1. Moving automorphic forms from domains to groups

Automorphic forms on domains such as the upper half-plane \( \mathfrak{H} \) are usually introduced first because it is easier to discuss them. However, this picture hides important features about the representation theory of \( SL_2(\mathbb{R}) \). It is easy to convert automorphic forms on \( \mathfrak{H} \), both waveforms and holomorphic modular forms, to automorphic forms on \( SL_2(\mathbb{R}) \).

1.1 \( \Gamma \)-invariant functions and \( SO_2(\mathbb{R}) \)-invariance

Choice of (maximal) compact subgroup \( SO(2, \mathbb{R}) \) in \( SL_2(\mathbb{R}) \) amounts to choice of basepoint \( i \in \mathfrak{H} \), as \( SO_2(\mathbb{R}) \) is the isotropy subgroup of \( i \). Via the isomorphism \( SL(\mathbb{R})/SO_2(\mathbb{R}) \to \mathfrak{H} \) by \( gSO_2(\mathbb{R}) \to g(i) \), and function \( f \) on \( \mathfrak{H} \) can be converted to a function \( F \) on \( SL_2(\mathbb{R}) \) by

\[
F(g) = f(g(i))
\]

[1] Despite contrary assertions in the literature, rewriting Eisenstein series, as opposed to more general automorphic forms, on adele groups does not use Strong Approximation. Strong Approximation does make precise the relation between general automorphic forms on adele groups and automorphic forms on \( SL_2 \) and even on \( SL_n \), but rewriting these Eisenstein series does not need this comparison. Indeed, Strong Approximation does not hold in the simplest form for general semi-simple or reductive groups, but this does harm anything. Strong approximation does show that there can be no other extension of the Eisenstein series to the adele group beyond that described here.

[2] This is an example of conversion to automorphic forms on reductive or semi-simple real Lie groups, of which \( SL_2(\mathbb{R}) \) is a small example.
The function $F$ is right $SO_2(\mathbb{R})$-invariant:

$$F(gk) = f(gk(i)) = f(g(i)) = F(g) \quad \text{(for } g \in SL_2(\mathbb{R}) \text{ and } k \in SO_2(\mathbb{R}))$$

Oppositely, any such right-$SO(\mathbb{R})$-invariant function $F$ on $SL_2(\mathbb{R})$ descends to a function $f$ on the quotient $\mathcal{H}$ by

$$f(z) = F(g_z) \quad \text{(for any } g_z \in SL_2(\mathbb{R}) \text{ with } g_z(i) = z)$$

For $f$ also left invariant by $\Gamma = SL_2(\mathbb{Z})$ on $\mathcal{H}$, the corresponding function $F$ on $SL_2(\mathbb{R})$ is left $\Gamma$-invariant, and vice-versa:

$$F(\gamma g) = f(\gamma g(i)) = f(g(i)) = F(g) \quad \text{(for } \gamma \in \Gamma \text{ and } g \in SL_2(\mathbb{R}))$$

**[1.2] Automorphy condition and $SO_2(\mathbb{R})$-equivariance** To motivate extension of the treatment of left $\Gamma$-invariant function on $\mathcal{H}$, of course holomorphic modular forms $f$ on $\mathcal{H}$ are not quite left $\Gamma$-invariant, but satisfy an automorphy condition

$$f(\gamma z) = j(\gamma, z) \cdot f(z) \quad \text{(where } j(\alpha\beta, z) = j(\alpha, \beta z) \cdot j(\beta, z))$$

Apart from the trivial cocycle $j(g, z) = 1$, the simplest example of cocycle is $j(g, z) = (cz + d)^2k$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This extends to be a cocycle defined not merely on $\Gamma \times \mathcal{H}$ but on $SL_2(\mathbb{R}) \times \mathcal{H}$. Associate the function $F$ on $SL_2(\mathbb{R})$ given by

$$F(g) = j(g, i)^{-1} \cdot f(g(i))$$

**[1.2.1] Claim:** The function $F$ is left $\Gamma$-invariant on $SL_2(\mathbb{R})$, and right $SO_2(\mathbb{R})$-equivariant by

$$F(gk) = j(k, i)^{-1} \cdot F(g)$$

and $k \to j(k, i)^{-1}$ is a group homomorphism $SO_2(\mathbb{R}) \to \mathbb{C}^\times$.

**Proof:** This is a direct computation. For $\gamma \in \Gamma$,

$$F(\gamma g) = j(\gamma g, i)^{-1} \cdot f(\gamma g(i)) = \left( j(\gamma, g(i)) \cdot j(g, i) \right)^{-1} \cdot f(g(i)) = j(g, i)^{-1} \cdot j(\gamma, g(i)) \cdot f(g(i)) = j(g, i)^{-1} \cdot f(g(i)) = F(g)$$

For $k \in SO_2(\mathbb{R})$,

$$F(gk) = j(gk, i)^{-1} \cdot f(gk(i)) = \left( j(g, k(i)) \cdot j(k, i) \right)^{-1} \cdot f(g(i)) = \left( j(g, i) \cdot j(k, i) \right)^{-1} \cdot f(g(i)) = j(k, i)^{-1} \cdot j(g, i)^{-1} \cdot f(g(i)) = j(k, i)^{-1} \cdot F(g)$$

Finally, for $k, h \in SO_2(\mathbb{R})$,

$$j(hk, i) = j(h, k(i)) \cdot j(k, i) = j(h, i) \cdot j(k, i)$$

proving that $k \to j(k, i)$ is a group homomorphism on $SO_2(\mathbb{R})$. //

**[1.2.2] Remark:** Half-integral weight automorphic forms have a cocycle on $\Gamma \times \mathcal{H}$ which does not extend to $SL_2(\mathbb{R}) \times \mathcal{H}$. The obstruction to this extension defines a two-fold covering group of $SL_2(\mathbb{R})$, the metaplectic group, the group where half-integral weight automorphic forms live. This and other complications account for our emphasis on integral-weight holomorphic modular forms, and on waveforms, in these notes.
[1.2.3] **Remark:** The seemingly different analytic conditions, *holomorphy* and being an *eigenfunction* for the invariant Laplacian, both become eigenfunction conditions on the group $SL_2(\mathbb{R})$. We will return to this a little later.

[1.3] **Conversion to automorphic forms on $GL_2(\mathbb{R})$** To eventually accommodate Hecke operators, and for many other reasons, the group $GL_2$ is better than $SL_2(\mathbb{R})$.

Already, the subgroup

$$GL_2^+(\mathbb{R}) = \{ g \in GL_2(\mathbb{R}) : \det g > 0 \}$$

preserves the upper half-plane $H$, and

$$GL_2^+(\mathbb{Z}) = \{ g \in GL_2(\mathbb{Z}) : \det g > 0 \} = SL_2(\mathbb{Z})$$

since $g \in GL_2(\mathbb{Z})$ has determinant $\pm 1$. Now the isotropy group of $i \in \mathcal{H}$ is

$$Z = \{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R}^\times \}$$

As with $SL_2(\mathbb{R})$, we have

$$\mathcal{H} \approx GL_2^+(\mathbb{R})/(\text{isotropy subgroup of } i) = GL_2^+(\mathbb{R})/Z^+GL_2^+(\mathbb{Z})$$

For left $GL_2^+(\mathbb{Z})$-invariant functions $f$ on $\mathcal{H}$, the associated function $F$ on $GL_2^+(\mathbb{R})$ is

$$F(g) = f(g(i))$$

and $F$ is left $GL_2^+(\mathbb{Z})$-invariant, right $SO_2(\mathbb{R})$-invariant, and $Z$-invariant. Since $Z$ is the *center* of $GL_2(\mathbb{R})$, the function $F$ is both right and left $Z$-invariant.

For $f$ not left $GL_2^+(\mathbb{Z})$-*invariant*, but only meeting a cocycle condition $f(\gamma z) = j(\gamma, z) \cdot f(z)$ with a cocycle extending to $GL_2^+(\mathbb{R}) \times \mathcal{H}$, such as

$$j(g, z) = (cz + d)^{2k} \cdot (\det g)^{-k} \quad (\text{with } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}))$$

for weight $2k$ holomorphic modular forms, the corresponding function $F$ on $GL_2^+(\mathbb{R})$ is again

$$F(g) = j(g, i)^{-1} \cdot f(g(i))$$

as for $SL_2(\mathbb{R})$. Some cocycles are trivial on $Z$, some are not.

