$vg \;=\; (c,d)g \;=\; (\,c \quad d\,) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \;=\; (c\sqrt{y},\ \frac{cx+d}{\sqrt{y}})$$

and thus

$$\sum_v |vg|^{-2s} \;=\; \sum_v |(c\sqrt{y},\frac{cx+d}{\sqrt{y}})|^{-2s} \;=\; \sum_v (c^2y + \frac{(cx+d)^2}{y})^{-s}$$

$$=\; \sum_v \frac{y^s}{(c^2y^2+(cx+d)^2)^s} \;=\; \sum_v \frac{y^s}{|ciy+cx+d|^{2s}} \;=\; \sum_v \frac{y^s}{|cz+d|^{2s}}$$

Letting $1 \le \delta = \gcd(c,d)$, this is

$$\sum_v \frac{y^s}{|cz+d|^{2s}} \;=\; \sum_\delta \frac{1}{\delta^{2s}} \sum_{\text{coprime } c,d} \frac{y^s}{|cz+d|^{2s}} \;=\; 2\,\zeta(2s)\cdot E_s(z)$$

The expression

$$2\,\zeta(2s)\,E_s(z) \;=\; \sum_{(c,d)\neq(0,0)} \frac{y^s}{|cz+d|^{2s}} \qquad\qquad \text{(summing } (c,d) \text{ over } \textit{all} \text{ non-zero vectors in } \mathbb{Z}^2\text{)}$$

is convenient, being a sum over a lattice with 0 removed.

Thus, the integral representation yields the Eisenstein series with a leading power of π, a gamma function, and a factor of $\zeta(2s)$:

$$\int_0^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} \;=\; \pi^{-s}\,\Gamma(s)\,\zeta(2s)\,E_s(g(i))$$

To prove the meromorphic continuation, use that integral representation as in Riemann's argument for $\zeta(s)$, first breaking the integral into two parts, one from 0 to 1, and the other from 1 to $+\infty$. Keep $g \in SL(2,\mathbb{R})$ in a compact subset of $SL(2,\mathbb{R})$. Since elementary estimates show the absolute convergence uniformly on compacts,

$$\int_1^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} \;=\; \text{entire in } s$$

Apply Poisson summation to the kernel: first, the Gaussian $\varphi(v) = e^{-\pi|v|^2}$ is its own Fourier transform, and

$$\text{Fourier transform of } (v \to \varphi(tvg)) \;=\; (v \to t^{-2}\det(g)^{-1}\cdot\varphi(t^{-1}v\,{}^\top g^{-1}))$$

where ${}^\top g$ is g-transpose. Then Poisson summation asserts

$$\Theta(tg) \;=\; t^{-2}\,\det(g)^{-1}\cdot\Theta(t^{-1}\,{}^\top g^{-1})$$

Noting $\det g = 1$, the modification for the kernel gives

$$\Theta(tg) - 1 \;=\; t^{-2}\cdot[\Theta(t^{-1}\,{}^\top g^{-1}) - 1] + t^{-2} - 1$$

Transform the integral from 0 to 1 into an integral from 1 to $+\infty$: at first only for $\mathrm{Re}\,(s) > 1$,

$$\int_0^1 t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} \;=\; \int_0^1 t^{2s}\,\Big(t^{-2}\cdot[\Theta(t^{-1}\,{}^\top g^{-1})-1] + t^{-2} - 1\Big)\,\frac{dt}{t}$$

Replacing t by $1/t$ turns this into

$$\int_1^\infty t^{-2s}\,\Big(t^2\cdot[\Theta(t\,{}^\top g^{-1})-1] + t^2 - 1\Big)\,\frac{dt}{t}$$

Explicitly evaluating the last two elementary integrals of powers of t from 1 to ∞, using $\mathrm{Re}\,(s) > 1$, this is

$$\int_1^\infty t^{2-2s}\,(\Theta(t\,{}^\top g^{-1})-1)\,\frac{dt}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

For g in $SL(2)$,

$${}^\top g^{-1} \;=\; wgw^{-1}$$

where w is the *long Weyl element*

$$w \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since $\mathbb{Z}^2 - (0,0)$ is stable under w, and since the length function $v \to |v|^2$ is invariant under w,

$$\Theta(g) \;=\; \Theta(wg) \;=\; \Theta(gw^{-1})$$

so

$$\Theta({}^\top g^{-1}) \;=\; \Theta(g)$$

Thus, the original integral from 0 to 1 becomes

$$\int_1^\infty t^{2-2s}\,(\Theta(tg)-1)\,\frac{dt}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

and the *whole* equality, with g of the special form above, is

$$\pi^{-s}\Gamma(s)\,\zeta(2s)\,E_s(z) \;=\; \int_1^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} + \int_1^\infty t^{2-2s}\,(\Theta(tg)-1)\,\frac{dt}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

The integrals from 1 to ∞ are nicely convergent for all $s \in \mathbb{C}$, uniformly for g in compacts. The elementary rational expressions of s have meromorphic continuations. Thus, the right-hand side gives a meromorphic continuation of the Eisenstein series, and is visibly invariant under $s \to 1-s$.

The only poles are at $s = 1, 0$. The residue at $s = 1$ is the constant function $\frac{1}{2}$, and at $s = 0$ the residue is the constant function $-\frac{1}{2}$. At $s = 1$ the factor $\pi^{-s}\Gamma(s)$ is holomorphic and has value $1/\pi$, so

$$\mathrm{Res}_{s=1}\,\zeta(2s)\,E_s \;=\; \frac{\pi}{2}$$

Now we recover the assertions for E_s itself. The convergence of the infinite product

$$\zeta(2s) \;=\; \sum_n \frac{1}{n^{2s}} \;=\; \prod_{p\text{ prime}} \frac{1}{1-p^{-2s}}$$

for $\mathrm{Re}(s) > 1/2$ assures that $\zeta(2s)$ is not zero for $\mathrm{Re}(s) > 1/2$. And $\zeta(2) = \pi^2/6$. These facts and the previous discussion give the result for E_s.

To see the moderate growth, first observe that on bounded vertical strips $s(s-1)\cdot\xi(2s)\cdot E_s$ easily admits a bound of the form $e^{|s|^N}$, from the integral representation in terms of Θ. Thus, Phragmén-Lindelöf is applicable.

Fix $\delta > 0$ and $y_o > 0$. On the vertical line $\mathrm{Re}(s) = 1+\delta$, the convergent series for $|y^{-s}\cdot\zeta(2s)\cdot E_s(z)|$ shows that it is bounded. The functional equation gives

$$y^{-(1-s)}\zeta(2-2s)\,E_{1-s}(z) \;=\; \frac{y^{-(1-s)}}{\pi^{-(1-s)}\,\Gamma(1-s)}\,\xi(2-2s)\,E_{1-s}(z) \;=\; \frac{y^{-(1-s)}}{\pi^{-(1-s)}\,\Gamma(1-s)}\,\xi(2s)\,E_s(z)$$

$$= \frac{y^{-(1-s)}\,\pi^{-s}\,\Gamma(s)}{\pi^{-(1-s)}\,\Gamma(1-s)}\,\zeta(2s)\,E_s(z) \;=\; \frac{y^{-(1-2s)}\,\pi^{-s}\,\Gamma(s)}{\pi^{-(1-s)}\,\Gamma(1-s)} \cdot y^{-s}\,\zeta(2s)\,E_s(z)$$

On the vertical line $\mathrm{Re}(s) = 1+\delta$, using asymptotics of $\Gamma(s)$ [6] this is bounded by

$$y^{1+2\delta}\,\pi^{-1-2\delta}\left|\frac{\Gamma(1+\delta+it)}{\Gamma(-\delta-it)}\right| \ll y^{1+2\delta}\,\pi^{-1-2\delta}\,t^{1+2\delta}$$

Thus, with large N, getting rid of the poles by multiplying by $s(s-1)$ and compensating via the Gamma function,

$$\frac{\Gamma(s+N)}{\Gamma(s+N+3+2\delta)} \times s(s-1) \times y^{-s}\,\zeta(2s)\,E_s(z)$$

[6] The asymptotic $\Gamma(a+it)/\Gamma(b+it) \sim t^{a-b}$ follows from Stirling's formula, but even more easily from asymptotics of integrals, sometimes called *Watson's lemma.*

is holomorphic on the strip $-\delta \le \mathrm{Re}(s) \le 1+\delta$, and bounded by a constant multiple of $y^{1+2\delta}$. By Phragmén-Lindelöf,

$$\Big|\frac{\Gamma(s+N)}{\Gamma(s+N+3+2\delta)} \times s(s-1) \times y^{-s}\,\zeta(2s)\,E_s(z)\Big| \;\le_{\delta,N}\; y^{1+2\delta} \qquad (\text{for } -\delta \le \mathrm{Re}(s) \le 1+\delta)$$

That is, away from poles,

$$\Big|y^{-s}\,\zeta(2s)\,E_s(z)\Big| \;\le_{\delta}\; y^{1+2\delta}\cdot t^{1+2\delta} \qquad (\text{for } -\delta \le \mathrm{Re}(s) \le 1+\delta)$$

Moderate growth $E_s(z) \ll y^{\mathrm{Re}(s)}$ is clear in $\mathrm{Re}(s) \ge 1+\delta$ for fixed $\delta > 0$, and from the previous discussion in $-\delta \le \mathrm{Re}(s) \le 1+\delta$. Similarly, in $-B \le \mathrm{Re}(s) \le -\delta$, the functional equations of E_s and $\xi(2s)$ give

$$E_s(z) \;=\; \frac{\xi(2s)\,E_s(z)}{\xi(2s)} \;=\; \frac{\xi(2-2s)\,E_{1-s}(z)}{\xi(1-2s)} \;\ll_s\; y^{1-\mathrm{Re}(s)} \qquad (\text{as } y \to +\infty)$$

Thus, $E_s(z)$ is of moderate growth for all s. ///

[1.4] Constant term of Eisenstein series By definition, the *constant term* is a sort of 0^{th} Fourier coefficient in $x = \mathrm{Re}(z)$. Literally, it is the 0^{th} Fourier *component* in x, a function of $y = \mathrm{Im}(z)$:

$$c_P E_s(iy) \;=\; \int_0^1 E_s(x+iy)\,dx$$

We will see later that the form of the constant term of E_s dictates the *functional equation* and other features of E_s.

