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# Review of metric spaces

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We review basics concerning metric spaces from a modern viewpoint, and prove the *Baire category theorem*, for both *complete metric* spaces and *locally compact Hausdorff*<sup>[1]</sup> spaces. The general idea of *metric space* appeared in [Fréchet 1906], and metric-space structures on vector spaces, especially *spaces of functions*, was developed by [Fréchet 1928] and [Hausdorff 1931]. See [Dieudonné 1981].

- Metric spaces, completeness
- Completions
- Baire category theorem

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## 1. Metric spaces, continuous maps, completeness

### [1.1] Abstract notion of metric space

The abstracted idea of a *metric space* is of a set with a notion of *distance* on it,

$$d(x, y) = \text{distance from } x \text{ to } y$$

which should conform to basic physical intuition about distance. Thus, first, the only point  $y$  at distance 0 from a point  $x$  is  $y = x$  itself. Second, distance does not depend on *direction*: the distance from  $x$  to  $y$  is the same as the distance from  $y$  to  $x$ . Third, thinking that the distance from  $x$  to  $y$  should correspond to a *shortest route* of some kind from  $x$  to  $y$ , the distance from  $x$  to  $y$  should be at most the sum of distances from  $x$  to any other intermediate point  $z$  and then from  $z$  to  $y$ , giving the *triangle inequality*.

That is, a *metric space*  $X, d$  is a set  $X$  with a *metric*  $d(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$ , a real-valued function such that, for  $x, y, z \in X$ ,

- (*Positivity*)  $d(x, y) \geq 0$  and  $d(x, y) = 0$  if and only if  $x = y$
- (*Symmetry*)  $d(x, y) = d(y, x)$
- (*Triangle inequality*)  $d(x, z) \leq d(x, y) + d(y, z)$

A *Cauchy sequence* in a metric space  $X$  is a sequence  $x_1, x_2, \dots$  with the property that for every  $\varepsilon > 0$  there is  $N$  sufficiently large such that for  $i, j \geq N$  we have  $d(x_i, x_j) < \varepsilon$ . A point  $x \in X$  is a **limit** of that Cauchy sequence if for every  $\varepsilon > 0$  there is  $N$  sufficiently large such that for  $i \geq N$  we have  $d(x_i, x) < \varepsilon$ . A subset  $X$  of a metric space  $Y$  is **dense** in  $Y$  if every point in  $Y$  is a limit of a Cauchy sequence in  $X$ .

### [1.2] Continuous maps

Generalizing the usual discussion of continuity on  $\mathbb{R}^n$ , a map  $f : X \rightarrow Y$  from one metric space to another is *continuous* at  $x \in X$  if, for every  $\varepsilon > 0$ , there is  $\delta > 0$  such that  $d_X(x, x') < \delta$  implies  $d_Y(f(x), f(x')) < \varepsilon$ .

The map  $f : X \rightarrow Y$  is *continuous* if it is continuous at every point of  $X$ .

[1.2.1] **Proposition:** A map  $f : X \rightarrow Y$  continuous if and only if, for every Cauchy sequence  $\{x_i\}$  in  $X$  converging to  $x \in X$ ,

$$\lim_i f(x_i) = f(\lim_i x_i)$$

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[1] Recall that a topological vector space is *locally compact* if every point has an open neighborhood with compact closure. A space is *Hausdorff* if for any two points  $x, y$  there are opens  $U, V$  such that  $x \in U, y \in V$ , and  $U \cap V = \emptyset$ .

*Proof:* On one hand, for continuous  $f$  and  $x_i \rightarrow x$ , given any ball at  $x$ , from some point onward all the  $x_i$  are inside that ball. Thus, eventually,  $f(x_i)$  is close to  $f(x)$ , so  $\lim_i f(x_i) = f(x)$ .

On the other hand, if  $f$  were *not* continuous at  $x$ , then for some  $\varepsilon > 0$ , for every  $\delta > 0$  there is some  $x'$  inside the  $\delta$ -ball at  $x$  such that  $|f(x) - f(x')| \geq \varepsilon$ . Take the sequence of  $\delta$ 's given by  $\delta_n = 1/n$ , and let  $x_n$  be the corresponding bad  $x'$ . Then  $x_n \rightarrow x$  but  $f(x_n) \not\rightarrow f(x)$ , so  $f$  is not continuous at  $x$ . ///

[1.2.2] **Remark:** The property of the proposition is *sequential* continuity. It is necessary to assume explicitly that  $\{x_i\}$  has a limit  $x \in X$ , or else make the blanket assumption that  $X$  is *complete*, as just below.