Further, the positive-determinant condition can be removed when the function $f$ on $\mathcal{H}$ is extended to be a function on $\mathcal{H} \cup \overline{\mathcal{H}}$, since $GL_2(\mathbb{R})$ stabilizes the union of both half-planes, producing functions $F$ on $GL_2(\mathbb{R})$ that are left $GL_2(\mathbb{Z})$-invariant, right $O_2(\mathbb{R})$-equivariant by some *representation* of $O_2(\mathbb{R})$, and $Z$-equivariant by some character $\mathbb{Z} \to \mathbb{C}^\times$. 

3
2. Iwasawa decompositions of $GL_2(\mathbb{R})$ and $GL_2(\mathbb{Q}_p)$

That $g \in GL_2(\mathbb{R})$ can be written as a product $g = pk$ of upper-triangular $p$ and orthogonal matrices $k$ was known for a long time before K. Iwasawa’s work in the 1940s, but after Iwasawa’s work this and other related classical facts are understood as instances of a very general pattern that applies, not only to real or complex matrix groups, but also to related $p$-adic matrix groups.

The pattern of an Iwasawa decomposition in a real or complex or $p$-adic matrix group is essentially\[3\]

whole group = (upper-triangular elements) · (maximal compact subgroup)

The subgroup $P$ of upper-triangular matrices is a parabolic subgroup. The general definition of parabolic is not essential here, nor is determination or certification that various subgroups are maximal compact. Rather, we are acknowledging that our present examples fit into a larger pattern.

[2.1] Iwasawa decomposition for $GL_2(\mathbb{R})$ Let $G_\infty = GL_2(\mathbb{R})$. Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty$, choose the element $k \in K_\infty = O_2(\mathbb{R})$ to right multiply by to put $gk^{-1}$ into the group $P_\infty$ of upper-triangular real matrices, by rotating the lower half $(c \ d)$ of $g$ in $\mathbb{R}^2$ into the form $(0 \ \ast)$. Indeed, realizing that matrices in $K_\infty$ are of the form

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

(for $\theta \in \mathbb{R}$)

in particular with the squares of the left column summing to 1, choose

$$k^{-1} = \begin{pmatrix} \\
\frac{d}{\sqrt{c^2+d^2}} & \ast \\
\frac{-c}{\sqrt{c^2+d^2}} & \ast
\end{pmatrix}$$

with the right column of course completely determined by the left, so that

$$gk^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d \\
\frac{-c}{\sqrt{c^2+d^2}} \\
\frac{d}{\sqrt{c^2+d^2}} \\
\ast
\end{pmatrix} = \begin{pmatrix} \ast \\
\frac{c}{\sqrt{c^2+d^2}} + \frac{d}{\sqrt{c^2+d^2}} \ast \\
\frac{d}{\sqrt{c^2+d^2}} \ast
\end{pmatrix} = \begin{pmatrix} \ast \\
0 \ast
\end{pmatrix}$$

as desired. For application to Eisenstein series, we only need the diagonal entries of $gk^{-1}$, and we easily further compute the lower-right entry

$$gk^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d \\
\frac{c}{\sqrt{c^2+d^2}} \\
\frac{d}{\sqrt{c^2+d^2}} \\
\ast
\end{pmatrix} = \begin{pmatrix} \ast \\
0 \sqrt{c^2+d^2}
\end{pmatrix}$$

For $g \in SL_2(\mathbb{R})$, the determinant-one condition gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & \ast \\ 0 & \frac{1}{\sqrt{c^2+d^2}} \end{pmatrix} \cdot K_\infty \subset P_\infty \cdot K_\infty$$

[3] Probably the group should be reductive or semi-simple, whose formal definitions do not concern us for the moment. Rather, we note that this class of groups includes important examples: $GL_n(\mathbb{R})$, $GL_n(\mathbb{Q}_p)$, $GL_n(\mathbb{C})$, $GL_n(\mathbb{C})$, as well as orthogonal groups, unitary groups, and other matrix groups defined by preservation of additional structure. The class does not include upper-triangular matrices in $GL_n(\mathbb{R})$, although this subgroup is important in its own right. This class of groups does also include $p$-adic versions of groups over $\mathbb{R}$ and $\mathbb{C}$. 
For general \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty, \)
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \left( \frac{\det g}{\sqrt{c^2 + d^2}} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right) \cdot K_\infty \subset P_\infty \cdot K_\infty
\]

**[2.2] Iwasawa decomposition for \( GL_2(\mathbb{Q}_p) \)**

Even though the substance of a \( p \)-adic Iwasawa decomposition is quite different from that of archimedean Iwasawa decompositions, the commonalities are very useful.

The standard maximal compact subgroup \( K_v \) of \( G_v = GL_2(\mathbb{Q}_v) \) with \( v \) a finite prime, in sharp contrast to the orthogonal group as maximal compact subgroup of \( GL_2(\mathbb{R}) \), is

\[
K_v = GL(\mathbb{Z}_v) = \{ \text{\textit{p}-adic integral matrices with determinants in } \mathbb{Z}_v^\times \} \quad (v \text{ is finite prime } p)
\]

It is not so hard to prove that this is compact, but a little harder to prove that it is maximal compact. But, for the moment, we don’t use either property, especially not the maximality.

The essential aspect of \( \mathbb{Q}_p \) is that, for any two \( x, y \in \mathbb{Q}_p^\times \), either \( x/y \in \mathbb{Z}_p \) or \( y/x \in \mathbb{Z}_p \), since

\[
\mathbb{Z}_p = \{ z \in \mathbb{Q} : |z|_p \leq 1 \}
\]

This contrasts violently with the classical situation of rational numbers \( x, y \), where the classical (=archimedean!) notion of "size" is very far from a sufficient determiner of divisibility. Thus, given \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p), \)

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\frac{d}{a} & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \text{ (for } |c|_v \leq |d|_v) \]
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \text{ (for } |c|_v \geq |d|_v) \]

In both cases, the right-multiplying matrix is in \( K_v = GL_2(\mathbb{Z}_v) \): in the first case \( |c/d|_v \leq 1 \) implies \( c/d \in \mathbb{Z}_v \), and in the second \( |d/c|_v \leq 1 \) implies \( d/c \in \mathbb{Z}_v \). In the first case, the resulting product is already in the group \( P_v \) of upper-triangular matrices in \( G_v \). In the second, further right-multiplication by the long Weyl element \( w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) puts the product into \( P_v \). This proves that the \( p \)-adic Iwasawa decomposition succeeds.

Specifics about the diagonal entries in the Iwasawa decomposition will be needed in rewriting Eisenstein series:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\frac{d}{a} & 1 \end{pmatrix} = \begin{pmatrix} a - \frac{bc}{d} & * \\ 0 & d \end{pmatrix} \text{ (for } |c|_v \leq |d|_v) \]
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{ad}{c} + b & * \\ 0 & c \end{pmatrix} \text{ (for } |c|_v \geq |d|_v) \]

That is,

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \left( \begin{pmatrix} a - \frac{bc}{d} & * \\ 0 & d \end{pmatrix} \right) \cdot K_v \text{ (for } |c|_v \leq |d|_v) \]
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \left( \begin{pmatrix} -\frac{ad}{c} + b & * \\ 0 & c \end{pmatrix} \right) \cdot K_v \text{ (for } |c|_v \geq |d|_v) \]

**[2.2.1] Remark:** It is very convenient that the shape of the parabolic \( P_v \) is the same for both archimedean and non-archimedean \( v \), while the shape of the (maximal) compact subgroup \( K_v \) is significantly different.
3. Rewriting the $GL_2$ Eisenstein series

With $\Gamma = SL_2(\mathbb{Z})$ and $\Gamma_\infty = \{ \left( \begin{array}{cc} * & * \\ \mathbf{0} & * \end{array} \right) \in \Gamma \}$, the simplest waveform-type Eisenstein series on the domain $\mathcal{H}$ is the usual

$$E_s(z) = \sum_{\Gamma_\infty \setminus \Gamma} (\text{Im } \gamma z)^s \quad (\text{for } z \in \mathcal{H} \text{ and } \text{Re}(s) > 1)$$

This Eisenstein series on the domain $\mathcal{H}$ is easily converted to a left $\Gamma$-invariant, right $SO_2(\mathbb{R})$-invariant function on the group $GL_2^+(\mathbb{R})$, still called an Eisenstein series, by

$$E_{s\text{group}}(g) = E_s(g(i)) \quad (\text{for } g \in GL_2^+(\mathbb{R}))$$

The basepoint $i \in \mathcal{H}$ is chosen since $K_\infty = SO_2(\mathbb{R})$ is its isotropy group, which immediately explains the right $SO_2(\mathbb{R})$-invariance.

