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Continuous spectrum for $SL_2(\mathbb{Z})\backslash\mathfrak{H}$

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

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We prove that the orthogonal complement in $L^2(\Gamma\backslash\mathfrak{H})$ to *cuspforms* is spanned by *pseudo-Eisenstein series*, in turn expressible as *integrals* or *wave packets* of Eisenstein series E_s. We prove a *Plancherel theorem* for this space. The essential harmonic analysis is *Fourier transform* on the real line, in coordinates on which it is called a *Mellin transform*. That is, the non-cuspidal part of harmonic analysis on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ reduces to harmonic analysis on a related, lower-dimensional group.

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1. *Pseudo-Eisenstein series, adjunction to constant term*

As usual, let N be the upper-triangular unipotent matrices in $G = SL_2(\mathbb{R})$, A the diagonal matrices, A^+ the diagonal matrices with positive diagonal entries, and $P = NA$ the parabolic subgroup of upper-triangular matrices, $P^+ = NA^+$, $\Gamma = SL_2(\mathbb{Z})$, $\Gamma_\infty = P^+ \cap \Gamma = N \cap \Gamma$, and $\mathfrak{H}$ the upper half-plane.

[1.1] Constant terms $c_P f$ In most-elementary terms, the *constant term* $c_P f$ along P of a reasonable function f on $\Gamma\backslash\mathfrak{H}$ is

$$\text{(constant term of } f)(z) \;=\; c_P f(z) \;=\; \int_{\Gamma_\infty\backslash N} f(n\cdot z)\, dn$$

Changing variables in the integral shows that $c_P f$ is left N-invariant. *Identify* left N-invariant functions on $\mathfrak{H}$ as functions of $y = im(z)$ alone, thus as functions on $(0,\infty)$.

[1.2] Cuspforms A reasonable function f on $\Gamma\backslash\mathfrak{H}$ is a *cuspform* when its constant term essentially vanishes:

$$f \text{ cuspform} \iff c_P f \;=\; 0$$

The cuspform condition is better recast as

$$f \text{ cuspform} \iff \int_{N\backslash\mathfrak{H}} c_P f(z)\cdot\varphi(z)\, dz \;=\; 0 \qquad (\text{for all } \varphi\in C_c^\infty(N\backslash\mathfrak{H}) \approx C_c^\infty(0,\infty))$$

That is, the cuspform condition is that the constant term vanishes *as a distribution* on $N\backslash\mathfrak{H}$.

[1.3] Pseudo-Eisenstein series Pseudo-Eisenstein series are the solutions to an *adjunction* problem, as follows. Identify $\mathfrak{H} \approx G/K$, where $K = SO(2,\mathbb{R})$, as usual. We need *asymmetrical* pairings of the form

$$\langle f, F\rangle_{H\backslash\mathfrak{H}} \;=\; \int_{H\backslash\mathfrak{H}} f\cdot F \qquad (\mathbb{C}\text{-bilinear, } \textit{not} \text{ hermitian})$$

even when f, F may not be in the same space of functions, and use the same notation for extensions of integral pairings to distributions. The problem is, given φ in $C_c^\infty(N\backslash\mathfrak{H})$, find $\Psi_\varphi \in C_c^\infty(\Gamma\backslash\mathfrak{H})$ such that

$$\langle c_P f, \varphi\rangle_{N\backslash\mathfrak{H}} \;=\; \langle f, \Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}} \qquad (\text{for } f \text{ on } \Gamma\backslash\mathfrak{H})$$

Exhibition of Ψ_φ will legitimize treatment of c_P as a map from distributions on $\Gamma\backslash\mathfrak{H}$ to distributions on $N\backslash\mathfrak{H}$, even though this generality is not needed.

[1.3.1] Remark: The measure $dx\,dy/y^2$ on $\mathfrak{H}\approx G/K$ is the usual left G-invariant measure $dx\,dy/y^2$ inherited from Haar measure on $G=NA^+K$. This descends to dy/y^2 on $N\backslash\mathfrak{H}\approx N\backslash G/K$, *not* to the measure dy/y from the Haar measure on A^+.