### [1.3] Completeness

A metric space is *complete* if every Cauchy sequence has a limit. <sup>[2]</sup> Unsurprisingly, *complete* metric spaces have the advantage that *infinitary* operations often make sense.

The following standard lemma makes some intuition explicit:

[1.3.1] **Lemma:** Let  $\{x_i\}$  be a Cauchy sequence in a metric space  $X, d$  converging to  $x$  in  $X$ . Given  $\varepsilon > 0$ , let  $N$  be sufficiently large such  $d(x_i, x_j) < \varepsilon$  for  $i, j \geq N$ . Then  $d(x_i, x) \leq \varepsilon$  for  $i \geq N$ .

*Proof:* Let  $\delta > 0$  and take  $j \geq N$  also large enough such that  $d(x_j, x) < \delta$ . Then for  $i \geq N$  by the triangle inequality

$$d(x_i, x) \leq d(x_i, x_j) + d(x_j, x) < \varepsilon + \delta$$

Since this holds for every  $\delta > 0$  we have the result. ///

### [1.4] Metric topologies

A metric space  $X$  has a *natural topology* with *basis* given by *open balls*

$$\{y \in X : d(x, y) < r\} = (\text{open ball of radius } r > 0 \text{ centered at } x \in X)$$

That is, a set  $U \subset X$  is *open* when around every point  $x \in U$  there is an open ball of positive radius contained in  $U$ .

## 2. Completions

A not-complete metric space presents the difficulty that Cauchy sequences may fail to converge. Reasonably, we want to repair this situation, as economically as possible. <sup>[3]</sup>

Our *intention* is that, for a not-complete metric space  $X, d_X$ , there should be a *complete* metric space  $\tilde{X}, d_{\tilde{X}}$  and *isometry* (distance-preserving)  $j : X \rightarrow \tilde{X}$ , such that every continuous  $f : X \rightarrow Y$  to *complete* metric

[2] Convergence of Cauchy *sequences* is more properly *sequential completeness*. For metric spaces, sequential completeness *implies* implies the strongest form of completeness, namely convergence of Cauchy *nets*, as we will observe more carefully later. This is *not* important at the moment, but will have importance for non-metrizable spaces, *rarely* complete (in the strongest sense), but in practice often at least sequentially complete. A useful form of completeness stronger than sequential completeness but weaker than outright completeness is *local completeness*, also called *quasi-completeness*, which will play a significant role later.

[3] One traditional approach is to immediately *construct* a *completion* as equivalence class of Cauchy sequences in  $X$ . This has the disadvantage that it does not address the possibility of other constructions yielding different outcomes.

space  $Y$  factors through  $j$  uniquely. That is, there are *commutative diagrams*<sup>[4]</sup> of continuous maps

$$\begin{array}{ccc}
 \tilde{X} & & \\
 \uparrow j & \searrow \exists! & \\
 X & \xrightarrow{\quad} & Y
 \end{array}
 \quad (\text{for every continuous } X \rightarrow Y \text{ ???})$$

By chance, this characterization accidentally asks impossibly much of the completion. The basic problem is that completeness is not a *topological property*, but *metric*, as merely-continuous maps can fail to map Cauchy sequences to Cauchy sequences.<sup>[5]</sup> That is, the universal property is plausibly only for some restricted class of continuous functions  $f : X \rightarrow Y$  preserving Cauchy sequences. An obvious class is *non-expanding* or *contractive*  $f$ , meaning  $d_Y(f(x), f(x')) \leq d_X(x, x')$  for all  $x, x' \in X$ . A somewhat weaker condition on  $f$  that promises preservation of Cauchy sequences is *uniform continuity* of  $f : X \rightarrow Y$ , meaning that, for any  $\varepsilon > 0$ , there is  $\delta > 0$  such that, for all  $x, x' \in X$ ,  $d_X(x, x') < \delta$  implies  $d_Y(f(x), f(x')) < \varepsilon$ . Uniformly continuous maps preserve Cauchy sequences.<sup>[6]</sup> Thus, one adequately corrected condition is

$$\begin{array}{ccc}
 \tilde{X} & & \\
 \uparrow j & \searrow \exists! & \\
 X & \xrightarrow{\quad} & Y
 \end{array}
 \quad (\text{for every } \textit{contractive} \ X \rightarrow Y)$$

Such an  $\tilde{X}$  is a *completion* of  $X$ . Thus, we *characterize* completions by certain of their interaction with other metric spaces, explaining our expectations, rather than giving a construction with unclear motivations.