In fact, $E_s$ can be extended to an automorphic form on the union of upper and lower half-planes by

$$E_s(-z) = E_s(z), \quad \text{noting that } \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) (z) = -z$$

Thus, now letting $\Gamma = GL_2(\mathbb{Z})$ and $\Gamma_\infty = \{ \left( \begin{array}{cc} * & * \\ \mathbf{0} & * \end{array} \right) \in \Gamma \}$, we have an expression for the extended Eisenstein series that is of the same form:

$$E_s(z) = \sum_{\Gamma_\infty \setminus \Gamma} |\text{Im } \gamma z|^s \quad (\text{for } z \in \mathcal{H} \cup \bar{\mathcal{H}})$$

The Eisenstein series on the whole group $G_\infty = GL_2(\mathbb{R})$ is

$$E_{s\text{group}}(g) = E_s(g(i)) \quad (\text{for } g \in GL_2(\mathbb{R}))$$

[3.1] The localized rewrite As above, let $v$ be an index for completions of $\mathbb{Q}$, with $\mathbb{Q}_v$ the $v^{th}$ completion, and $\mathbb{Z}_v$ the $v$-adic integers for $v$ a prime, with $\mathbb{Q}_\infty = \mathbb{R}$, and $| \cdot |_\infty$ the usual real absolute value. The latter is distinguished from genuine primes by the convention of calling it the infinite prime, in analogy with a more legitimate use of this in the function-field setting.

For each place $v$, finite and infinite, using the $v^{th}$ Iwasawa decomposition $G_v = P_v K_v$, define a function $\varphi_v$ on $G_v$ by

$$\varphi_v\left( \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \cdot k \right) = \frac{1}{|d|_v} \left| \frac{a}{d} \right|^s \begin{cases} \text{for } k \in GL_2(\mathbb{Z}_v) \\ \text{for } k = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \in O(2) \end{cases}$$

where in all cases $a, d \in \mathbb{Q}_v^\times$ and $b \in \mathbb{Q}_v$. By design, each $\varphi_v$ is invariant under the center $Z_v$ of $G_v$. Define a function on the adele group $G_\mathbb{A} = GL_2(\mathbb{A})$ by

$$\varphi = \bigotimes_v \varphi_v \quad \text{(meaning } \varphi(\{g_v\}) = \prod_v \varphi_v(g_v), \text{ where } g_v \in G_v)$$

[4] Another tradition for referring to $p$-adic completions, as well as to $\mathbb{R}$, is as places of $\mathbb{Q}$. Also, sometimes $\infty$ is called the infinite prime, while the actual primes $p$ are finite primes.
[3.1.1] Claim: (The product formula) For $x \in \mathbb{Q}^\times$, 
\[
\prod_{v \leq \infty} |x|_v = 1 \quad \text{(product over all } p\text{-adic norms as well as } \mathbb{R})
\]

Proof: Since the assertion is multiplicative, it suffices to prove the product formula for units and for primes. All norms of ±1 are 1, and all norms but the archimedean and $p^{th}$ of a prime $p$ are 1. Since $|p|_\infty = p$ and $|p|_p = \frac{1}{p}$, we have the product formula. ///

The product formula shows that $\varphi$ is left $P_\mathbb{Q}$-invariant: using the Iwasawa decomposition locally everywhere, with $k \in \prod_v K_v$, 
\[
\varphi\left(\begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \begin{pmatrix} a & * \\ 0 & d \end{pmatrix}^s k \right) = |Aa/Dd|^s = |A/D|^s \cdot |a/d|^s = |A/D|^s \cdot \varphi\left(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix}^s k \right)
\]

Let 
\[
Z_\mathbb{A} = \{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a, a^{-1} \in \mathbb{A} \}
\]

[3.1.2] Theorem: For $g_\infty \in GL_2(\mathbb{R})$, acting as usual on $\mathcal{H} \cup \overline{\mathcal{H}}$, 
\[
E_s(g_\infty \cdot i) = \sum_{\gamma \in P_\mathbb{Q} \setminus G_\mathbb{Q}} \varphi(\gamma \cdot g_\infty)
\]

This expression gives a left $G_\mathbb{Q}$-invariant, $Z_\mathbb{A}$-invariant function (still denoted $E_s$) on the adele group $G_\mathbb{A} = GL_2(\mathbb{A})$:
\[
E_s^{\text{adelic}}(g) = \sum_{\gamma \in P_\mathbb{Q} \setminus G_\mathbb{Q}} \varphi(\gamma \cdot g)
\]

Proof of this will occupy the rest of this section.

[3.1.3] Remark: The notations $E_s^{\text{group}}$ and $E_s^{\text{adelic}}$ are not standard, but have the obvious descriptive utility. In fact, we will revert to writing $E_s$ for the Eisenstein series on $G_\mathbb{A} = GL_2(\mathbb{A})$.

[3.2] Disambiguation The immediate question arises of evaluation of $\varphi(\gamma \cdot g_\infty)$. One point is that $G_\mathbb{Q} = GL_2(\mathbb{Q})$ should not be considered as only a subgroup of $G_\infty = GL_2(\mathbb{R})$, but also a subgroup of every $G_v = GL_2(\mathbb{Q}_v)$. Thus, $G_\mathbb{Q}$ is best considered as imbedded on the diagonal in $G_\mathbb{A}$. Adding a temporary notational burden for clarity, for each place $v$ let $j_v : GL_2(\mathbb{Q}) \to GL_2(\mathbb{Q}_v)$ be the natural injective map, and let 
\[
j = \prod_v j_v : GL_2(\mathbb{Q}) \longrightarrow \prod_v GL_2(\mathbb{Q}_v)
\]

be the natural diagonal map to the product. Then 
\[
\varphi(\gamma \cdot g_\infty) = \varphi_\infty(j_\infty(\gamma) \cdot g_\infty) \cdot \prod_{v < \infty} \varphi_\infty(j_v(\gamma))
\]

That is, in the definition of $\varphi$, the archimedean $g_\infty \in G_\infty$ does not interact with the non-archimedean groups $G_v = GL_2(\mathbb{Q}_v)$.

[3.3] Comparison to classical formulation The familiar Eisenstein series $E_s(z)$ can be obtained from the above by reverting to a form that does not refer to anything $p$-adic or adelic. That is, we claim that 
\[
\varphi_\infty(g_\infty) = |\text{Im}(g_\infty \cdot i)|^s \quad \text{(for } g_\infty \in GL_2(\mathbb{R}))
\]
The description of $\phi$. The argument is about the Iwasawa decomposition, namely, that any element of $GL_2(\mathbb{R})$ can be written as a product of upper-triangular and orthogonal matrices. Indeed, given a matrix in $GL_2(\mathbb{R})$, right multiplication by an orthogonal matrix can be viewed as rotating the bottom row. This suggests the appropriate orthogonal group element: formulaically,

$$(a \ b) \cdot \begin{pmatrix} \frac{d}{\sqrt{c^2 + d^2}} & \frac{c}{\sqrt{c^2 + d^2}} \\ \frac{c}{\sqrt{c^2 + d^2}} & \frac{d}{\sqrt{c^2 + d^2}} \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{\sqrt{c^2 + d^2}} & \frac{ac+bd}{\sqrt{c^2 + d^2}} \\ 0 & \frac{1}{\sqrt{c^2 + d^2}} \end{pmatrix} = \begin{pmatrix} 1 & \ast \\ 0 & \sqrt{c^2 + d^2} \end{pmatrix}$$