[1.4.1] Claim: The constant term of E_s is

$$c_P E_s(iy) \;=\; y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}$$

Proof: By direct computation,

$$c_P\Big(2\zeta(2s)\cdot E_s(x+iy)\Big) \;=\; \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2} \int_0^1 \frac{y^s\,dx}{|cz+d|^{2s}}$$

$$= \sum_{d\neq 0}\int_0^1 \frac{y^s\,dx}{|d|^{2s}} \;+\; \sum_{c\neq 0}\sum_{d \bmod c}\sum_{\ell\in\mathbb{Z}}\int_0^1 \frac{y^s\,dx}{|c|^{2s}\cdot|z+\frac{d}{c}+\ell|^{2s}}$$

$$= \quad 2\zeta(2s)\,y^s \;+\; y^s\sum_{c\neq 0}\frac{1}{|c|^{2s}}\sum_{d \bmod c}\sum_{\ell\in\mathbb{Z}}\int_0^1 \frac{1\,dx}{\big((x+\frac{d}{c}+\ell)^2+y^2\big)^s}$$

Unwinding the sum over ℓ and integral over $[0,1]$ to make an integral over $\mathbb{R}$, this becomes

$$2\zeta(2s)\,y^s \;+\; y^s\sum_{c\neq 0}\frac{1}{|c|^{2s}}\sum_{d \bmod c}\int_{\mathbb{R}} \frac{1\,dx}{\big((x+\frac{d}{c})^2+y^2\big)^s}$$

The $|c|$ translates by d/c for $d \bmod c$ do not alter the integral, so this is

$$2\zeta(2s)\,y^s \;+\; y^s\sum_{c\neq 0}\frac{1}{|c|^{2s-1}}\int_{\mathbb{R}} \frac{dx}{(x^2+y^2)^s}$$

Replacing x by xy in the integral gives

$$2\zeta(2s)\,y^s \;+\; y^{1-s}\sum_{c\neq 0}\frac{1}{|c|^{2s-1}}\int_{\mathbb{R}}\frac{dx}{(x^2+1)^s} \;=\; 2\zeta(2s)\,y^s \;+\; 2y^{1-s}\zeta(2s-1)\int_{\mathbb{R}}\frac{dx}{(x^2+1)^s}$$

The remaining integral is evaluated in terms of $\Gamma(s)$ by a standard device, as follows. At first just for *real* $s>0$, but then for complex s by the identity principle,

$$\int_0^\infty e^{ty}\,t^s\,\frac{dt}{t} \;=\; y^{-s}\int_0^\infty e^t\,t^s\,\frac{dt}{t} \;=\; y^{-s}\,\Gamma(s)$$

Thus,

$$\int_{\mathbb{R}}\frac{dx}{(x^2+1)^s} \;=\; \frac{1}{\Gamma(s)}\int_{\mathbb{R}}\int_0^\infty e^{-t(x^2+1)}\,t^s\,\frac{dt}{t}\,dx \;=\; \frac{1}{\Gamma(s)}\sqrt{\pi}\int_{\mathbb{R}}\int_0^\infty \frac{1}{\sqrt{t}}\,e^{-(\pi x^2+t)}\,t^s\,\frac{dt}{t}\,dx$$

$$=\; \frac{1}{\Gamma(s)}\sqrt{\pi}\int_0^\infty e^{-t}\,t^{s-\frac{1}{2}}\,\frac{dt}{t} \;=\; \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sqrt{\pi}$$

Thus, the whole constant term of $2\zeta(2s)\,E_s$ is

$$2\zeta(2s)\,y^s \;+\; 2y^{1-s}\zeta(2s-1)\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sqrt{\pi}$$

Dividing by $2\zeta(2s)$ and re-attributing powers of π gives

$$y^s \;+\; y^{1-s}\,\frac{\xi(2s-1)}{\xi(2s)}$$

as claimed. ///

[1.4.2] Remark: In $0<\mathrm{Re}(s)<\frac{1}{2}$, the Eisenstein series has poles at $\rho/2$ for all non-trivial zeros ρ of $\zeta(s)$.

[Epstein 1903] studied $s\to E_s(z_o)$ for fixed $z_o\in\mathfrak{H}$ as a generalized *zeta function*, subsequently called an *Epstein zeta function*. For generic $z_o\in\mathfrak{H}$ the function $s\to E_s(z_o)$ does *not* have an *Euler product*, although it can be construed as a generalized Dirichlet series. Indeed, in a great variety of cases, these Epstein zeta functions substantially *violate* a Riemann Hypothesis: their zeros *do not* all lie on $\mathrm{Re}(s)=\frac{1}{2}$. See [Potter-Titchmarsh 1935], [Davenport-Heilbronn 1936], [Stark 1967], [Chowla-Selberg 1967], [Voronin 1976], [Bombieri-Hejhal 1995].

---

2. *Heegner point periods of Eisenstein series*

Linear combinations or *integrals* of functions $s\to E_s(z_o)$ are called *periods* of the Eisenstein series E_s. Those periods admitting *Euler products* are of special interest. [7]

Hecke and Maaß and others were aware of facts such as

$$E_s(i) \;=\; 2\,\frac{\zeta_{\mathbb{Q}(i)}(s)}{\zeta(2s)}$$

[7] Although some such computations can be done in a seemingly elementary context, as here, they are significantly more intelligible and persuasive when done using the Iwasawa-Tate adele-group description of L-functions, and using a description of Eisenstein series as functions on adele groups $GL_2(\mathbb{A})$. We return to these methods a little later.

where $\zeta_{\mathbb{Q}(i)}(s)$ is the Dedekind zeta function of the Gaussian integers $\mathbb{Z}[i]$: the ring of Gaussian integers $\mathbb{Z}[i]$ is a principal ideal domain with just four units $\{\pm 1, \pm i\}$, and its Dedekind zeta function can be correctly described in a naive fashion:

$$\zeta_{\mathbb{Q}(i)}(s) \;=\; \sum_{\alpha \in \mathbb{Z}[i]^\times \backslash (\mathbb{Z}[i]-0)} \frac{1}{|\alpha|^{2s}} \;=\; \frac{1}{\#\mathbb{Z}[i]^\times} \sum_{0\neq\alpha\in\mathbb{Z}[i]} \frac{1}{|\alpha|^{2s}} \;=\; \tfrac{1}{4} \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2} \frac{1}{(c^2+d^2)^s} \;=\; \tfrac{1}{2}\cdot\zeta(2s)\cdot E_s(i)$$

Because $\mathbb{Z}[i]$ is *Euclidean*, it is a principal ideal domain, and has unique factorization, so $\zeta_{\mathbb{Q}(i)}(s)$ has an Euler product, analogous to $\zeta(s)$:

$$\zeta_{\mathbb{Q}(i)}(s) \;=\; \sum_{\alpha \in \mathbb{Z}[i]^\times \backslash (\mathbb{Z}[i]-0)} \frac{1}{|\alpha|^{2s}} \;=\; \prod_{\text{Gaussian primes } \varpi} \frac{1}{1-|\varpi|^{-2s}}$$

A similar straightforward result holds for rings of algebraic integers $\mathbb{Z}[\omega]$ for ω complex quadratic whenever $\mathbb{Z}[\omega]$ happens to be a principal ideal domain. For example, rings $\mathbb{Z}[\omega]$ with

$$\omega \;=\; \sqrt{-1},\; \sqrt{-2},\; \frac{-1+\sqrt{-3}}{2},\; \frac{-1+\sqrt{-7}}{2},\; \frac{-1+\sqrt{-11}}{2}$$

are demonstrably *Euclidean*, so are principal ideal domains.

[2.1] Heegner points and complex-quadratic periods

The ring of algebraic integers $\mathfrak{o} = \mathbb{Z}[\sqrt{-5}]$, inside the complex quadratic field $k = \mathbb{Q}(\sqrt{-5})$, is *not* a principal ideal domain: [8]

$$2\cdot 3 \;=\; (1+\sqrt{-5})\cdot(1-\sqrt{-5})$$

The zeta function $\zeta_k(s)$ of the ring of algebraic integers $\mathfrak{o} = \mathfrak{o}_k = \mathbb{Z}[\sqrt{-5}]$ is

$$\zeta_k(s) \;=\; \sum_{0\neq\mathfrak{a}\subset\mathfrak{o}} \frac{1}{N\mathfrak{a}^s}$$

where $\mathfrak{a}$ is summed over non-zero ideals in $\mathfrak{o}$, and $N\mathfrak{a} = \#\mathfrak{o}/\mathfrak{a}$. Since $\mathfrak{o}$ is *not* a principal ideal domain, $\zeta_k(s)$ is *not* a *single* value $E_s(z_o)$, but happnes to be essentially the sum of *two* values of Epstein zetas: quite unobviously,

$$|\sqrt{-5}|^s \cdot \frac{\zeta_k(s)}{\zeta(2s)} \;=\; E_s(\sqrt{-5}) + E_s\Big(\frac{1+\sqrt{-5}}{2}\Big)$$

$$=\; |\sqrt{-5}|^s \cdot \Big(\sum_{\text{coprime } m,n \bmod \pm 1} \frac{1}{(m^2+5n^2)^s} \;+\; \sum_{\text{coprime } m,n \bmod \pm 1} \frac{1}{(2m^2+2mn+3n^2)^s} \Big)$$

Note that the discriminants of both quadratics x^2+5 and $2x^2+2x+3$ are -20.

[2.1.1] Theorem: For a complex quadratic field extension $k = \mathbb{Q}(\sqrt{-D})$ with ring of algebraic integers $\mathfrak{o} = \mathfrak{o}_k$ having h different $\mathfrak{o}$-isomorphism classes of non-zero ideals, [9] there are $z_1, \ldots, z_h$ in a *strict* version of the fundamental domain,

$$F^{\text{str}} \;=\; \{z = x+iy \in \mathfrak{H} \;:\; |z|\geq 1,\; -\tfrac{1}{2} < x \leq \tfrac{1}{2},\; \text{and } |z|>1 \text{ if } x<0\}$$

---

[8] The Galois norm $N(a+b\sqrt{-5}) = a^2+5b^2$ is *multiplicative*, and (easily checked) $N(\alpha)=1$ for $\alpha\in\mathbb{Z}[\sqrt{-5}]$ if and only α is a unit. For integers a,b, we have $a^2+5b^2=1$ exactly for $b=0$, $a=\pm 1$. Since $N(2)=4$, 2 is irreducible since there is no $\alpha\in\mathfrak{o}$ with $N(\alpha)=2$. Similarly, 3 and $1\pm\sqrt{-5}$ are irreducible. Similarly, none of the other three factors appearing can be further factored. The left-hand factors $2,3$ do *not* differ by units from the right-hand factors $1\pm\sqrt{-5}$, so 6 has two different factorizations.