We will prove that completions *exist* (by constructing one) and are essentially *unique*.

Before any *constructions* of completions, we can prove some things about the behavior of any *possible* completion. In particular, that any two completions are *naturally isometric* to each other. *Thus, the outcome will be independent of construction.*

**[2.0.1] Proposition:** (*Uniqueness*) Let  $i : X \rightarrow Y$  and  $j : X \rightarrow Z$  be two completions of a metric space  $X$ . Then there is a unique *isometric isomorphism*  $h : Y \rightarrow Z$  such that  $j = h \circ i$ . That is, we have commutative diagram

$$\begin{array}{ccc}
 Y & \xrightarrow{\quad \exists! \quad} & Z \\
 \swarrow i & & \nearrow j \\
 & X &
 \end{array}$$

*Proof:* First, take  $Y = Z$  and  $f : X \rightarrow Y$  to be the inclusion  $i$ , in the characterization of  $i : X \rightarrow Y$ . The defining characterization of  $i : X \rightarrow Y$  shows that there is *unique* contractive  $f : Y \rightarrow Y$  fitting into a

<sup>[4]</sup> A diagram of maps is *commutative* when the composite map from one object to another within the diagram does not depend on the route taken within the diagram.

<sup>[5]</sup> The interval  $[1, +\infty)$  and the half-open interval  $(0, 1]$  are homeomorphic by  $x \rightarrow 1/x$ , but with the usual metric the former is complete while the latter is not. The Cauchy sequence  $1, \frac{1}{2}, \frac{1}{3}, \dots$  in  $(0, 1]$  maps back to the non-Cauchy sequence  $1, 2, 3, \dots$  in  $[1, \infty)$ .

<sup>[6]</sup> For *uniformly* continuous  $f : X \rightarrow Y$  and Cauchy  $\{x_i\}$  in  $X$ , given  $\varepsilon > 0$ , choose  $\delta > 0$  so that  $d_X(x, x') < \delta$  implies  $d_Y(f(x), f(x')) < \varepsilon$ . Choose  $N$  large enough so that  $d_X(x_i, x_j) < \delta$  for  $i, j \geq N$ . Then  $d_Y(f(x_i), f(x_j)) < \varepsilon$  for all  $i, j \geq N$ , so  $\{f(x_i)\}$  is Cauchy in  $Y$ .

commutative diagram

$$\begin{array}{ccc} & Y & \\ & \uparrow & \searrow \exists! f \\ X & \xrightarrow{i} & Y \end{array}$$

Since the *identity* map  $Y \rightarrow Y$  is contractive and fits into this diagram, the *only* contractive  $f$  fitting into the diagram is the identity on  $Y$ .

Next, applying the characterizations of both  $i : X \rightarrow Y$  and  $j : X \rightarrow Z$ , we have unique contractive  $f : Y \rightarrow Z$  and  $g : Z \rightarrow Y$  fitting into

$$\begin{array}{ccc} Y & & Z \\ \uparrow i & \searrow \exists! f & \uparrow j \\ X & \xrightarrow{j} & Z \end{array} \qquad \begin{array}{ccc} Z & & Y \\ \uparrow j & \searrow \exists! g & \uparrow i \\ X & \xrightarrow{i} & Y \end{array}$$

Then  $f \circ g : Y \rightarrow Y$  and  $g \circ f : Z \rightarrow Z$  fit into

$$\begin{array}{ccc} Y & & Y \\ \uparrow i & \searrow f \circ g & \uparrow i \\ X & \xrightarrow{i} & Y \end{array} \qquad \begin{array}{ccc} Z & & Z \\ \uparrow j & \searrow g \circ f & \uparrow j \\ X & \xrightarrow{j} & Z \end{array}$$

By the first observation, this means that  $f \circ g$  is the identity on  $Y$ , and  $g \circ f$  is the identity on  $Z$ . Since the composition of two contractive maps can be an isometry only when both are isometries,  $f$  and  $g$  are mutual inverses, and  $Y$  and  $Z$  are *isometrically isomorphic*. ///

[2.0.2] **Remark:** A virtue of the characterization of completion is that it does not refer to the *internals* of any completion.

Next, prove that the mapping-property characterization of a completion does not introduce more points than absolutely necessary:

[2.0.3] **Proposition:** Every point in a completion  $\tilde{X}$  of  $X$  is the limit of a Cauchy sequence in  $X$ . That is,  $X$  is *dense* in  $\tilde{X}$ .