Thus,

$$\varphi^\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varphi^\infty \begin{pmatrix} 1 & \ast \\ \frac{c}{\sqrt{c^2 + d^2}} & \frac{d}{\sqrt{c^2 + d^2}} \end{pmatrix} = \frac{1}{\sqrt{c^2 + d^2}} = \frac{1}{c^2 + d^2}$$

On the other hand, a familiar computation gives

$$\Im \begin{pmatrix} a & b \\ c & d \end{pmatrix}(i) = \frac{1}{2i} (ai+b - ai+b) = \frac{ad-bc}{c^2 + d^2} = \frac{1}{c^2 + d^2}$$

Since $\gamma \in GL_2(\mathbb{Z})$ maps to $GL_2(\mathbb{Z}_v)$ at all finite places $v$,

$$\varphi_v(\gamma) = 1 \quad \text{(for } \gamma \in GL_2(\mathbb{Z}) \text{ and finite place } v)$$

Thus,

$$\sum_{\gamma \in P^*_v \backslash GL_2(\mathbb{Z})} \varphi(\gamma \cdot g_\infty) = \sum_{\gamma \in P^*_v \backslash GL_2(\mathbb{Z})} \varphi^\infty(\gamma \cdot g_\infty) \cdot 1 = \sum_{\gamma \in P^*_v \backslash GL_2(\mathbb{Z})} \left( \Im g_\infty \cdot i \right)^s$$

Taking the popular choice

$$g_\infty = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{v} & 0 \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix} \quad \text{(with } x \in \mathbb{R} \text{ and } y > 0)$$

produces $E_v(x + iy)$ on $\mathfrak{H}$, as claimed.

[3.4] Well-definedness on $P^*_v \backslash G_Q$ We should show that $\varphi(\gamma g_\infty)$ depends only upon the coset $P^*_v \gamma$. Any $\gamma \in GL_2(\mathbb{Q})$ is in $GL_2(\mathbb{Z}_v)$ for almost all $v$, since the entries are in $\mathbb{Z}_v$ for almost all $v$, and the determinant is a $v$-adic unit for almost all $v$, so the inverse is $v$-integral also. Thus, in an infinite product $\prod_{v < \infty} \varphi_v(\gamma)$, all but finitely-many factors are 1.

Let $\chi_v$ be the character on upper-triangular $v$-adic matrices $P_v$ given by

$$\chi_v \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \left| \frac{a}{d} \right|_v^s \quad \text{(with } a, d \in \mathbb{Q}_v^* \text{ and } b \in \mathbb{Q}_v)$$

The usual maximal compact [5] subgroups $K_v$ of the groups $GL_2(\mathbb{Q}_v)$ are

$$K_v = \begin{cases} GL_2(\mathbb{Q}_v) & \text{(for } v \text{ finite)} \\ O(2) & \text{(for } v \text{ real)} \end{cases}$$

The description of $\varphi_v$ can be rewritten more succinctly as

$$\varphi_v(pk) = \chi_v(p) \quad \text{(for } p \in P_v \text{ and } k \in K_v)$$

[5] We will not use the fact that these are maximal among compact subgroups, despite referring to them as such.
For \( g_\infty \in G_\mathbb{R}, \gamma \in G_\mathbb{Q}, \) and \( \beta \in P_\mathbb{Q}, \) keeping in mind that \( G_\mathbb{Q} \) maps to all groups \( G_v, \) not just to \( G_\infty, \)

\[
\varphi(\beta \cdot \gamma \cdot g_\infty) = \varphi_\infty(\beta \cdot \gamma \cdot g_\infty) \cdot \prod_{v < \infty} \varphi_v(\beta \cdot \gamma)
\]

At the archimedean place, let \( \gamma g_\infty = pk \) be an Iwasawa decomposition in \( G_v, \) with \( p \in P_v \) and \( k \in K_v. \) We see the left equivariance of \( \varphi_v \) by \( \chi_v, \) namely,

\[
\varphi_v(\beta \gamma g_\infty) = \varphi_v(\beta pk) = \chi_v(\beta \cdot p) = \chi_v(\beta) \cdot \chi_v(p) = \chi_v(\beta) \cdot \varphi_v(pk) = \chi_v(\beta) \cdot \varphi_v(\gamma g_\infty)
\]

Similarly, but now without \( g_\infty \) playing any role, at a finite place \( v, \) let \( \gamma = pk \) be an Iwasawa decomposition in \( G_v, \) with \( p \in P_v \) and \( k \in K_v. \) We see the left equivariance of \( \varphi_v \) by \( \chi_v: \)

\[
\varphi_v(\beta \gamma) = \varphi_v(\beta pk) = \chi_v(\beta \cdot p) = \chi_v(\beta) \cdot \chi_v(p) = \chi_v(\beta) \cdot \varphi_v(\gamma)
\]

Putting all these local equivariances together,

\[
\varphi(\beta \cdot \gamma \cdot g_\infty) = \prod_v \chi_v(\beta) \cdot \varphi_v(\gamma \cdot g_\infty) \quad (\text{for } \beta \in P_\mathbb{Q}, \gamma \in G_\mathbb{Q}, \text{ and } g_\infty \in G_\infty)
\]

By the product formula,

\[
\prod_v \chi_v \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \prod_v \left| \frac{a}{d} \right|_v = 1 \quad (\text{for } a/d \in \mathbb{Q}^\times)
\]

That is, we have the left invariance

\[
\varphi(\beta \cdot \gamma \cdot g_\infty) = \varphi(\gamma \cdot g_\infty) \quad (\text{for } \beta \in P_\mathbb{Q}, \gamma \in G_\mathbb{Q}, \text{ and } g_\infty \in G_\infty)
\]

### [3.5] Bijection of cosets

Changing from the \( \Gamma \) and \( \Gamma_\infty \) notation, let \( G_\mathbb{Z} = GL_2(\mathbb{Z}) \) and \( P_\mathbb{Z} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G_\mathbb{Z} \right\}, \) we claim that

\[
P_\mathbb{Q} \backslash G_\mathbb{Q} \approx P_\mathbb{Z} \backslash G_\mathbb{Z}
\]

More precisely, we claim that the obvious map \( P_\mathbb{Z} \backslash G_\mathbb{Z} \to P_\mathbb{Q} \backslash G_\mathbb{Q} \) by \( P_\mathbb{Z} g \to P_\mathbb{Q} g \) is a surjection. It is an injection because \( P_\mathbb{Z} = G_\mathbb{Z} \cap P_\mathbb{Q}. \) That is, we claim that that every coset \( P_\mathbb{Q} h \) with \( h \in G_\mathbb{Q} \) has a representative in \( G_\mathbb{Z} = GL_2(\mathbb{Z}). \) The argument attaches meaning to both these coset spaces, and will thereby give the bijection.

The coset space \( P_\mathbb{Q} \backslash G_\mathbb{Q} \) is in bijection with the set of lines in \( \mathbb{Q}^2, \) respecting the right multiplication by \( G_\mathbb{Q}, \) because \( G_\mathbb{Q} \) is transitive on these lines, and \( P_\mathbb{Q} \) is the stabilizer of the line \( \{(0, *)\}. \) Next, each line in \( \mathbb{Q}^2 \) meets \( \mathbb{Z}^2 \) in a free rank-one \( \mathbb{Z} \)-module generated by a primitive vector \((x, y), \) meaning that \( \gcd(x, y) = 1. \) Call such a \( \mathbb{Z} \)-module a primitive \( \mathbb{Z} \)-line in \( \mathbb{Q}^2. \) The collection of lines in \( \mathbb{Q}^2 \) is thus in bijection with primitive \( \mathbb{Z} \)-lines in \( \mathbb{Z}^2, \) by sending a line to its intersection with \( \mathbb{Z}^2. \) The group \( SL_2(\mathbb{Z}) \subset GL_2(\mathbb{Z}) \) is already transitive on primitive \( \mathbb{Z} \)-lines: for \( \gcd(x, y) = 1 \), let \( b, d \in \mathbb{Z} \) be such that

\[
bx + dy = \gcd(x, y) = 1
\]

Then

\[
(x \ y) \cdot \begin{pmatrix} y & b \\ -x & d \end{pmatrix} = (0 \ 1)
\]

That is, any primitive vector can be mapped to \((0, 1), \) so the action of \( SL_2(\mathbb{Z}) \) is transitive on primitive \( \mathbb{Z} \)-vectors, hence on primitive \( \mathbb{Z} \)-lines. Thus, certainly the slightly larger group \( GL_2(\mathbb{Z}) \) is transitive on primitive vectors. The stabilizer subgroup of the primitive \( \mathbb{Z} \)-line spanned by \((0, 1)\) in \( GL_2(\mathbb{Z}) \) is

\[
P_\mathbb{Z} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{Z}^\times, \ b \in \mathbb{Z} \right\}
\]
This proves the bijection of coset spaces.