[9] The number of $\mathfrak{o}$-isomorphism classes of non-zero ideals in $\mathfrak{o}$ is the *class number* of $\mathfrak{o}$.

for $SL_2(\mathbb{Z})$ on $\mathfrak{H}$, such that

$$\frac{(|\sqrt{-D}|/2)^s}{\zeta(2s)}\,\zeta_k(s) \;=\; E_s(z_1) + E_s(z_2) + \ldots + E_s(z_h)$$

[2.1.2] Remark: This characterizes the *Heegner points* $z_1, \ldots, z_h$ associated to $k = \mathbb{Q}(\sqrt{-D})$. [10]

Proof: Such expressions are obtained as follows. The sum expression for the usual zeta-function of $\mathfrak{o}$ breaks into subsums according to $\mathfrak{o}$-isomorphism classes of ideals:

$$\zeta_k(s) \;=\; \sum_{\text{classes } \mathfrak{a}} \sum_{\mathfrak{b}\approx\mathfrak{a}} \frac{1}{(N\mathfrak{b})^s} \;=\; \sum_{\text{classes } \mathfrak{a}} \sum_{\mathfrak{b}\approx\mathfrak{a}} \frac{1}{N(\mathfrak{b}\mathfrak{a}^{-1})^s \cdot (N\mathfrak{a})^s} \;=\; \sum_{\text{classes } \mathfrak{a}} \frac{1}{(N\mathfrak{a})^s} \sum_{\mathfrak{b}\approx\mathfrak{a}} \frac{1}{N(\mathfrak{b}\mathfrak{a}^{-1})^s}$$

An $\mathfrak{o}$-isomorphism $f : \mathfrak{a} \to \mathfrak{b}$ from one non-zero $\mathfrak{o}$-ideal to another is given by some $\theta \in k^\times$, since the k-linear extension to $f : k \to k$ is a k-linear automorphism of a one-dimensional k-vectorspace. That is, $\mathfrak{b} = \theta \cdot \mathfrak{a}$. Since $\theta \cdot \mathfrak{a} = \mathfrak{b} \subset \mathfrak{o}$, necessarily $\theta \in \mathfrak{a}^{-1}$, where we adopt the standard convention

$$\mathfrak{a}^{-1} \;=\; \{\beta \in k \;:\; \beta \cdot \mathfrak{a} \subset \mathfrak{o}\}$$

In such terms,

$$\zeta_k(s) \;=\; \sum_{\text{classes } \mathfrak{a}} \; \sum_{0\neq\theta\in\mathfrak{a}^{-1},\ \text{mod } \mathfrak{o}^\times} \frac{1}{N(\theta\cdot\mathfrak{a})^s}$$

The quadratic form

$$Q_\mathfrak{a}(\theta) \;=\; N(\theta \cdot \mathfrak{a}) \qquad\qquad (\text{on } \theta \in \mathfrak{a}^{-1})$$

on the $\mathbb{Z}$-module $\mathfrak{a}^{-1}$ is positive-definite and turns out to have *discriminant* $-D$, in the following sense. Fix a $\mathbb{Z}$-basis e_1, e_2 of $\mathfrak{a}^{-1}$, and take integers A, B, C so that for all $m, n \in \mathbb{Z}$

$$N\big((me_1 + ne_2) \cdot \mathfrak{a}^{-1}\big) \;=\; Am^2 + Bmn + Cn^2$$

It turns out that $B^2 - 4AC = -D$: proof of this point requires a little algebraic number theory, which we postpone. Put

$$z \;=\; z_\mathfrak{a} \;=\; \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

The quadratic form $Q_\mathfrak{a}$ is recovered from $z = x + iy$:

$$\frac{|m + zn|^2}{y} \;=\; \frac{m^2 + 2xmn + |z|^2 n^2}{y} \;=\; \frac{m^2 + 2\frac{B}{2A}mn + \frac{4AC}{4A^2}n^2}{|\sqrt{-D}|/2A} \;=\; \frac{Am^2 + Bmn + Cn^2}{|\sqrt{-D}|/2} \;=\; \frac{Q_\mathfrak{a}(m,n)}{|\sqrt{-D}|/2}$$

That is,

$$\Big(\frac{|\sqrt{-D}|}{2}\Big)^s \sum_{0\neq\theta\in\mathfrak{a}^{-1}\ \text{mod } \pm 1} \frac{1}{N(\theta\cdot\mathfrak{a})^s} \;=\; E_s(z_\mathfrak{a}) \qquad\qquad (\text{with } z_\mathfrak{a} = \tfrac{B\pm\sqrt{B^2-4AC}}{2A})$$

Change-of-basis in the $\mathbb{Z}$-module $\mathfrak{a}^{-1}$ changes the coefficients A, B, C of the quadratic form $Q_\mathfrak{a}$, which changes $z_\mathfrak{a} \in \mathfrak{H}$ by the action of $SL_2(\mathbb{Z})$ on $\mathfrak{H}$: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

---

[10] We should restrict attention to *fundamental discriminants* $-D < 0$, meaning that *square-free* $-d < 0$ is replaced by $-D = -4d$ for $-d = 2, 3 \bmod 4$. Thus, for example, -1 is replaced by -4, -2 is replaced by -8, and -5 is replaced by -20. In terms of algebraic number theory, this makes the discriminant the square of the volume of the quotient $\mathbb{C}/\mathfrak{o}$, where $\mathfrak{o}$ is the ring of algebraic integers in $\mathbb{Q}(\sqrt{-D})$. The ring $\mathfrak{o} = \mathfrak{o}_k$ of *algebraic integers* in k is the collection of $\alpha \in k$ satisfying a *monic* equation $\alpha^2 + b\alpha + c = 0$ with $a, b \in \mathbb{Z}$.

$$\frac{Q_{\mathfrak{a}}((m,n)\cdot\gamma)}{|\sqrt{-D}|/2} \;=\; \frac{Q_{\mathfrak{a}}(ma+nc,\, mb+nd)}{|\sqrt{-D}|/2} \;=\; \frac{|(ma+nc)+z_{\mathfrak{a}}(mb+nd)|^2}{y_{\mathfrak{a}}} \;=\; \frac{|m(a+bz_{\mathfrak{a}})+(c+dz_{\mathfrak{a}})n|^2}{y_{\mathfrak{a}}}$$

$$=\; \frac{|m+\frac{dz_{\mathfrak{a}}+c}{bz_{\mathfrak{a}}+a}\,n|^2}{y_{\mathfrak{a}}/|bz_{\mathfrak{a}}+a|^2} \;=\; \frac{|m+\frac{dz_{\mathfrak{a}}+c}{bz_{\mathfrak{a}}+a}\,n|^2}{\mathrm{Im}\left(\frac{dz_{\mathfrak{a}}+c}{bz_{\mathfrak{a}}+a}\right)}$$

In particular, by changing the $\mathbb{Z}$-basis of $\mathfrak{a}^{-1}$ we can put $z_{\mathfrak{a}}$ into the strict fundamental domain F^{str} for Γ, and uniquely so for each isomorphism class $\mathfrak{a}$ of non-zero $\mathfrak{o}$-ideals. These points $z_{\mathfrak{a}}$ are the *Heegner points* attached to $-D<0$. For

$$z_{\mathfrak{a}} \;=\; \frac{B\pm\sqrt{B^2-4AC}}{2A} \;\in\; F^{\mathrm{str}}$$

the corresponding quadratic form is said to be *reduced*. In terms of the coefficients A,B,C in the quadratic form $Am^2+Bmn+Cn^2$, the *reduced* condition is

$$-A<B\le A,\;\; A\le C,\;\; \text{and}\;\; A<C \;\;\text{if}\;\; B<0 \qquad\qquad (\text{all integers, with } B^2-4AC=-D)$$

From $-A<B\le A\le C$ and $B^2-4AC=-D$ follows

$$D \;=\; 4AC-B^2 \;\ge\; 3AC \;\ge\; A^2$$

Thus, there are only finitely-many values of A, hence also of B,C, for given $-D$.

This proves that there are only finitely-many Heegner points $z_{\mathfrak{a}}\in F^{\mathrm{str}}$ for given $-D<0$. This is *finiteness of class number*, that is, finiteness of the $\mathfrak{o}$-isomorphism classes of non-zero ideals in $\mathfrak{o}$. ///

[2.1.3] Remark: Proof that the discriminants of all quadratic forms $\theta\to N(\theta\cdot\mathfrak{a})$ for $\theta\in\mathfrak{a}^{-1}$ are $-D$ was omitted in the above discussion.

---

3. *Closed geodesic periods of Eisenstein series*

For *real* quadratic fields $k=\mathbb{Q}(\sqrt{D})$ with discriminants $D>0$ and ring of algebraic integers $\mathfrak{o}=\mathfrak{o}_k$, the Dedekind zeta function

$$\zeta_k(s) \;=\; \sum_{0\ne\mathfrak{a}\subset\mathfrak{o}} \frac{1}{|N\mathfrak{a}|^s}$$

defined in terms of the non-zero ideals in $\mathfrak{o}$ are *integrals* of $E_s(z)$ over z in a finite sum of *closed geodesic curves* on $\Gamma\backslash\mathfrak{H}$. For real quadratic fields, the norm can be negative, hence the necessity of the absolute value. The number of geodesics proves to be the *class number* of $\mathfrak{o}$. Apparently, the Heegner points for the complex quadratic case become closed geodesics for the real quadratic case. To isolate the new phenomenon, take $k=\mathbb{Q}(\sqrt{D})$ with ring of integers $\mathfrak{o}=\mathfrak{o}_k$ a principal ideal domain, that is, with class number 1. For example, $k=\mathbb{Q}(\sqrt{2})$ with $\mathfrak{o}=\mathbb{Z}[\sqrt{2}]$ is provably *Euclidean*, so is a principal ideal domain.