*Proof:* Write  $d(\cdot, \cdot)$  for both the metric on  $X$  and its extension to  $\tilde{X}$ . Let  $Y \subset \tilde{X}$  be the collection of limits of Cauchy sequences of points in  $X$ . We claim that  $Y$  itself is *complete*. Indeed, given a Cauchy sequence  $\{y_i\}$  in  $Y$  with limit  $z \in \tilde{X}$ , let  $x_i \in X$  such that  $d(x_i, y_i) < 2^{-i}$ . It will suffice to show that  $\{x_i\}$  is Cauchy with limit  $z$ . Indeed, given  $\varepsilon > 0$ , take  $N$  large enough so that  $d(y_i, z) < \varepsilon/2$  for all  $i \geq N$ , and increase  $N$  if necessary so that  $2^{-i} < \varepsilon/2$ . Then, by the triangle inequality,  $d(x_i, z) < \varepsilon$  for all  $i \geq N$ . Thus,  $Y$  is complete.

By the defining property of  $\tilde{X}$ , every contractive  $f : X \rightarrow Z$  to complete  $Z$  has a unique extension to a contractive  $F : \tilde{X} \rightarrow Z$  fitting into

$$\begin{array}{ccc} Y & \xrightarrow{c} & \tilde{X} \\ & \searrow & \uparrow j \\ & & X \\ & \swarrow & \xrightarrow{f} Z \\ & & \end{array}$$

Since  $Y$  is already complete and  $j(X) \subset Y$ , the restriction of  $F$  to  $Y$  gives a diagram

$$\begin{array}{ccc} & Y & \\ & \uparrow j & \searrow F \\ X & \xrightarrow{f} & Z \end{array}$$

That is,  $Y$  fits the characterization of a completion of  $X$ . By uniqueness,  $Y \subset \tilde{X}$  is an isometric isomorphism, so  $Y = \tilde{X}$ . ///

Now prove *existence* of completions by the standard *construction*.

The intention is that every Cauchy sequence has a limit, so we should (somehow!) *adjoin* points as needed for these limits. However, different Cauchy sequences may happen to have the same limit. Thus, we want an equivalence relation on Cauchy sequences that says they should have the same limit, even without knowing the limit exists or having somehow constructed or adjoined the limit point.

Define an equivalence relation  $\sim$  on the set  $C$  of Cauchy sequences in  $X$ , by

$$\{x_s\} \sim \{y_t\} \iff \lim_s d(x_s, y_s) = 0$$

*Attempt* to define a metric on the set  $C/\sim$  of equivalence classes by

$$d(\{x_s\}, \{y_t\}) = \lim_s d(x_s, y_s)$$

We must verify that this is well-defined on the quotient  $C/\sim$  and gives a metric. We have an injection  $j : X \rightarrow C/\sim$  by

$$x \rightarrow \{x, x, x, \dots\} \text{ mod } \sim$$

[2.0.4] **Proposition:**  $j : X \rightarrow C/\sim$  is a completion of  $X$ .

*Proof:* Grant for the moment that the distance function on  $\tilde{X} = C/\sim$  is well-defined, and is complete, and show that it has the property of a completion of  $X$ . To this end, let  $f : X \rightarrow Y$  be a contractive map to a complete metric space  $Y$ .

Given  $z \in \tilde{X}$ , choose a Cauchy sequence  $x_k$  in  $X$  with  $j(x_k)$  converging to  $z$ , and *try* to define  $F : \tilde{X} \rightarrow Y$  in the natural way, by

$$F(z) = \lim_k f(x_k)$$

Since  $f$  is contractive,  $f(x_k)$  is Cauchy in  $Y$ , and by completeness of  $Y$  has a limit, so  $F(z)$  *exists*, at least if well-defined.

For well-definedness of  $F(z)$ , for  $x_k$  and  $x'_k$  two Cauchy sequences whose images  $j(x_k)$  and  $j(x'_k)$  approach  $z$ , since  $j$  is contractive eventually  $x_k$  is close to  $x'_k$ , so  $f(x_k)$  is eventually close to  $f(x'_k)$  in  $Y$ , showing  $F(z)$  is well-defined.

We saw that every element of  $\tilde{X}$  is a limit of a Cauchy sequence  $j(x_k)$  for  $x_k$  in  $X$ , and *any* continuous  $\tilde{X} \rightarrow Y$  respects limits, so  $F$  is the only possible extension of  $f$  to  $\tilde{X}$ .