[3.6] The essential conclusion  From

\[ E_s(g_\infty \cdot i) = \sum_{\gamma \in P_\mathbb{Z} \setminus G_\mathbb{Z}} \varphi_\infty(\gamma \cdot g_\infty) \quad \text{(with } g_\infty \in GL_2(\mathbb{R})) \]

from the bijection of coset spaces, and from the well-definedness of \( \varphi \) left modulo \( P_Q \), we have the desired re-expression of the Eisenstein series

\[ E_s(g_\infty \cdot i) = \sum_{\gamma \in P_Q \setminus G_Q} \varphi(\gamma \cdot g_\infty) \quad \text{(with } g_\infty \in GL_2(\mathbb{R})) \]

This is the first main point, and there are further advantages to the viewpoint. Computation of the constant term is the first illustration, after the preparation of the next section.

4. Bruhat decomposition for \( GL_2 \)

Expressing the coset space \( P_\mathbb{Z} \setminus G_\mathbb{Z} \) in terms of rational matrices \( P_Q \setminus G_Q \), rather than integral matrices, simplifies natural choices of representatives immediately, as we will see. Further, more significantly, this makes visible the unwinding and Euler factorization of the Bruhat-cell summands of the constant term, as we see in the next section.

[4.1] Bruhat decomposition  The Bruhat decomposition for \( GL_2(k) \) for any field \( k \) is\(^6\)

\[ G_k = P_k \sqcup P_k w N_k \quad \text{(with } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}) \]

This purely algebraic fact has natural extensions to \( GL_n \) and the classical groups. For \( G = GL_2 \), the Bruhat decomposition is easy to prove: of course, the little cell \( P_Q \) consists of matrices with \( c = 0 \), and we must claim that the big cell \( P_k w N_k \) is exactly

\[ P_k w N_k = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k) : c \neq 0 \} \]

Indeed, given \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( c \neq 0 \),

\[ g \begin{pmatrix} 1 & -d/c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -d/c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \in P_k w \]

That proves the Bruhat decomposition in this simple case.

This expression for \( G_k \) exhibits a remarkably simple set of representatives for \( P_k \setminus G_k \) when rational rather than only integral entries are allowed:

\[ P_k \setminus G_k = P_k \setminus P_k \cup P_k \setminus (P_k \setminus (P_k w N_k)) \approx \{ 1 \} \cup w \cdot (w^{-1}P_k w \cap N_k) \setminus N_k \approx \{ 1 \} \cup w N_k \]

\[^{[6]}\] The validity of this decomposition for \( GL_2(k) \) or \( GL_2(k) \), and other specific groups, was known long before F. Bruhat’s work in the 1950s. Nevertheless, it is good practice to refer to the \( GL_2 \) case in terms indicated the extension to the general case.
where the bijection of cosets is

\[ P_{wn} = w \cdot (w^{-1}P_kw \cap N_k) \cdot n \]

Indeed, with \( H, N \) subgroups of a larger group, \( H \cdot n = H \cdot n' \) for \( n \in N \), if and only if \( n'n^{-1} \in H \). Also, \( n'n^{-1} \in N \), so

\[ H \cdot n = H \cdot n' \iff (H \cap N) \cdot n = (H \cap N) \cdot n' \quad \text{(for } n, n' \in N \text{)} \]

[4.2] Another comparison

Another computation verifies our rewrite of the Eisenstein series, paying attention to the Bruhat-cell parametrization. In principle, this computation is unnecessary, but it is informative.

Using the Bruhat decomposition, the sum defining the Eisenstein series is

\[ E_s(g_{\infty}) = \sum_{\gamma \in PQ \setminus GQ} \varphi(\gamma \cdot g_{\infty}) = \varphi(g_{\infty}) + \sum_{\gamma \in wNQ} \varphi(\gamma \cdot g_{\infty}) \]

That is, the summands can be parametrized by \( \{1\} \) and \( NQ \approx Q \). This will be important in computing the constant term in the next section.

As \( \varphi_{\infty} \) is right \( O(2) \)-invariant and center-invariant, by the Iwasawa decomposition \( G_{\infty} = P_{\infty} \cdot K_{\infty} \) we can take

\[ g_{\infty} = \begin{pmatrix} 1 & x & y & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \]

(with \( x \in \mathbb{R} \) and \( y > 0 \))

With this \( g_{\infty} \), the term \( \gamma = 1 \) is

\[ \varphi(g_{\infty}) = \varphi(1) \cdot \prod_{v < \infty} \varphi_v(1) = |y|^s \cdot 1 = |y|^s \]

For the big cell contribution to the sum, we need to compute the archimedean part

\[ \varphi_{\infty}(w \begin{pmatrix} 1 & t & y & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}) \]

(for \( t \in \mathbb{Q} \))

and the non-archimedean

\[ \varphi_v(w \begin{pmatrix} 1 & t & y & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}) \]

(for finite \( v \), with \( t \in \mathbb{Q} \))

In the archimedean case,

\[ w \begin{pmatrix} 1 & t & y & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ y & x + t \end{pmatrix} \]

Right multiplying by a suitable orthogonal matrix rotates the bottom row to put the result in \( P_{\mathbb{R}} \), namely,

\[ \begin{pmatrix} 0 & -1 \\ y & x + t \end{pmatrix} \begin{pmatrix} \frac{x+t}{\sqrt{(x+t)^2+y^2}} & \frac{y}{\sqrt{(x+t)^2+y^2}} \\ \frac{y}{\sqrt{(x+t)^2+y^2}} & \frac{x+t}{\sqrt{(x+t)^2+y^2}} \end{pmatrix} = \begin{pmatrix} \frac{y}{\sqrt{(x+t)^2+y^2}} & \frac{x}{\sqrt{(x+t)^2+y^2}} \\ 0 & 1 \end{pmatrix} \]

Thus,

\[ \varphi_{\infty}(w \begin{pmatrix} 1 & t & y & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}) = |\frac{y}{(x+t)^2+y^2}|^s \]

11
For finite $v$, adjust the given matrix by right multiplication by $GL_2(\mathbb{Z}_v)$ to make the result upper-triangular. For $t \in \mathbb{Z}_v$, the matrix is already in $GL_2(\mathbb{Z}_v)$, so

$$
\varphi_v\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = 1 \quad \text{(for finite $v$, with $t \in \mathbb{Q} \cap \mathbb{Z}_v$)}
$$

For $t \not\in \mathbb{Z}_v$, necessarily $t^{-1} \in \mathbb{Z}_v$. Thus, the matrix

$$
\begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix}
$$

can be multiplied by

$$
\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}
$$

in $GL_2(\mathbb{Z}_v)$ to obtain

$$
\begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} = \begin{pmatrix} t^{-1} & * \\ 0 & t \end{pmatrix}
$$

Thus,

$$
\varphi_v\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = |t|_v^{-2s} \quad \text{(for finite $v$, with $t \not\in \mathbb{Q} \cap \mathbb{Z}_v$)}
$$

That is,

$$
\varphi_v\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} 
1 & \text{(for $|t|_v \leq 1$)} \\
|t|_v^{-2s} & \text{(for $|t|_v > 1$)}
\end{cases}
$$

Combining the archimedean and non-archimedean,

$$
\varphi\left(w \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \left|\frac{y}{(x+t)^2 + y^2}\right|_\infty^s \cdot \prod_{v<\infty} \begin{cases} 
1 & \text{(for $|t|_v \leq 1$)} \\
|t|_v^{-2s} & \text{(for $|t|_v > 1$)}
\end{cases}
$$