As a very special case of Dirichlet's Units Theorem: in real-quadratic rings of integers the units group is *infinite*. Specifically, modulo torsion (which is just ±1) the units group is free on one generator. For example,

$$\mathbb{Z}[\sqrt{2}]^\times \;=\; \{\pm \text{ powers of } 1+\sqrt{2}\}$$

The useful description of the relevant geodesic curves is *not* in *metric-geometry* terms,[11] but *group-theoretic*, more essentially so than the complex-quadratic case. Imbed $k^\times$ in $GL_2(\mathbb{Q})$ by

$$a+b\sqrt{D} \;\longrightarrow\; \begin{pmatrix} a & Db \\ b & a\end{pmatrix} \qquad\qquad (\text{for } a,b\in\mathbb{Q})$$

[11] An elementary description of all full geodesics on $\mathfrak{H}$ is that they are arcs of circles that are orthogonal to the real axis, together with vertical lines orthogonal to the real axis.

Let

$$H \;=\; \{\begin{pmatrix} a & Db \\ b & a \end{pmatrix} \;:\; a,b\in\mathbb{R},\; a^2-Db^2=1\} \;=\; \{\begin{pmatrix} \cosh t & \sqrt{D}\sinh t \\ \frac{\sinh t}{\sqrt{D}} & \cosh t \end{pmatrix} \;:\; t\in\mathbb{R}\}$$

For brevity, write

$$h_t \;=\; \{\begin{pmatrix} \cosh t & \sqrt{D}\sinh t \\ \frac{\sinh t}{\sqrt{D}} & \cosh t \end{pmatrix}$$

The fact that there are non-trivial units in $\mathfrak{o}$ is equivalent to the *compactness* of the quotient $(\Gamma\cap H)\backslash H$. Noting that

$$h_t\cdot i\sqrt{D} \;=\; \frac{\cosh t\cdot i\sqrt{D}+\sqrt{D}\sinh t}{\frac{\sinh t}{\sqrt{D}}\cdot i\sqrt{D}+\cosh t} \;=\; \sqrt{D}\cdot\frac{i\cosh t+\sinh t}{i\sinh t+\cosh t}$$

the $(\Gamma\cap H)\backslash H$-orbit of the point $z_o=i\sqrt{D}\in\mathfrak{H}$ in $\Gamma\backslash\mathfrak{H}$ is

$$Y \;=\; (\Gamma\cap H)\backslash\{h\cdot z_o : h\in H\} \;=\; (\Gamma\cap H)\Big\backslash\Big\{\sqrt{D}\cdot\frac{i\cosh t+\sinh t}{i\sinh t+\cosh t} \;:\; t\in\mathbb{R}\Big\}$$

In fact, for $t_o>0$ such that e^{t_o} is the smallest non-trivial unit $a+b\sqrt{D}>1$, the corresponding matrix h_{t_o} is in $\Gamma\cap H$, and so *closes-up* the line swept out by H in its action on $z_o\in\mathfrak{H}$.

[3.0.1] Theorem: Take $D>0$ square-free, $D=2 \bmod 4$ or $D=3 \bmod 4$, so that the ring of algebraic integers $\mathfrak{o}=\mathfrak{o}_k$ in $k=\mathbb{Q}(\sqrt{D})$ is $\mathfrak{o}=\mathbb{Z}[\sqrt{D}]$. Suppose for simplicity that $\mathfrak{o}$ is a principal ideal domain. Then the geodesic period is

$$\int_{(\Gamma\cap H)\backslash H} E_s(h\cdot z_o)\,dh \;=\; 2^s\,\frac{\sqrt{D}^{\,s}\,\Gamma(\frac{s}{2})\,\Gamma(\frac{s}{2})}{\Gamma(s)}\cdot\frac{\zeta_k(s)}{\zeta(2s)}$$

Proof: The *period* is

$$\int_{(\Gamma\cap H)\backslash H} E_s(h\cdot z_o)\,dh \;=\; \int_{(\Gamma\cap H)\backslash H}\;\sum_{\gamma\in(\Gamma\cap P)\backslash\Gamma} \mathrm{Im}\,(\gamma h(z_o))^s\,dh$$

$$=\; \sum_{\alpha\in(\Gamma\cap P)\backslash\Gamma/(\Gamma\cap H)} \int_{(\Gamma\cap H)\backslash H}\;\sum_{\beta\in(\alpha^{-1}(\Gamma\cap P)\alpha\cap H)\backslash(\Gamma\cap H)} \mathrm{Im}\,(\alpha\beta h(z_o))^s\,dh$$

Since eigenvalues of elements of $\Gamma\cap P$ are integers, and eigenvalues of (non-trivial) elements of $\Gamma\cap H$ are (non-trivial) units in $\mathfrak{o}$,

$$\alpha^{-1}(\Gamma\cap P)\alpha\cap H \;=\; \{\pm 1\}$$

and

$$\int_{(\Gamma\cap H)\backslash H} E_s(h\cdot z_o)\,dh \;=\; \sum_{\alpha\in(\Gamma\cap P)\backslash\Gamma/(\Gamma\cap H)} \int_{(\Gamma\cap H)\backslash H}\;\sum_{\beta\in\{\pm1\}\backslash(\Gamma\cap H)} \mathrm{Im}\,(\alpha\beta h(z_o))^s\,dh$$

$$=\; \sum_{\alpha\in(\Gamma\cap P)\backslash\Gamma/(\Gamma\cap H)} \int_{\{\pm1\}\backslash H} \mathrm{Im}\,(\alpha h(z_o))^s\,dh \;=\; \sum_{\text{coprime }(c,d)/(\Gamma\cap H)} \int_{\{\pm1\}\backslash H} \frac{\mathrm{Im}\,(hz_o)^s}{|c\cdot hz_o+d|^{2s}}\,dh \qquad \text{(unwinding!)}$$

$$=\; \frac{1}{2\zeta(2s)} \sum_{\{(c,d)\neq(0,0)\}/(\Gamma\cap H)} \int_{\{\pm1\}\backslash H} \frac{\mathrm{Im}\,(hz_o)^s}{|c\cdot hz_o+d|^{2s}}\,dh \qquad \text{(removing coprimality condition on } c,d\text{)}$$

Identifying $(c,d)\in\mathbb{Z}^2$ with $c\sqrt{D}+d$ is compatible with the identification $a+b\sqrt{D}\to\begin{pmatrix} a & Db \\ b & a\end{pmatrix}$, in the sense that

$$(c\sqrt{D}+d)\times(a+b\sqrt{D}) \;\longrightarrow\; (\,c \quad d\,)\begin{pmatrix} a & Db \\ b & a\end{pmatrix}$$

$$=\;(\,ca+db \quad cbD+da\,) \;\longleftarrow\; (ca+db)\sqrt{D}+(cbD+da) \;=\; (c\sqrt{D}+d)\cdot(a+b\sqrt{D})$$

Thus, $\{(c,d)\neq(0,0)\}/(\Gamma\cap H)$ is identified with $(\mathfrak{o}-0)/\mathfrak{o}^1$, where $\mathfrak{o}^1\subset\mathfrak{o}^\times$ is the group of units with norm 1 (as opposed to -1). Rearrange a little:

$$\frac{\mathrm{Im}\,(hz_o)^s}{|c\cdot hz_o+d|^{2s}} \;=\; \frac{\sqrt{D}^s}{\Big|c\sqrt{D}(i\cosh t+\sinh t)+d(i\sinh t+\cosh t)\Big|^{2s}}$$

and

$$\Big|c\sqrt{D}(i\cosh t+\sinh t)+d(i\sinh t+\cosh t)\Big|^2 \;=\; (c\sqrt{D}\cosh t+d\sinh t)^2+(c\sqrt{D}\sinh t+d\cosh t)^2$$

$$=\; c^2D(\cosh^2 t+\sinh^2 t)+4cd\sqrt{D}\cosh t\sinh t+d^2(\sinh^2 t+\cosh^2 t)$$

$$=\; \tfrac{1}{2}\cdot\Big((c^2D+2cd\sqrt{D}+d^2)e^{2t}+(c^2D-2cd\sqrt{D}+d^2)e^{-2t}\Big)$$

The integral is the seemingly-daunting

$$\int_{\mathbb{R}} \frac{2^s\sqrt{D}^s\;dt}{\Big((c^2D+2cd\sqrt{D}+d^2)e^{2t}+(c^2D-2cd\sqrt{D}+d^2)e^{-2t}\Big)^s}$$

Up to constants, this is of the form

$$\int_{\mathbb{R}} \frac{dt}{(Ae^{2t}+Be^{-2t})^s}$$

Recall the trick

$$y^{-s}\cdot\Gamma(s) \;=\; y^{-s}\int_0^\infty u^s\,e^{-u}\,\frac{du}{u} \;=\; \int_0^\infty u^s\,e^{-u\cdot y}\,\frac{du}{u} \qquad\qquad (\text{for } y>0)$$

Thus,

$$\Gamma(s)\int_{\mathbb{R}} \frac{dt}{(Ae^{2t}+Be^{-2t})^s} \;=\; \int_{\mathbb{R}}\int_0^\infty u^s\;e^{-u(Ae^{2t}+Be^{-2t})}\;\frac{du}{u}\;dt$$

Letting $t=\frac{\log v}{2}$ makes this

$$\tfrac{1}{2}\int_0^\infty\int_0^\infty u^s\;e^{-u(Av+Bv^{-1})}\;\frac{du}{u}\;\frac{dv}{v} \;=\; \tfrac{1}{2}\int_0^\infty\int_0^\infty u^s\;v^s\;e^{-u(Av^2+B)}\;\frac{du}{u}\;\frac{dv}{v}$$

by replacing u by uv. Replacing v by $\sqrt{v}$ makes this

$$\frac{1}{4}\int_0^\infty\int_0^\infty u^s\;v^{\frac{s}{2}}\;e^{-u(Av+B)}\;\frac{du}{u}\;\frac{dv}{v} \;=\; \frac{1}{4}\int_0^\infty\int_0^\infty u^{\frac{s}{2}}\;v^{\frac{s}{2}}\;e^{-(Av+Bu)}\;\frac{du}{u}\;\frac{dv}{v}$$

by replacing v by v/u. Replacing v by v/A and u by u/B gives

$$\int_{\mathbb{R}} \frac{dt}{(Ae^{2t}+Be^{-2t})^s} \;=\; \frac{\Gamma(\frac{s}{2})\,\Gamma(\frac{s}{2})}{4\cdot\Gamma(s)}\,A^{-\frac{s}{2}}\,B^{-\frac{s}{2}}$$