The obvious argument will show that  $F$  is continuous. Namely, let  $z, z' \in \tilde{X}$ , with Cauchy sequences  $x_t$  and  $x'_t$  approaching  $z$  and  $z'$ . Given  $\varepsilon > 0$ , by contractiveness of  $F$ , there is  $N$  large enough such that  $d_Y(F(j(x_r)), F(j(x_s))) < \varepsilon$  and  $d_Y(F(j(x'_r)), F(j(x'_s))) < \varepsilon$  for  $r, s \geq N$ . From the lemma above (!), for such  $r$  even in the limit the strict inequalities are at worst non-strict inequalities:

$$d_Y(f(x_r), F(z)) \leq \varepsilon \quad \text{and} \quad d_Y(f(x'_r), F(z')) \leq \varepsilon$$

By the triangle inequality, since  $f : X \rightarrow Y$  is continuous, we can increase  $r$  to have  $d_X(x_r, x'_r)$  small enough so that  $d_Y(f(x_r), f(x'_r)) < \varepsilon$ , and then

$$d_Y(F(z), F(z')) \leq d_Y(F(z), f(x_r)) + d_Y(f(x_r), f(x'_r)) + d_Y(f(x'_r), F(z')) \leq \varepsilon + \varepsilon + \varepsilon$$

Since  $j : X \rightarrow \tilde{X}$  is an isometry,

$$d_X(x_r, x'_r) = d_{\tilde{X}}(j(x_r), j(x'_r)) \leq d_{\tilde{X}}(j(x_r), z) + d_{\tilde{X}}(z, z') + d_{\tilde{X}}(j(x'_r), z')$$

so

$$d_X(x_r, x'_r) \leq d_{\tilde{X}}(z, z') + 2\varepsilon$$

Thus,

$$d_Y(F(z), F(z')) \leq d_{\tilde{X}}(z, z') + 4\varepsilon \quad (\text{for all } \varepsilon > 0)$$

Thus,  $F$  is continuous. Granting that  $\tilde{X} = C/\sim$  is complete, etc., it is a completion of  $X$ .

It remains to prove that the apparent metric on  $\tilde{X}$  truly is a metric, and that  $\tilde{X}$  is complete.

First, the limit in attempted definition

$$d(\{x_s\}, \{y_t\}) = \lim_s d(x_s, y_s)$$

does exist: given  $\varepsilon > 0$ , take  $N$  large enough so that  $d(x_i, x_j) < \varepsilon$  and  $d(y_i, y_j) < \varepsilon$  for  $i, j \geq N$ . By the triangle inequality,

$$d(x_i, y_i) \leq d(x_i, x_N) + d(x_N, y_N) + d(y_N, y_i) < \varepsilon + d(x_N, y_N) + \varepsilon$$

Similarly,

$$d(x_i, y_i) \geq -d(x_i, x_N) + d(x_N, y_N) - d(y_N, y_i) > -\varepsilon + d(x_N, y_N) - \varepsilon$$

Thus, unsurprisingly,

$$\left| d(x_i, y_i) - d(x_N, y_N) \right| < 2\varepsilon$$

and the sequence of real numbers  $d(x_i, y_i)$  is Cauchy, so convergent.

Similarly, when  $\lim_i d(x_i, y_i) = 0$ , then  $\lim_i d(x_i, z_i) = \lim_i d(y_i, z_i)$  for any other Cauchy sequence  $z_i$ , so the distance function is *well-defined* on  $C/\sim$ .

The positivity and symmetry for the alleged metric on  $C/\sim$  are immediate. For triangle inequality, given  $x_i, y_i, z_i$  and  $\varepsilon > 0$ , let  $N$  be large enough so that  $d(x_i, x_j) < \varepsilon$ ,  $d(y_i, y_j) < \varepsilon$ , and  $d(z_i, z_j) < \varepsilon$  for  $i, j \geq N$ . As just above,

$$\left| d(\{x_s\}, \{y_s\}) - d(x_i, y_i) \right| < 2\varepsilon$$

Thus,

$$d(\{x_s\}, \{y_s\}) \leq 2\varepsilon + d(x_N, y_N) \leq 2\varepsilon + d(x_N, z_N) + d(z_N, y_N) \leq 2\varepsilon + d(\{x_s\}, \{z_s\}) + 2\varepsilon + d(\{z_s\}, \{y_s\}) + 2\varepsilon$$

This holds for all  $\varepsilon > 0$ , so we have the triangle inequality.