Since $\mathfrak{o}$ is a principal ideal domain with units $\pm 1$, we can easily parametrize $t \in \mathbb{Q}$ in a fashion conforming to evaluation of the displayed expression. Namely, write $t = d/c$ with $c, d$ relatively prime, modulo $\pm 1$. Note that for relatively prime integers $c, d$

$$
\begin{cases} 
1 & \text{(for $|d/c|_v \leq 1$)} \\
|d/c|_v^{-2s} & \text{(for $|d/c|_v > 1$)}
\end{cases} = \begin{cases} 
1 & \text{(for $p_v$ not dividing $c$)} \\
|d/c|_v^{-2s} & \text{(for $p_v$ dividing $c$)}
\end{cases}
$$

That is, by the product formula,

$$
\prod_{v<\infty} \begin{cases} 
1 & \text{(for $|d/c|_v \leq 1$)} \\
|d/c|_v^{-2s} & \text{(for $|d/c|_v > 1$)}
\end{cases} = \prod_{v<\infty} \frac{|c|_v^{2s}}{|d/c|_v^{2s}} = 1
$$

Then, once again, we recover the expected:

$$
\left|\frac{y}{(x+t)^2 + y^2}\right|_\infty^s \cdot \frac{1}{|c|_\infty^{2s}} = \left|\frac{y}{(cz+d)^2 + (cy)^2}\right|_\infty^s = \frac{y^s}{|cz+d|^{2s}}
$$

Again, in principle the above computation is unnecessary, but it is informative to see the details of the reversion to a classical form.
5. Application: constant term of $GL_2$ Eisenstein series

An immediate use of the localized rewrite of the Eisenstein series is computation of the constant term presenting each Bruhat cell’s contribution as an Euler product, by unwinding the integral defining the constant term.

[5.1] Transition for the constant term In these simplest situations, the constant term of any kind of modular/automorphic form is the zero-th Fourier component, separating variables in the $x + iy$ coordinates:

\[ \text{constant term of } f(iy) = \int_0^1 f(x + iy) \, dx \]

Since $f(x + iy)$ is periodic in $x$, we obtain the same outcome integrating over any interval $[a, a + 1]$ in place of $[0, 1]$. In fact, the integral is over the quotient $\mathbb{R}/\mathbb{Z}$

\[ c_p f(iy) = \int_{\mathbb{R}/\mathbb{Z}} f(x + iy) \, dx \]

Further, the integral can be written as an integral over a quotient of a subgroup of $G_\infty$, namely, with $N = \{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \}$,

\[ c_p f(iy) = \int_{N_\mathbb{Z}\backslash N_\mathbb{R}} f(n \cdot iy) \, dn \]

where the Haar measure on $N_\mathbb{R}$ is really just the usual measure on $\mathbb{R}$.

[5.1.1] Claim: $\mathbb{R} + q^{\Delta} + \mathbb{Z} = A$.

Proof: An adele $x$ fails to be in $\mathbb{Z}_v$ only for $v$ in a finite set $S$ of finite places $v = p$. At such $v$, we can write

\[ x_v = \frac{a_\ell}{p^\ell} + \ldots + \frac{a_1}{p} + a_o + a_1 p^1 + \ldots \quad \text{(with } a_i \in \mathbb{Z}) \]

The truncated sum

\[ y_v = \frac{a_\ell}{p^\ell} + \ldots + \frac{a_1}{p} + a_o \]

is a rational number, and is $v'$-integral at all other finite $v'$. Thus,

\[ y = \sum_{v \in S} y_v \]

is a rational number, and $x - y$ is everywhere locally integral, where $y \in q^{\Delta}$. That is, $x - y \in \mathbb{R} + \mathbb{Z}$. //

[5.1.2] Remark: The point of the corollary is not to get from $A/\mathbb{Q}$ back to $\mathbb{R}/\mathbb{Z}$, but to get from the classical quotient $\mathbb{R}/\mathbb{Z}$ to $A/\mathbb{Q}$.

[5.1.3] Remark: In fact, (additive approximation) asserts that $\mathbb{R} + q^{\Delta}$ is dense in $A$, where $q^{\Delta}$ is the diagonal copy. Equivalently, the diagonal copy of $\mathbb{Q}$ in the finite adeles $A_{\text{fin}}$ is dense.

Thus, $A/\mathbb{Q}$ has representatives in $\mathbb{R} + \mathbb{Z}$. Two elements $r, r' \in \mathbb{R}$ have the same image in $A/\mathbb{Q}$ if and only if $r - r' \in \mathbb{Q} \cap (\mathbb{R} + \mathbb{Z})$. The latter intersection is the diagonal copy of $\mathbb{Z}$, since a rational number that is in $\mathbb{Z}_v$ for all $v < \infty$ is in $\mathbb{Z}$. Thus, $\mathbb{R}/\mathbb{Z}$ injects to $A/\mathbb{Q}$. 

13
Applying this to the coordinate in the subgroup \( N = GL_2(\mathbb{A}) \),
\[
N_\mathbb{Z} \cdot \left( N_\mathbb{R} \cdot (N_\mathbb{A} \cap K_{\text{fin}}) \right) = N_\mathbb{A}
\]
and
\[
N_\mathbb{Z} \setminus (N_\mathbb{R} \cdot (N_\mathbb{A} \cap K_{\text{fin}})) = N_\mathbb{Q} \setminus N_\mathbb{A}
\]
For right \( K_{\text{fin}} \)-invariant \( f \) on the adele group, for \( n_\infty \in N_\infty \) and \( n_o \in N_{\text{fin}} \),
\[
f(n_\infty \cdot g_\infty) = f(n_\infty \cdot g_\infty \cdot n_o) = f(n_\infty \cdot n_o \cdot g_\infty)
\]
because \( G_\infty \) and \( G_{\text{fin}} \) commute as subgroups of \( G_\mathbb{A} \). Thus, evaluating the constant term on \( g_\infty \),
\[
c_P f(g_\infty) = \int_{N_\mathbb{Z} \setminus N_\mathbb{R}} f(n_\infty \cdot g_\infty) \, dn_\infty = \int_{N_\mathbb{Z} \setminus N_\mathbb{R}} \int_{N_\mathbb{A} \cap K_{\text{fin}}} f(n_\infty \cdot g_\infty \cdot n_o) \, dn_o \, dn_\infty
\]
\[
= \int_{N_\mathbb{Z} \setminus N_\mathbb{R}} \int_{N_\mathbb{A} \cap K_{\text{fin}}} f(n_\infty \cdot n_o \cdot g_\infty) \, dn_o \, dn_\infty = \int_{N_\mathbb{Q} \setminus N_\mathbb{A}} f(n \cdot g_\infty) \, dn
\]

[5.2] Unwinding With
\[
E_s(g_\infty) = \sum_{\gamma \in P_\mathbb{Q} \setminus G_\mathbb{Q}} \varphi(\gamma \cdot g_\infty)
\]
the constant term of \( E_s \) along \( P \) is the adelic integral
\[
c_P E_s(g) = \int_{N_\mathbb{Q} \setminus N_\mathbb{A}} E_s(ng) \, dn
\]
Parametrizing \( P_\mathbb{Q} \setminus G_\mathbb{Q} \) via the Bruhat decomposition makes computation nearly trivial:
\[
\int_{N_\mathbb{Q} \setminus N_\mathbb{A}} E_s(ng) \, dn = \int_{N_\mathbb{Q} \setminus N_\mathbb{A}} \sum_{\gamma \in P_\mathbb{Q} \setminus G_\mathbb{Q}} \varphi(\gamma ng) \, dn = \sum_{w \in P_\mathbb{Q} \setminus G_\mathbb{Q}} \int_{N_\mathbb{Q} \setminus N_\mathbb{A}} \sum_{\gamma \in P_\mathbb{Q} \setminus P_\mathbb{Q} w N_\mathbb{Q}} \varphi(\gamma ng) \, dn
\]
By the Bruhat decomposition, \( P_\mathbb{Q} \setminus G_\mathbb{Q} / N_\mathbb{Q} \) has exactly two representatives, \( 1, w \), and the constant term becomes, upon unwinding the second sum-and-integral,
\[
\int_{N_\mathbb{Q} \setminus N_\mathbb{A}} \varphi(ng) \, dn + \int_{N_\mathbb{Q} \setminus N_\mathbb{A}} \sum_{\gamma \in N_\mathbb{Q}} \varphi(w \gamma ng) \, dn = \int_{N_\mathbb{Q} \setminus N_\mathbb{A}} \varphi(ng) \, dn + \int_{N_\mathbb{A}} \varphi(wng) \, dn
\]
Because \( \varphi \) is left \( N_\mathbb{A} \)-invariant, the first of the two summands is
\[
\int_{N_\mathbb{Q} \setminus N_\mathbb{A}} \varphi(ng) \, dn = \varphi(g) \cdot \text{vol} \left( (N_\mathbb{Q} \setminus N_\mathbb{A}) \right) \quad \text{(the small Bruhat cell contribution)}
\]
Since the integral in the second summand unwound, it factors over primes
\[
\int_{N_\mathbb{A}} \varphi(wng) \, dn = \prod_{\nu \leq \infty} \int_{N_\nu} \varphi_\nu(wng_\nu) \, dn
\]