Thus, the seemingly-daunting integral is

$$2^s\,\sqrt{D}^s\,\frac{\Gamma(\frac{s}{2})\,\Gamma(\frac{s}{2})}{4\cdot\Gamma(s)}\,(c^2D+2cd\sqrt{D}+d^2)^{-\frac{s}{2}}\,(c^2D-2cd\sqrt{D}+d^2)^{-\frac{s}{2}} \;=\; 2^s\,\sqrt{D}^s\,\frac{\Gamma(\frac{s}{2})\,\Gamma(\frac{s}{2})}{4\cdot\Gamma(s)}\,\frac{1}{|c^2D-d^2|^s}$$

Thus, up to a constant factor, the geodesic period is

$$\int_{(\Gamma\cap H)\backslash H} E_s(h\cdot z_o)\,dh \;=\; 2^s\,\sqrt{D}^s\,\frac{\Gamma(\frac{s}{2})\,\Gamma(\frac{s}{2})}{4\cdot\Gamma(s)}\,\frac{1}{\zeta(2s)}\sum_{c,d}\frac{1}{|c^2D-d^2|^s}$$

where the sum is, in effect, over integers modulo units, as in the Dedekind zeta for a principal ideal domain. That is,

$$\int_{(\Gamma\cap H)\backslash H} E_s(h\cdot z_o)\,dh \;=\; 2^s\,\frac{\sqrt{D}^{\,s}\,\Gamma(\frac{s}{2})\,\Gamma(\frac{s}{2})}{\Gamma(s)}\cdot\frac{\zeta_k(s)}{\zeta(2s)}$$

The Gamma factors in the numerator are the appropriate ones for the real quadratic field k, and the Gamma factor in the denominator is appropriate for $\zeta(2s)$. The corresponding powers of π happen to cancel completely in the ratio. ///

4. *Cuspidal waveforms, Fourier expansions*

The *idea* of a waveform *cuspform* f for $\Gamma = SL_2(\mathbb{Z})$ is that f is Γ-invariant and vanishing *constant term*: [12]

$$\int_0^1 f(x+iy)\,dx \;=\; 0$$

We may also require that $f\in L^2(\Gamma\backslash\mathfrak{H})$ with respect to the invariant measure $\frac{dx\,dy}{y^2}$, or, alternatively, be *of moderate growth*. Any similar conditions characterize vector spaces of cuspforms with additional properties. Relations among the additional properties are typically non-trivial, so should not be expected to be obvious or elementary.

We may also require that f be a $\Delta = \Delta^{\mathfrak{H}}$ eigenfunction. Linear combinations of eigenfunctions with different eigenvalues usually fail to produce eigenfunctions, so the eigenfunction requirement is often weakened to a condition that does allow finite sums, namely, Δ-*finiteness*: f is Δ-finite when the *ideal* $I_f \subset \mathbb{C}[\Delta]$ of polynomials in Δ annihilating f is not $\{0\}$, equivalently, $\mathbb{C}[\Delta]/I_f$ is finite-dimensional over $\mathbb{C}$.

Exactly which properties are assumed of a *cuspform* can only be determined from context. The essential point is vanishing of the constant term $c_Pf(iy) = \int_0^1 f(x+iy)\,dx$.

[4.0.1] Theorem: Let $f\in C^\infty(\Gamma\backslash\mathfrak{H})$ be a Δ-eigenfunction with eigenvalue λ, and of moderate growth in the sense that $f(x+iy)\ll y^A$ as $y\to+\infty$, for some exponent A. Assume the cuspidal condition $c_Pf=0$. Then f has a Fourier expansion

$$f(x+iy) \;=\; \sum_{n\neq 0} c_n\,u_\lambda(|n|y)\,e^{2\pi inx}$$

where $y\to u_\lambda(y)$ is the unique (up to scalar multiples) solution of $u''-\Big(\frac{\lambda}{y^2}+4\pi^2n^2\Big)\cdot u=0$ going to 0 as $y\to+\infty$. [13] Such a cuspform f is *of exponential decrease* as $y\to+\infty$, in the sense that $f(x+iy)\ll_f e^{-2\pi y}$ as $y\to+\infty$. The eigenvalue λ is non-positive *real*. The numerical coefficients c_n are *bounded*.

[12] A minor technical issue is that for f merely *measurable*, for example, the integral defining the constant term c_Pf is not defined everywhere. That is, the sense of $c_Pf=0$ may require elaboration, depending on context.

[13] This function u_λ is known in the special-functions literature as essentially a *Bessel function*, and the differential equation it satisfies is essentially a *Bessel equation*. Many special things are known about these functions, but it is preferable to develop their properties from more-standard ideas, as we attempt.

Proof: The smoothness assumption guarantees a nicely-converging Fourier expansion in x, since f is invariant under $z \to z+1$:

$$f(x+iy) \;=\; \sum_{n\in\mathbb{Z}} C_n(y)\, e^{2\pi i n x}$$

for some functions $C_n(y)$. The cuspidal condition is $C_0(y) = 0$. The assumption of moderate growth $f(x+iy) \ll y^A$ implies that the coefficient functions C_n are of moderate growth:

$$C_n(y) \;=\; \int_0^1 e^{-2\pi i n x}\, f(x+iy)\, dx \;\ll\; \int_0^1 y^A\, dx \;=\; y^A$$

The condition $\Delta f = \lambda \cdot f$ gives

$$\sum_{n\neq 0} y^2 \Big(C_n''(y)\, e^{2\pi i n x} \;+\; C_n(y)\, (2\pi i n)^2\, e^{2\pi i n x}\Big) \;=\; \sum_{n\neq 0} \lambda \cdot C_n(y)\, e^{2\pi i n x}$$

By uniqueness of Fourier expansions, this is equivalent to

$$y^2 \Big(C_n''(y) \;-\; 4\pi^2 n^2\, C_n(y)\Big) \;=\; \lambda \cdot C_n(y) \qquad\qquad (\text{for all } n \in \mathbb{Z})$$

That is, C_n is a solution to [14]

$$u'' - \Big(\frac{\lambda}{y^2} + 4\pi^2 n^2\Big) \cdot u \;=\; 0$$

By changing variables, we see that for u_1 a solution with $n = 1$, $y \to u_1(|n|y)$ is a solution for general $n \neq 0$. Thus, to understand solutions for general $n \neq 0$ it suffices to treat $n = 1$.

To understand the behavior of solutions of $u'' - \Big(\frac{\lambda}{y^2} + 4\pi^2\Big) \cdot u = 0$ as $y \to +\infty$, a natural heuristic is to replace the coefficients of the equation by their limiting values as $y \to +\infty$, sometimes termed *freezing* the equation (at $y = +\infty$). As discussed in detail in the supplement on differential equations with *irregular singular points*, this heuristic is correct: for $n \neq 0$, there is a unique (up to scalars) solution u_λ on $(0, +\infty)$ with asymptotic

$$u_\lambda(y) \;\sim\; e^{-2\pi y}$$

and another solution v_λ with $v_\lambda(y) \sim e^{2\pi y}$. Any linear combination of u_λ, v_λ non-trivially involving v_λ blows up exponentially. Since $C_n(y)$ is of moderate growth, up to scalars it must be the rapidly-decreasing solution $u_\lambda(|n|y)$.

Thus, there are constants c_n such that

$$f(x+iy) \;=\; \sum_{n\neq 0} c_n\, u_\lambda(|n|y)\, e^{2\pi i n x}$$

The moderate growth condition gives

$$c_n\, u_\lambda(|n|y) \;=\; \int_0^1 e^{-2\pi i n x}\, f(x+iy)\, dx \;\ll\; \int_0^1 y^A\, dx \;=\; y^A \qquad\qquad (\text{as } y \to +\infty)$$

and the asymptotics of u_λ give a bad initial estimate for large y_o:

$$|c_n| \;\ll_{y_o}\; \frac{y_o^A}{u_\lambda(|n|y_o)} \;\ll_\lambda\; \frac{y_o^A}{e^{-2\pi|n|y_o}} \;\ll\; y_o^A\, e^{2\pi|n|y_o}$$

[14] For $n = 0$ the differential equation becomes $u'' - \frac{\lambda}{y^2} \cdot u = 0$. This is *Eulerian*, in the sense that it has solutions of the form $u(y) = y^s$, where $s \in \mathbb{C}$ can be found by substituting and solving the resulting equation $s(s-1) - \lambda = 0$ for s in terms of λ.

Then

$$|f(x+iy)| \ll \sum_{n\neq 0} y_o^A\, e^{2\pi|n|y_o} \cdot e^{-2\pi|n|y} \ll \sum_{n\neq 0} e^{-2\pi|n|(y-y_o)} = \frac{2\, e^{-2\pi(y-y_o)}}{1-e^{-2\pi(y-y_o)}} \ll e^{-2\pi y}$$

Since f is exponentially decreasing as $y \to +\infty$, integration by parts twice leaves no boundary terms, and

$$\lambda \cdot \int_{\Gamma\backslash\mathfrak{H}} f\cdot\overline{f}\ \frac{dx\,dy}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \Delta f\cdot\overline{f}\ \frac{dx\,dy}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} f\cdot\overline{\Delta f}\ \frac{dx\,dy}{y^2} = \overline{\lambda}\cdot\int_{\Gamma\backslash\mathfrak{H}} f\cdot\overline{f}\ \frac{dx\,dy}{y^2}$$

Thus, $\lambda \in \mathbb{R}$. Further, integration by parts once gives

$$\lambda \cdot \int_{\Gamma\backslash\mathfrak{H}} f\cdot\overline{f}\ \frac{dx\,dy}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} -\Big(\frac{\partial f}{\partial x}\Big)^2 - \Big(\frac{\partial f}{\partial y}\Big)^2\ dx\,dy \le 0$$

so $\lambda \le 0$. Since f is *bounded* on a fundamental domain and Γ-invariant, it is bounded on $\mathfrak{H}$, and

$$|c_n\, u_\lambda(|n|y)| = \Big|\int_0^1 e^{-2\pi i n x}\, f(x+iy)\, dx\Big| \ll \int_0^1 1\, dx = 1$$

and $|c_n| \ll 1/|u_\lambda(|n|y)|$ for all $y>0$. Taking $y = y_o/|n|$ for some y_o such that $u_\lambda(y_o)\neq 0$ gives

$$|c_n| \ll \frac{1}{|u_\lambda(|n|\cdot\frac{y_o}{|n|})|} = \frac{1}{u_\lambda(y_o)} \ll_\lambda 1$$

---

5. *Fourier expansions of Eisenstein series*

The explicit and relatively elementary nature of Eisenstein series gives further information about the solution u_λ to the eigenfunction equation $y^2(u''-4\pi^2 u) = \lambda\cdot u$, in the course of determining the Fourier expansion of $E_s(x+iy)$ in x.