Finally, perhaps anticlimactically, the completeness. Given Cauchy sequences  $c_s = \{x_{sj}\}$  in  $X$  such that  $\{c_s\}$  is Cauchy in  $C/\sim$ , for each  $s$  we will choose large-enough  $j(s)$  such that the diagonal-ish sequence  $y_\ell = x_{\ell, j(\ell)}$  is a Cauchy sequence in  $X$  to which  $\{c_s\}$  converges.

Given  $\varepsilon > 0$ , take  $i$  large enough so that  $d(c_s, c_t) < \varepsilon$  for all  $s, t \geq i$ . For each  $i$ , choose  $j(i)$  large enough so that  $d(x_{ij}, x_{ij'}) < \varepsilon$  for all  $j, j' \geq j(i)$ . Let  $c = \{x_{i, j(i)} : i = 1, 2, \dots\}$ . For  $s \geq i$ ,

$$d(c_s, c) = \lim_\ell d(x_{s\ell}, x_{\ell, j(\ell)}) \leq \sup_{\ell \geq i} d(x_{s\ell}, x_{\ell, j(\ell)}) \leq \sup_{\ell \geq i} \left( d(x_{s\ell}, x_{s, j(\ell)}) + d(x_{s, j(\ell)}, x_{\ell, j(\ell)}) \right) \leq 2\varepsilon$$

This holds for all  $\varepsilon > 0$ , so  $\lim_s c_s = c$ , and  $C/\sim$  is complete. ///

### 3. The Baire category theorem

This standard result is both indispensable and mysterious.

A set  $E$  in a topological space  $X$  is *nowhere dense* if its closure  $\bar{E}$  contains no non-empty open set. A *countable union* of nowhere dense sets is said to be *of first category*, while every other subset (if any) is *of second category*. The idea (not at all clear from this traditional terminology) is that first category sets are *small*, while second category sets are *large*. In this terminology, the theorem's assertion is equivalent to the assertion that (non-empty) *complete metric spaces* and *locally compact Hausdorff spaces* are *of second category*.

Further, a  $G_\delta$  set is a countable intersection of open sets. Concomitantly, an  $F_\sigma$  set is a countable union of closed sets. Again, the following theorem can be paraphrased as asserting that, in a complete metric space, *a countable intersection of dense  $G_\delta$ 's is still a dense  $G_\delta$* .

**[3.0.1] Theorem:** (*Baire category*) Let  $X$  be a set with metric  $d$  making  $X$  a *complete metric space*. Or let  $X$  be a locally compact Hausdorff topological space. The intersection of a *countable collection*  $U_1, U_2, \dots$  of *dense open subsets*  $U_i$  of  $X$  is still *dense* in  $X$ .

*Proof:* Let  $B_o$  be a non-empty open set in  $X$ , and show that  $\bigcap_i U_i$  meets  $B_o$ . Suppose an open ball  $B_{n-1}$  inductively chosen. By denseness of  $U_n$ , there is an open ball  $B_n$  whose closure  $\bar{B}_n$  satisfies

$$\bar{B}_n \subset B_{n-1} \cap U_n$$

For complete metric spaces, take  $B_n$  to have radius less than  $1/n$  (or any other sequence of reals going to 0), and in the locally compact Hausdorff case take  $B_n$  to have compact closure.

Let

$$K = \bigcap_{n \geq 1} \bar{B}_n \subset B_o \cap \bigcap_{n \geq 1} U_n$$

For complete metric spaces, the centers of the nested balls  $B_n$  form a Cauchy sequence (since they are nested and the radii go to 0). By completeness, this Cauchy sequence *converges*, and the limit point lies inside each *closure*  $\bar{B}_n$ , so lies in the intersection. In particular,  $K$  is non-empty. For locally compact Hausdorff spaces, the intersection of a nested family of non-empty compact sets is non-empty, so  $K$  is non-empty, and  $B_o$  necessarily meets the intersection of the  $U_n$ . ///

### Bibliography

[Dieudonné 1981] J. Dieudonné, *History of Functional Analysis*, Mathematical Studies 49, North Holland, 1981.

[Fréchet 1906] M. Fréchet, *Sur quelques points du Calcul fonctionnel*, Rend. Circ. mat. Palermo **22** (1906), 1-74.

[Fréchet 1928] M. Fréchet, *Les espaces abstraits topologiquement affine*, Acta Math. **27** (1928), 25-52.

[Hausdorff 1931] F. Hausdorff, *Zur Theorie der linearen metrischen Räume*, J. de Crelle **167** (1931), 294-311.