[5.3] Essential features of \( p \)-adic integrals We do not need many formulaic details about integrals on \( \mathbb{Q}_p \), since we will only consider \( \mathbb{Z}_p^\times \)-invariant integrands \( f(x) \), that is, \( f(\eta \cdot x) = f(x) \) for all \( \eta \in \mathbb{Z}_p^\times \) and \( x \in \mathbb{Q}_p \).
It is reasonable to normalize the additive Haar measure on \( \mathbb{Q}_p \) so that the compact, open subgroup \( \mathbb{Z}_p \) has total measure 1.

The \( p^n \) distinct compact, open subgroups \( a + p^n \mathbb{Z}_p \subset \mathbb{Z}_p \) are translates of each other, and mutually disjoint, so the total measure of \( a + p^n \mathbb{Z}_p \) is \( p^{-n} \) for \( a \in \mathbb{Z}_p \). By translation-invariance, the total measure of \( a + p^n \mathbb{Z}_p \) is \( p^{-n} \) for any \( a \in \mathbb{Q}_p \).

For \( \eta \in \mathbb{Z}_p^\times \),
\[
\eta \cdot (a + p^n \mathbb{Z}_p) = \eta a + p^n \mathbb{Z}_p
\]
Thus, multiplication by units preserves Haar measure.\[\text{[7]}\] The expression
\[
\mathbb{Z}_p^\times = \mathbb{Z}_p - p \mathbb{Z}_p
\]
shows that the measure of \( \mathbb{Z}_p^\times \) is \( 1 - \frac{1}{p} \). Similarly, the measure of \( p^n \mathbb{Z}_p^\times \) is \( (1 - \frac{1}{p}) p^{-n} \). Thus, with continuous \( \mathbb{Z}_p^\times \)-invariant \( f \),
\[
\int_{\mathbb{Z}_p} f(x) \, dx = \sum_{\ell=0}^{\infty} \int_{p^{\ell} \mathbb{Z}_p} f(x) \, dx = \sum_{\ell=0}^{\infty} f(p^\ell) \cdot \int_{p^{\ell} \mathbb{Z}_p^\times} 1 \, dx = \sum_{\ell=0}^{\infty} f(p^\ell) \cdot (1 - \frac{1}{p}) p^{-\ell}
\]

[5.4] Evaluation of local factors: non-archimedean case For \( g \in G_\infty \), so that \( g_v = 1 \), the finite-prime local factors in the Euler product for the big Bruhat cell are readily evaluated, as follows. Above, we computed
\[
\varphi_v \left( \begin{array}{c} 1 \\ t \\ 1 \end{array} \right) = \begin{cases} 1 & \text{(for } |t|_v \leq 1) \\ |t|_v^{2s} & \text{(for } |t|_v > 1) \end{cases}
\]
With the \( v \)-adic factor corresponding to prime \( p \), the \( v \)-adic local factor is
\[
\int_{\mathbb{Z}_p} 1 \, dt = \int_{|t|_v \leq 1} |t|_v^{2s} \, dt = 1 + \sum_{\ell=1}^{\infty} \frac{|p^{-\ell}|_v^{2s}}{p^{-\ell}} = 1 + \sum_{\ell=1}^{\infty} \frac{(p^\ell)^{-2s}}{p^{\ell}} = \frac{1 - p^{-2s}}{1 - p^{-1}} = \frac{\zeta_v(2s) - 1}{\zeta_v(2s)}
\]
where \( \zeta_v(s) \) is the \( v \)th Euler factor of the zeta function. Thus, the finite-prime part of the big-cell summand is \( \zeta(2s - 1)/\zeta(2s) \).

[5.5] Evaluation of local factors: archimedean case The archimedean factor of the big-cell summand of the constant term is
\[
\int_{\mathbb{R}} \frac{y}{(x + t)^2 + y^2} \, dt = \int_{\mathbb{R}} \frac{y^{-s}}{(x + t)^2 + y^2} \, dt = \frac{1}{\Gamma(s)} \int_{\mathbb{R}} \frac{1}{(t^2 + y^2)^s} \, dt = y^{1-s} \cdot \frac{1}{\Gamma(s)} \int_{\mathbb{R}} \frac{1}{(ty)^s} \, dt = y^{1-s} \cdot \frac{1}{\Gamma(s)} \int_{\mathbb{R}} e^{u(t^2 + y^2)} \frac{du}{u} \, dt = y^{1-s} \cdot \frac{1}{\Gamma(s)} \int_{\mathbb{R}} e^{u + t^2} u s^{-\frac{1}{2}} \frac{du}{u} \, dt
\]
\[
= y^{1-s} \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} = y^{1-s} \frac{\pi^{-\frac{1}{2}} \Gamma(s - \frac{1}{2})}{\Gamma(s)} = y^{1-s} \frac{\zeta_\infty(2s - 1)}{\zeta_\infty(2s)} \quad \text{(with } \zeta_\infty(s) = \pi^{-s/2} \Gamma(s/2))
\]
\[\text{[7]}\] Quite generally, a compact group of automorphisms of a topological group must preserve the Haar measure on the latter.
[5.6] Conclusion of constant-term computations Thus, with $\xi(s)$ the completed zeta function $\xi(s) = \zeta_\infty(s) \cdot \zeta(s)$, the constant term of $E_s$ is

$$c_pE_s(x + iy) = y^s + \frac{\xi(2s - 1)}{\xi(2s)} \cdot y^{1-s}$$

The present point is that rewriting the Eisenstein series as an automorphization of a product of local data makes the computation of the constant term far more natural, and more genuinely representative of the corresponding computation for larger groups.

6. Application: Hecke operators on $GL_2$ Eisenstein series

The rewritten Eisenstein series will show that the Hecke operators are not global things, but are local, just acting on the local components $\varphi_v$. Indeed, the local components $\varphi_v$ are eigenfunctions for the local version of Hecke operators, with eigenvalues depending on the parameter $s$.

[6.1] Classical description of Hecke operators The $p^{th}$ Hecke operator $T_p$ on weight-0 automorphic forms $f$ for $\Gamma = GL_2(\mathbb{Z})$ is

$$T_p f(z) = \sum_{\gamma \in \Gamma \setminus \Theta_p} f(\gamma \cdot z) \quad \text{(where } \Theta_p = \text{integer matrices with det } = p)$$

with action by linear fractional transformations $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) : z \to \frac{az + b}{cz + d}$.