[5.1] Fourier expansion of E_s We already computed the 0^{th} Fourier component: it is

$$\int_0^1 E_s(x+iy)\, dx = y^s + \frac{\xi(2s-1)}{\xi(2s)}\, y^{1-s} \qquad\qquad (\text{with } \xi(2s) = \pi^{-s}\,\Gamma(s)\,\zeta(2s))$$

As on other occasions, it is easiest to treat

$$2\,\zeta(2s)\, E_s(z) = \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2} \frac{y^s}{|cz+d|^{2s}}$$

For $0\neq n\in\mathbb{Z}$, the n^{th} Fourier component is

$$2\,\zeta(2s)\cdot\int_0^1 e^{-2\pi i n x}\ E_s(x+iy)\, dx$$

Since $n\neq 0$, the $c=0$ subsum does not contribute to this. In the region of convergence of the sum expression for E_s, we break the sum into fragments stable under $x\to x+1$:

$$2\,\zeta(2s)\cdot E_s(x+iy) = y^s\sum_{c\neq 0}\frac{1}{|c|^{2s}}\sum_{d\in\mathbb{Z}}\frac{1}{|x+\frac{d}{c}+iy|^{2s}} = y^s\sum_{c\neq 0}\frac{1}{|c|^{2s}}\sum_{d \bmod c}\ \sum_{\ell\in\mathbb{Z}}\frac{1}{|x+\ell+\frac{d}{c}+iy|^{2s}}$$

The inner sum over $\ell \in \mathbb{Z}$ is periodic in x. Its n^{th} Fourier component is

$$\int_0^1 e^{-2\pi inx} \sum_{\ell\in\mathbb{Z}} \frac{1}{|x+\ell+\frac{d}{c}+iy|^{2s}}\,dx \;=\; \int_{\mathbb{R}} e^{-2\pi inx}\,\frac{1}{|x+\frac{d}{c}+iy|^{2s}}\,dx \qquad\qquad \text{(by unwinding)}$$

The sum over d mod c is

$$\sum_{d \bmod c} \int_{\mathbb{R}} e^{-2\pi inx}\,\frac{1}{|x+\frac{d}{c}+iy|^{2s}}\,dx \;=\; \sum_{d \bmod c} e^{2\pi ind/c} \int_{\mathbb{R}} e^{-2\pi inx}\,\frac{1}{|x+iy|^{2s}}\,dx$$

The sum over d is $|c|$ for $c|n$ and 0 otherwise, by the cancellation lemma.

Unlike the holomorphic Eisenstein series, the latter integral is not an elementary function for $n \neq 0$. Nevertheless, it can be rearranged to give an expression that meromorphically continues in s, and shows other symmetries. First, replacing x by xy,

$$\int_{\mathbb{R}} e^{-2\pi inx}\,\frac{1}{|x+iy|^{2s}}\,dx \;=\; y^{1-2s}\int_{\mathbb{R}} e^{-2\pi inxy}\,\frac{1}{|x+i|^{2s}}\,dx \;=\; y^{1-2s}\int_{\mathbb{R}} e^{-2\pi inxy}\,\frac{1}{(x^2+1)^s}\,dx$$

Using the trick

$$A^{-s}\cdot\Gamma(s) \;=\; A^{-s}\cdot\int_0^\infty t^s\,e^{-t}\,\frac{dt}{t} \;=\; \int_0^\infty t^s\,e^{-tA}\,\frac{dt}{t} \qquad\qquad \text{(for } A>0)$$

we have

$$\int_{\mathbb{R}} e^{-2\pi inxy}\,\frac{1}{(x^2+1)^s}\,dx \;=\; \frac{1}{\Gamma(s)}\int_{\mathbb{R}} e^{-2\pi inxy}\int_0^\infty t^s\,e^{-t(x^2+1)}\,\frac{dt}{t}\,dx$$

Changing the order of integration, and using the good behavior of suitable Gaussians under Fourier transform,

$$\int_{\mathbb{R}} e^{-2\pi i\xi x}\,e^{-\pi x^2}\,dx \;=\; e^{-\pi\xi^2}$$

replace x by $x\cdot\frac{\sqrt{\pi}}{\sqrt{t}}$ and take the Fourier transform, to obtain

$$\frac{\sqrt{\pi}}{\Gamma(s)}\int_0^\infty t^{s-\frac{1}{2}}\,e^{-t}\int_{\mathbb{R}} e^{-2\pi inxy\frac{\sqrt{\pi}}{\sqrt{t}}}\,e^{-\pi x^2}\,dx\,\frac{dt}{t} \;=\; \frac{\sqrt{\pi}}{\Gamma(s)}\int_0^\infty t^{s-\frac{1}{2}}\,e^{-t-\pi\cdot\frac{\pi}{t}(ny)^2}\,\frac{dt}{t}$$

Replacing t by $t\cdot\pi|n|y$ gives

$$\frac{\sqrt{\pi}\cdot(\pi|n|y)^{s-\frac{1}{2}}}{\Gamma(s)}\int_0^\infty t^{s-\frac{1}{2}}\,e^{-(t+\frac{1}{t})\pi|n|y}\,\frac{dt}{t}$$

Restoring the factors of y^s and y^{1-2s} dropped along the way, this is

$$\frac{\pi^s\,|n|^{s-\frac{1}{2}}\,\sqrt{y}}{\Gamma(s)}\int_0^\infty t^{s-\frac{1}{2}}\,e^{-(t+\frac{1}{t})\pi|n|y}\,\frac{dt}{t}$$

The integral is invariant under $s \to 1-s$, by replacing t by $1/t$, and is nicely convergent for all $s\in\mathbb{C}$.

Thus, the n^{th} Fourier component of $2\,\zeta(2s)\,E_s(x+iy)$ is

$$n^{th}\text{ Fourier component } 2\,\zeta(2s)\,E_s(x+iy) \;=\; \Big(\sum_{c|n}\frac{1}{|c|^{2s-1}}\Big)\frac{\pi^s\,|n|^{s-\frac{1}{2}}\,\sqrt{y}}{\Gamma(s)}\int_0^\infty t^{s-\frac{1}{2}}\,e^{-(t+\frac{1}{t})\pi|n|y}\,\frac{dt}{t}$$

Summing over just the positive divisors of n, replacing c by $|n|/c$, and dividing through by $2\,\zeta(2s)$ gives

$$n^{th} \text{ Fourier component } E_s(x+iy) \;=\; \frac{\sigma_{2s-1}(|n|)}{\zeta(2s)\,|n|^{s-\frac{1}{2}}} \cdot \frac{\pi^s \sqrt{y}}{\Gamma(s)} \int_0^\infty t^{s-\frac{1}{2}}\; e^{-(t+\frac{1}{t})\pi|n|y}\, \frac{dt}{t}$$

where $\sigma_{2s-1}(|n|)$ is the sum of $(2s-1)^{th}$ powers of positive divisors. Altogether,

$$E_s(x+iy) \;=\; y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} \;+\; \frac{1}{\pi^{-s}\,\Gamma(s)\,\zeta(2s)} \sum_{n\neq 0} \frac{\sigma_{2s-1}(|n|)}{|n|^{s-\frac{1}{2}}} \cdot \sqrt{y} \int_0^\infty t^{s-\frac{1}{2}}\; e^{-(t+\frac{1}{t})\pi|n|y}\, \frac{dt}{t} \cdot e^{2\pi i n x}$$

[5.1.1] Remark: The Fourier expansion can be used to give another proof of the meromorphic continuation and functional equation of E_s.

[5.2] Integral representation of u_λ Let $\lambda = \lambda_s = s(s-1)$. Crude estimates show that

$$v_\lambda(|n|y) \;=\; \sqrt{y} \int_0^\infty t^{s-\frac{1}{2}}\; e^{-(t+\frac{1}{t})\pi|n|y}\, \frac{dt}{t}$$

is exponentially decreasing as $y \to +\infty$. Since $\Delta E_s = s(s-1) \cdot E_s$, separation of variables in the Fourier expansion shows that v_λ integral is a solution of the differential equation

$$y^2(u'' - 4\pi^2 u) \;=\; \lambda \cdot u$$

The earlier discussion of the asymptotics at ∞ of the solutions of this differential equation observed that up to scalars there is a unique solution asymptotic to $e^{-2\pi y}$, while all other solutions blow up like $e^{2\pi y}$. Thus, up to a normalizing constant, this integral expression v_λ is the function u_λ appearing in Fourier expansions of cuspforms with eigenvalue λ.

---

6. *Standard L-functions attached to cuspforms*

A natural, naive normalization $L^{\text{nf}}(s,f)$ of L-functions attached to cuspidal waveforms f is proven to have analytic continuation and functional equation. With hindsight, the set-up is renormalized for compatibility with other conventions.

[6.1] Naive normalization, analytic continuation and functional equation The naive form of the standard L-function attached to a cuspidal waveform f has two descriptions. When f is presumed a λ-eigenfunction for Δ, it has an expansion

$$f(x+iy) \;=\; \sum_{n\neq 0} c_n\, u_\lambda(|n|y)\, e^{2\pi i n x} \qquad\qquad (\text{where } u_\lambda \text{ solves } y^2(u''-4\pi^2 u) = \lambda\cdot u)$$

Assume for simplicity that f is *even* in x, meaning $f(-x+iy) = f(x+iy)$, so $c_n = c_{|n|}$, and no information is lost in putting

$$L^{\text{nf}}(s,f) \;=\; \sum_{n\geq 1} \frac{c_n}{n^s}$$

For $\lambda < -\frac{1}{4}$, the bound $c_n \ll_\varepsilon 1$ proven above guarantees convergence of this Dirichlet series in every right half-plane $\mathrm{Re}(s) > 1+\varepsilon$. The superscript indicates that this normalization is *naive*, although arguably natural.

On the other hand, if we can justify application of the usual integral transform, then

$$\int_0^\infty y^s\, f(iy)\, \frac{dy}{y} \;=\; \sum_{n\geq 1} c_n \int_0^\infty y^s\, u_\lambda(n\,y)\, \frac{dy}{y} \;=\; \sum_{n\geq 1} \frac{c_n}{n^s} \cdot \int_0^\infty y^s\, u_\lambda(y)\, \frac{dy}{y} \;=\; L^{\text{nf}}(s,f) \cdot \int_0^\infty y^s\, u_\lambda(y)\, \frac{dy}{y}$$

By now, we know that a cuspidal Δ-eigenfunction f is rapidly decreasing, so the integral behaves well at $+\infty$. To be sure of sufficiently good behavior as $y \to 0^+$, note that the differential equation $y^2 u'' - (\lambda + 4\pi^2 y^2)u = 0$ for u_λ has a *regular singular point* at $y = 0$, meaning that it is of the form $y^2 u'' + yb(y)u' + c(y)u = 0$ with b, c holomorphic at $y = 0$. To express $u_\lambda(y) = y^s \cdot \varphi(y)$ with φ holomorphic at $y = 0$, solve

$$0 \;=\; s(s-1) + b(0)s + c(0) \;=\; s(s-1) - \lambda$$

for the two values of s which make this possible. For $\lambda \le \frac{1}{4}$, $s = \frac{1}{2} \pm i\nu$ for some $\nu \in \mathbb{R}$. Thus, for some constants A, B,

$$|u_\lambda(y)| \;\sim\; |Ay^{\frac{1}{2}+i\nu} + By^{\frac{1}{2}-i\nu}| \;=\; \sqrt{y} \cdot |Ay^{i\nu} + By^{-i\nu}| \qquad\qquad (\text{as } y \to 0^+)$$

Thus, for example,

$$|u_\lambda(y)| \;\ll\; \sqrt{y} \qquad\qquad (\text{for } 0 < y \le 1)$$

and $|u_\lambda(y)| \ll e^{-2\pi y}$ for $y \ge 1$, from above. Also, from above, $|c_n| \ll 1$. Thus,

$$\Big|\sum_n c_n\, u_\lambda(|n|y)\Big| \;\ll\; \sum_{|n| \le \frac{1}{y}} \sqrt{|n|y} \;+\; \sum_{|n| > \frac{1}{y}} e^{-2\pi|n|y} \;\ll\; \sqrt{y} \int_1^{\frac{1}{y}} \sqrt{t}\, dt \;+\; e^{-2\pi y}$$

$$\ll\; \sqrt{y} \cdot \frac{1}{y^{3/2}} \;=\; \frac{1}{y} \qquad\qquad (\text{as } y \to 0^+)$$

Thus, the integral converges nicely for $\mathrm{Re}\,(s) > 1$.

Below, we will show that the resulting integral of $u_\lambda(y)$ is a product of Gamma functions: after renormalizing by constants depending only on λ, not s,

$$\int_0^\infty y^s\, u_\lambda(y)\, \frac{dy}{y} \;=\; \pi^{-s}\, \Gamma\Big(\frac{s+\mu}{2}\Big)\, \Gamma\Big(\frac{s+1-\mu}{2}\Big)$$

Granting that, the form of the functional equation disclosed by the following theorem provides motivation to renormalize:

[6.1.1] Theorem: For an *even* cuspidal waveform f with $\Delta^{\mathfrak{H}} f = \lambda \cdot f$ with $\lambda = \mu(\mu - 1)$, the completed L-function

$$\Lambda^{\mathrm{nf}}(f, s) \;=\; \pi^{-s}\, \Gamma\Big(\frac{s+\mu}{2}\Big)\, \Gamma\Big(\frac{s+1-\mu}{2}\Big) \cdot L^{\mathrm{nf}}(s, f)$$

has an analytic continuation to an *entire* function, with functional equation

$$\Lambda^{\mathrm{nf}}(-s, f) \;=\; \Lambda^{\mathrm{nf}}(s, f)$$

Proof: As in Riemann's argument with $\theta(iy)$ in place of $f(iy)$, the integral representation will yield the analytic continuation, beginning by breaking the integral into two pieces:

$$\Lambda^{\mathrm{nf}}(s, f) \;=\; \int_0^\infty y^s\, f(iy)\, \frac{dy}{y} \;=\; \int_1^\infty y^s\, f(iy)\, \frac{dy}{y} \;+\; \int_0^1 y^s\, f(iy)\, \frac{dy}{y}$$

The estimate $f(x + iy) \ll_f e^{-2\pi y}$ from the previous section shows that the integral from 1 to ∞ converges nicely for all $s \in \mathbb{C}$, and gives an entire function.

The functional equation $f(-1/z) = f(z)$ gives $f(i\frac{1}{y}) = f(iy)$, and converts the integral from 0 to 1 to an integral from 1 to ∞: replacing y by $1/y$,

$$\int_0^1 y^s\, f(iy)\, \frac{dy}{y} \;=\; \int_1^\infty y^{-s}\, f(i\frac{1}{y})\, \frac{dy}{y} \;=\; \int_1^\infty y^{-s}\, f(iy)\, \frac{dy}{y}$$

The latter integral is entire, by the decay estimate $f(iy) \ll_f e^{-2\pi y}$ as $y \to +\infty$. Thus,

$$\Lambda^{\rm nf}(s,f) \;=\; \int_1^\infty \big(y^s + y^{-s}\big)\, f(iy)\, \frac{dy}{y}$$

expresses $\Lambda^{\rm nf}(s,f)$ as an entire function with visible symmetry under $s \to -s$. ///

[6.2] Standard normalization and functional equation We might prefer the functional equation to be $s \to 1-s$ rather than $s \to -s$, especially if this can be arranged so that the correct notion of *critical strip* for these L-functions is $0 \le \mathrm{Re}(s) \le 1$. As we verify later, to achieve this effect put

$$L(f,s) \;=\; \sum_{n\ge 1} \frac{c_n}{n^{s-\frac{1}{2}}} \;=\; \sum_{n\ge 1} \frac{c_n \cdot \sqrt{n}}{n^s}$$

and, correspondingly,

$$\Lambda(s,f) \;=\; \int_0^\infty y^{s-\frac{1}{2}}\, f(iy)\, \frac{dy}{y} \;=\; \pi^{-s}\,\Gamma\Big(\frac{s-\frac{1}{2}+\mu}{2}\Big)\,\Gamma\Big(\frac{s+\frac{1}{2}-\mu}{2}\Big)\, L(s,f)$$

With this normalization, the functional equation is $s \to 1-s$.

[6.3] The gamma factor The explicit nature of the Eisenstein series suggests an approach to understanding the Gamma factor

$$\int_0^\infty y^s\, u_\lambda(y)\, \frac{dy}{y}$$

by using the *integral representation* for u_λ obtained in the course of determining the Fourier expansion of $E_s(z)$. Proceeding directly, putting $\lambda = \mu(\mu-1)$, up to a normalizing constant depending on μ, invoking the Gamma-function trick,

$$\int_0^\infty y^s\, u_\lambda(y)\, \frac{dy}{y} \;=\; \int_0^\infty y^s \int_{\mathbb{R}} e^{-2\pi i x}\, \frac{y^\mu}{(x^2+y^2)^\mu}\, dx\, \frac{dy}{y}$$

$$= \frac{1}{\Gamma(\mu)} \int_0^\infty \int_{\mathbb{R}} \int_0^\infty y^{s+\mu}\, e^{-2\pi i x}\, t^\mu\, e^{-t(x^2+y^2)}\, dx\, \frac{dy}{y}\, \frac{dt}{t} \;=\; \frac{\sqrt{\pi}}{\Gamma(\mu)} \int_0^\infty \int_0^\infty y^{s+\mu}\, t^{\mu-\frac{1}{2}}\, e^{-(\frac{\pi^2}{t}+ty^2)}\, \frac{dy}{y}\, \frac{dt}{t}$$

Replacing y by $\sqrt{y}$, and then y by y/t, gives

$$\frac{\sqrt{\pi}}{2\,\Gamma(\mu)} \int_0^\infty \int_0^\infty y^{\frac{s+\mu}{2}}\, t^{\mu-\frac{1}{2}}\, e^{-(\frac{\pi^2}{t}+ty)}\, \frac{dy}{y}\, \frac{dt}{t} \;=\; \frac{\sqrt{\pi}}{2\Gamma(\mu)} \int_0^\infty \int_0^\infty y^{\frac{s+\mu}{2}}\, t^{\mu-\frac{1}{2}-\frac{s+\mu}{2}}\, e^{-(\frac{\pi^2}{t}+y)}\, \frac{dy}{y}\, \frac{dt}{t}$$

$$= \frac{\sqrt{\pi}}{2\Gamma(\mu)}\Gamma\Big(\frac{s+\mu}{2}\Big) \int_0^\infty t^{\frac{\mu}{2}-\frac{1}{2}-\frac{s}{2}}\, e^{-\frac{\pi^2}{t}}\, \frac{dt}{t} \;=\; \frac{\sqrt{\pi}}{2\Gamma(\mu)}\Gamma\Big(\frac{s+\mu}{2}\Big)\, \pi^{-(s+1-\mu)}\, \Gamma\Big(\frac{s+1-\mu}{2}\Big)$$

Since we are at liberty to adjust by things depending on μ but not on s, we can renormalize u_λ by constants depending only on $\lambda = \mu(\mu-1)$ to achieve the effect that

$$\int_0^\infty y^s\, u_\lambda(y)\, \frac{dy}{y} \;=\; \pi^{-s}\,\Gamma\Big(\frac{s+\mu}{2}\Big)\,\Gamma\Big(\frac{s+1-\mu}{2}\Big) \qquad \text{(naive normalization)}$$

Renormalizing by replacing s by $s-\frac{1}{2}$, and adjusting by a constant, makes this

$$\int_0^\infty y^{s-\frac{1}{2}}\, u_\lambda(y)\, \frac{dy}{y} \;=\; \pi^{-s}\,\Gamma\Big(\frac{s-\frac{1}{2}+\mu}{2}\Big)\,\Gamma\Big(\frac{s+\frac{1}{2}-\mu}{2}\Big) \qquad \text{(renormalization)}$$

[6.3.1] Remark: The above computation is sensible when all the integrals converge nicely. Unfortunately, that regime is *not* where values of $\lambda = \mu(\mu - 1)$ are non-positive real, corresponding to $L^2(\Gamma\backslash\mathfrak{H})$-eigenfunctions. The region of interest is reached only by *analytic continuation*.