[6.2] Hecke operators on rewritten Eisenstein series Directly computing, with $g_\infty \in G_\infty$,

$$T_p E_s(g_\infty) = \sum_{\delta \in \Gamma \setminus \Theta_p} E_s(j_\infty(\delta) \cdot g_\infty) = \sum_{\delta \in \Gamma \setminus \Theta_p} \sum_{\gamma \in P_\mathbb{Q} \setminus G_\mathbb{Q}} \varphi(\gamma \cdot j_\infty(\delta) \cdot g_\infty)$$

Let $j_0 = \prod_{v<\infty} j_v$. Replace $\gamma$ by $\gamma \cdot \delta^{-1}$ in $G_\mathbb{Q}$, to obtain

$$\sum_{\delta \in \Gamma \setminus \Theta_p} \sum_{\gamma \in P_\mathbb{Q} \setminus G_\mathbb{Q}} \varphi(\gamma \cdot j_0(\delta^{-1}) \cdot g_\infty)$$

The finite-prime factor $j_v(\delta^{-1})$ commutes with the archimedean-prime factor $g_\infty$, so this is

$$\sum_{\delta \in \Gamma \setminus \Theta_p} \sum_{\gamma \in P_\mathbb{Q} \setminus G_\mathbb{Q}} \varphi(\gamma \cdot g_\infty \cdot j_0(\delta^{-1})) = \sum_{\delta \in \Gamma \setminus \Theta_p} \sum_{\gamma \in P_\mathbb{Q} \setminus G_\mathbb{Q}} \varphi_\infty(j_\infty(\gamma) \cdot g_\infty) \cdot \prod_{v<\infty} \varphi_v(j_v(\gamma \cdot \delta^{-1}))$$

At all finite places $v'$ but $v \sim p$, $\delta^{-1}$ is in the local maximal compact $K_{v'} = GL_{v'}(\mathbb{Z}_{v'})$, so $\varphi_{v'}(\gamma \cdot \delta^{-1}) = \varphi_{v'}(\gamma)$ for $v' \neq v$. Thus, suppressing $j_v$ in the notation,

$$T_p E_s(g_\infty) = \sum_{\gamma \in P_\mathbb{Q} \setminus G_\mathbb{Q}} \varphi(\gamma \cdot g_\infty) \cdot \sum_{\delta \in \Gamma \setminus \Theta_p} \varphi_v(\gamma \cdot \delta^{-1})$$

[9] That is, the weight-0 situation allows us to avoid worry over what to do with the determinant in $GL_2$. In the classical holomorphic case, the so-called slash operator is a normalization that accommodates this, usually without explanation or motivation.
Thus, the \( p \)-th Hecke operator’s effect is \textit{local} at \( v \sim p \). Further, this situation correctly suggests that we should hope that \( \varphi_v \) is an \textit{eigenfunction} for the effect of \( T_p \), with eigenvalue depending on the complex parameter \( s \).

\[ \textbf{[6.3] Hecke operators as integral operators} \]

Continue to let \( v \) correspond to prime \( p \). The function \( \varphi_v \) on \( G_v = GL_2(\mathbb{Q}_v) \) is left \( P_v \)-equivariant by \( \chi_v \), and right \( K_v \)-invariant. Each of the right translates \( g \to \varphi_v(g \cdot \delta^{-1}) \) with \( g \in G_v \) retains the left \( P_v, \chi_v \)-equivariance, but cannot be expected to retain right \( K_v \)-invariance.

Nevertheless, we claim that the sum over \( \delta \in \Gamma \setminus \Theta_p \) recovers the right \( K_v \)-invariance. That is, apparently, \( \Gamma \setminus \Theta_p \), or its image projected to \( G_v \), is stable under right multiplication by \( K_v \). As it stands, this doesn’t make sense, since \( \Theta_p \) itself (projected to \( G_v \)) is not literally stable under right multiplication by \( K_v \).

As on other occasions, the necessary claim suggests itself: let \( \widetilde{\Theta}_v \) be the \( \nu \)-adic analogue of \( \Theta_p \), namely, elements of \( G_v = GL_2(\mathbb{Q}_v) \) with entries in \( \mathbb{Z}_v \) and determinant of \( p \)-adic ord 1. Then we \textit{must} claim that the natural map \( \Theta_p^{-1} \to \widetilde{\Theta}_v^{-1} / K_v \) induces a \textit{bijection}

\[ \Theta_p^{-1} / \Gamma \to \widetilde{\Theta}_v^{-1} / K_v \]

Equivalently, inverting, \( \Theta_p \to K_v \setminus \widetilde{\Theta}_v \) induces a bijection

\[ \Gamma \setminus \Theta_p \to K_v \setminus \widetilde{\Theta}_v \]

Indeed, \( K_v \setminus \widetilde{\Theta}_v \) has the same representatives as \( \Gamma \setminus \Theta_p \), namely, \(^{[9]}\)

\[
\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \quad \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \quad \text{ (with } b \in \mathbb{Z}, 0 \leq b < p) \]

Thus, giving \( K_v \) total measure 1, using the right \( K_v \)-invariance of \( \varphi_v \),

\[
\sum_{\delta \in \Gamma \setminus \Theta_p} \varphi_v(g \cdot \delta^{-1}) = \sum_{h \in \widetilde{\Theta}_v^{-1} / K_v} \varphi_v(g \cdot h) = \int_{\widetilde{\Theta}_v^{-1}} \varphi_v(g \cdot h) \, dh
\]

Letting \( \eta \) be the characteristic function of \( \widetilde{\Theta}_v^{-1} \), this integral is an integral operator attached to the right translation action of \( G_v \) on functions on \( G_v \):

\[
\sum_{\delta \in \Gamma \setminus \Theta_p} \varphi_v(g \cdot \delta^{-1}) = \int_{G_v} \eta(h) \varphi_v(g \cdot h) \, dh = (\eta \cdot \varphi_v)(g)
\]

The integral expression makes right \( K_v \)-invariance clear, by changing variables in the integral:

\[
\int_{G_v} \eta(h) \varphi_v(gk \cdot h) \, dh = \int_{G_v} \eta(k^{-1}h) \varphi_v(g \cdot h) \, dh = \int_{G_v} \eta(h) \varphi_v(g \cdot h) \, dh \quad \text{ (for } k \in K_v)\]

\(^{[9]}\) The \( \nu \)-adic argument is easier than that over \( \mathbb{Z} \): given \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) in \( \widetilde{\Theta}_v \), if \( \gcd(a, c) = 1 \), then either \( a \) or \( c \) is in \( \mathbb{Z}_v \). Thus, left multiplication by either \( \begin{pmatrix} 1 & 0 \\ -c/a & 1 \end{pmatrix} \) or \( \begin{pmatrix} 0 & 1 \\ 1 & -a/c \end{pmatrix} \) puts \( g \) into the form \( \begin{pmatrix} 1 & b' \\ 0 & d' \end{pmatrix} \). Necessarily \( \text{ord}_v d' = 1 \), so further left multiplication gives the form \( \begin{pmatrix} 1 & b'' \\ 0 & a \end{pmatrix} \). Since \( \mathbb{Z}_v / p\mathbb{Z}_v \approx \mathbb{Z} / p\mathbb{Z} \), further left multiplication by \( \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in K_v \) gives the indicated representatives parametrized by \( b \). When \( \gcd(a, c) = p \), a similar argument gives the form \( \begin{pmatrix} p & b' \\ 0 & 1 \end{pmatrix} \), and now the \( b' \) entry can be made 0.
since $\eta$ is left and right $K_v$-invariant.

[6.4] Hecke eigenvalues Now we can prove that $\varphi_v$ is an eigenvector for the localized version of $T_p$ (with $v \sim p$), and compute its eigenvalue. The $v$-adic Iwasawa decomposition is $G_v = P_v \cdot K_v$. Thus, up to constant multiples, there is a unique left $P_v, \chi_v$-equivariant, right $K_v$-invariant function on $G_v$. Thus, every such is a multiple of $\varphi_v$.

In particular, with $\eta$ the characteristic function of $\tilde{\Theta}_v^{-1}$, necessarily $\eta \cdot \varphi_v = \lambda_s \cdot \varphi_v$ for some $\lambda_s \in \mathbb{C}$. To determine $\lambda_s$, it suffices to evaluate at $g = 1$, using $\varphi_v(1) = 1$. Thus,

$$
\lambda_s = \int_{G_v} \eta(h) \varphi_v(h) \, dh = \int_{\tilde{\Theta}_v^{-1}/K_v} \varphi_v(h) \, dh = \sum_{\delta \in \Gamma \backslash \Theta_p} \varphi_v(\delta^{-1})
$$

$$
= \sum_b \chi_v\left(\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}^{-1}\right) + \chi_v\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}\right) = p \cdot \chi_v\left(\begin{pmatrix} 1 & * \\ 0 & p^{-1} \end{pmatrix}\right) + \chi_v\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)
$$

$$
= p \cdot \left|\frac{1}{p^{-1}}\right|_v^s + \left|\frac{p^{-1}}{1}\right|_v^s = p^{1-s} + p^s
$$

This is the $p^{th}$ Hecke eigenvalue of $E_s$:

$$
T_p E_s = (p^{1-s} + p^s) \cdot E_s
$$