[6.3.2] Remark: Cuspforms whose L-functions have Euler products are distinguished by *Hecke operators*, in a fashion essentially identical to holomorphic modular forms.

[6.3.3] Remark: Hecke operators on waveforms commute with $\Delta = \Delta^{\mathfrak{H}}$, so stabilize each λ-eigenspace for Δ in $L^2(\Gamma\backslash\mathfrak{H})$. Conjecturally, for $\Gamma = SL_2(\mathbb{Z})$, for each λ appearing as an eigenvalue, the λ-eigenspace is *one-dimensional*. Thus, conjecturally, every square-integrable Δ-eigenfunction is also a Hecke eigenfunction.

---

7. *Good argument for non-vanishing of $\zeta(1+it)$*

Non-vanishing of $\zeta(s)$ on the edge $\mathrm{Re}(s) = 1$ of the critical strip $0 \le \mathrm{Re}(s) \le 1$ is essentially equivalent to the *prime number theorem*. Similarly, non-vanishing of other Euler products on the edges of the critical strip is critical in many number-theoretic applications.

Clever *ad hoc* arguments for this non-vanishing, using trigonometric identities, can succeed for the simplest classes of Euler products, but are inadequate in general. Even in simple cases, the old non-vanishing arguments are unilluminating.

Instead, the appearance of $\zeta(2s)$ in the denominators of the Fourier components of the Eisenstein series E_s for $SL_2(\mathbb{Z})$ gives a meaningful argument that generalizes to many other situations. From the Fourier expansion obtained above,

$$E_s(x+iy) \;=\; y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} \;+\; \frac{1}{\pi^{-s}\,\Gamma(s)\,\zeta(2s)} \sum_{n\neq 0} \frac{\sigma_{2s-1}(|n|)}{|n|^{s-\frac{1}{2}}} \cdot \sqrt{y} \int_0^\infty t^{s-\frac{1}{2}}\, e^{-(t+\frac{1}{t})\pi|n|y}\, \frac{dt}{t} \cdot e^{2\pi i n x}$$

a zero $2s_o = 1 + 2it_o$ of $\zeta(s)$ on $\mathrm{Re}(s) = 1$ would give a zero $s_o = \frac{1}{2} + it_o$ of the $\zeta(2s)$ appearing in the denominators. In the constant term, because of the functional equation,

$$\xi(2s_o - 1) \;=\; \xi(2 - 2s_o) \;=\; \xi(1 - 2it_o) \;=\; \overline{\xi(1-2it_o)} \;=\; 0$$

so the constant term *has no pole* at s_o. Thus, for $\zeta(1+2it_o) = 0$, the residue of E_s at s_o would be a *cuspform* f, and an eigenfunction with $\Delta f = s_o(s_o - 1)\cdot f$. Thus, f would be of rapid decay, and its integral against any Eisenstein series E_s would converge nicely, even for E_s meromorphically continued, since the meromorphic continuation is still of moderate growth, meaning $E_s(x+iy) \ll y^A$ for some exponent A as $y \to +\infty$. On one hand, for $\mathrm{Re}(s) > 1$, unwinding E_s,

$$\int_{\Gamma\backslash\mathfrak{H}} E_s \cdot \overline{f}\; \frac{dx\,dy}{y^2} \;=\; \int_{\Gamma_\infty\backslash\mathfrak{H}} y^s\, \overline{f}\; \frac{dx\,dy}{y^2} \;=\; \int_{\Gamma_\infty\backslash\mathfrak{H}} y^s \sum_{n\neq 0} c_n(y)\, e^{2\pi i n x}\; \frac{dx\,dy}{y^2}$$

where we write the n^{th} Fourier component as $e^{2\pi i n x}$ with coefficient a function $c_n(y)$ of y. A great virtue of the unwinding to an integral over $\Gamma_\infty\backslash\mathfrak{H}$ is that the latter quotion has very nice representatives allowing a *separation of variables*: a fundamental domain for Γ_∞ is $[0,1]\times(0,+\infty)$ in the x,y coordinates. The basic

$$\int_0^1 e^{2\pi i m x}\, dx \;=\; \begin{cases} 1 & (\text{for } m = 0) \\ 0 & (\text{for } m \neq 0) \end{cases}$$

and the vanishing of the 0^{th} Fourier component of a cuspform f gives

$$\int_{\Gamma\backslash\mathfrak{H}} E_s \cdot \overline{f}\; \frac{dx\,dy}{y^2} \;=\; \int_{N\backslash\mathfrak{H}} y^s\, 0 \frac{dy}{y^2} \;=\; 0$$

Yet, on the other hand, presuming that evaluation-of-residue commutes with the integral [15]

$$\mathrm{res}_{s=s_o} \int_{\Gamma\backslash\mathfrak{H}} E_s \cdot \overline{f} \;=\; \int_{\Gamma\backslash\mathfrak{H}} \mathrm{res}_{s=s_o} E_s \cdot \overline{f} \;=\; \int_{\Gamma\backslash\mathfrak{H}} f \cdot \overline{f}$$

Thus, $f = 0$, there is no such residue, and no such zero $s_o = 1 + 2it_o$ on the edge of the critical strip.

[7.0.1] Remark: The *residue* of E_s at $s = s_o$ can be understood as the collection of residues *pointwise* $\mathrm{res}_{s=s_o} E_s(z_o)$ for all values z_o, since the function $s \to E_s(z_o)$ is meromorphic in s for every fixed z_o. This would *not* promise anything about the collection of pointwise residues. However, we have further information about the collection of pointwise residues from the explicit Fourier expansion. Even in situations without explicit formulas, we can use the notion of *function-valued* meromorphic function $s \to E_s$, and rely upon the good behavior of *residue*, that is, that taking residues commutes with application of continuous linear functionals of the form

$$E_s \longrightarrow \int_{\Gamma\backslash\mathfrak{H}} E_s \cdot F$$

with F of rapid decay on the fundamental domain to give a (continuous) linear functional on moderate-growth functions such as E_s. This viewpoint is indispensable in more complicated, less-formulaic situations.

[15] Yes, evaluation-of-residue commutes with such integrals. This is a special case of the general good behavior of holomorphic function-valued functions. This idea was more-than-fully developed by Grothendieck, building upon Schwartz' distribution theory. The happy news is that things turn out as well as possible. The lack of dramatic tension, and opaque exposition, may account for the unfortunate lack of wider awareness of these results. Thanks to Bill Casselman some years ago for bringing the Grothendieck papers to my attention!

Bibliography

[Bombieri-Hejhal 1995] E. Bombieri, D. Hejhal, *On the distribution of zeros of linear combinations of Euler products*, Duke J. Math. **80** (1995), 821-862.

[Chowla-Selberg 1967] S. Chowla, A. Selberg, *On Epstein's zeta-function*, J. reine und angew. Math. **227** (1967), 86-110.

[Davenport-Heilbronn 1936] H. Davenport, H. Heilbronn, *On the zeros of certain Dirichlet series, I, II*, J. London Math. Soc. **11** (1936), 181-185 and 307-312.

[Epstein 1903] P. Epstein, *Zur Theorie allgemeiner Zetafunktionen, I*, Math. Ann. **56** (1903), 614-644.

[Godement 1966a] R. Godement, *Decomposition of* $L^2(\Gamma\backslash G)$ *for* $\Gamma = SL_2(\mathbb{Z})$, in Proc. Symp. Pure Math. **9** (1966), AMS, 211-24.

[Grothendieck 1953a,b] A. Grothendieck, *Sur certaines espaces de fonctions holomorphes, I*, J. Reine Angew. Math. **192** (1953), 35-64; *II*, **192** (1953), 77-95.

[Hecke 1918/20] E. Hecke, *Eine neue Art von Zetafunktionen und ihre Beziehungen der Verteilung der Primzahlen, I, II*, Math. Z. **1** no. 4 (1918), 357-376; *ibid* **6** no. 1-2 (1920), 11-51.

[Hecke 1937] E. Hecke, *Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklungen, I, II*, Math. Ann. **114** (1937) no. 1, 1-28, 316-351.

[Maaß 1949] H. Maaß, *Über eine neue Art von nicht analytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **121** (1949), 141-183.

[Potter-Titchmarsh 1935] H.S. Potter, E.C. Titchmarsh, *The zeros of Epstein's zeta functions*, Proc. London Math. Soc. **39** (1935), 372-384.

[Rankin 1939] R. Rankin, *Contributions to the theory of Ramanujan's function* $\tau(n)$ *and similar arithmetic functions, I*, Proc. Cam. Phil. Soc. **35** (1939), 351-372.

[Schwartz 1950/51] L. Schwartz, *Théorie des Distributions*, I,II Hermann, Paris, 1950/51, 3rd edition, 1965.

[Selberg 1940] A. Selberg, *Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist*, Arch. Math. Naturvid **43** (1940), pp. 47-50.

[Stark 1967] H.M. Stark, *On the zeroes of Epstein's zeta function*, Mathematika **14** (1967), 47-55.

[Voronin 1976] M. Voronin, *On the zeros of zeta functions of quadratic forms*, Trudy Mat. Inst. Steklov **142** (1976), 135-147.