Direct computation yields a canonical *expression* for the desired Ψ_φ, using the left N-invariance of φ and the left Γ-invariance of f, as follows. Note that $P\cap\Gamma$ differs from $N\cap\Gamma$ only by $\pm 1_2$, which act trivially on $\mathfrak{H}\approx G/K$. Abuse notation by writing $\varphi(z)=\varphi(\mathrm{Im}\,z)$:

$$\langle c_P f,\varphi\rangle_{N\backslash\mathfrak{H}} \;=\; \int_{N\backslash\mathfrak{H}} c_P f(z)\,\varphi(z)\,\frac{dy}{y^2} \;=\; \int_{\Gamma_\infty\backslash\mathfrak{H}} f(z)\,\varphi(z)\,\frac{dx\,dy}{y^2}$$

Winding up,

$$\int_{\Gamma_\infty\backslash\mathfrak{H}} f(z)\,\varphi(z)\,\frac{dx\,dy}{y^2} \;=\; \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} f(\gamma z)\,\varphi(\gamma z)\,\frac{dx\,dy}{y^2} \;=\; \int_{\Gamma\backslash\mathfrak{H}} f(z)\Big(\sum_{\gamma\in P\cap\Gamma\backslash\Gamma}\varphi(\gamma z)\Big)\,\frac{dx\,dy}{y^2}$$

The inner sum in the last integral is the pseudo-Eisenstein series [1] attached to φ:

$$\Psi_\varphi(z) \;=\; \sum_{\gamma\in P\cap\Gamma\backslash\Gamma}\varphi(\gamma z)$$

designed to fit into the *adjunction*

$$\langle c_P f,\varphi\rangle_{N\backslash\mathfrak{H}} \;=\; \langle f,\Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}}$$

While cuspforms are mysterious, these pseudo-Eisenstein series are completely explicit.

[1.3.2] Claim: The series for a pseudo-Eisenstein series Ψ_φ is *locally finite*, meaning that for z in a fixed compact in $\mathfrak{H}$, there are only finitely-many non-zero summands in $\Psi_\varphi(z)=\sum_\gamma\varphi(\gamma z)$. Thus, $\Psi_\varphi\in C_c^\infty(\Gamma\backslash\mathfrak{H})$.

Proof: Identify $N\backslash\mathfrak{H}\approx N\backslash G/K$, so $C_c^\infty(N\backslash\mathfrak{H})$ can be identified with right K-invariant functions in $C_c^\infty(N\backslash G)$ since K is compact. Given $\varphi\in C_c^\infty(N\backslash\mathfrak{H})$, let $C\subset G$ be compact so that $N\cdot C$ contains the support of φ. Fix compact $C_o\subset G$ in which $g\in G$ is constrained to lie. Then a summand $\varphi(\gamma g)$ is non-zero only if $\gamma g\in N\cdot C$, which holds only if

$$\gamma\;\in\;NC\cdot g^{-1}$$

so

$$\gamma\;\in\;\Gamma\;\cap\;NC\cdot C_o^{-1}$$

In the quotient $G\to(P\cap\Gamma)\backslash G$, the image of Γ is discrete. The image of the compact set $N\cdot C\cdot C_o^{-1}$ under the continuous quotient map is *compact*, since $(N\cap\Gamma)\backslash N$ is compact, and continuous images of compacts are compact. Thus, left modulo $P\cap\Gamma$, that intersection is the intersection of a (closed) discrete set and a compact set, so *finite*. Therefore, the series is *locally finite*, and defines a smooth function on $\Gamma\backslash G$. Summing over left translates certainly retains right K-invariance.

To show that Ψ_φ has compact support in $\Gamma\backslash G$, proceed similarly. That is, for a summand $\varphi(\gamma g)$ to be non-zero, it must be that $g\in\Gamma\cdot C$. The image $\Gamma\backslash(\Gamma\cdot C)$ is compact, being the continuous image of the compact set C under the continuous map $G\to\Gamma\backslash G$, proving the compact support. ///

[1] In 1966 Godement called these *incomplete theta series*, but more recently Moeglin-Waldspurger strengthened the precedent of calling them *pseudo-Eisenstein series*

[1.3.3] Corollary: The square-integrable cuspforms are the orthogonal complement of the (closed) space subspace of $L^2(\Gamma\backslash\mathfrak{H})$ spanned by the pseudo-Eisenstein series Ψ_φ with $\varphi \in C_o^\infty(N\backslash\mathfrak{H}) \approx C_o^\infty(0,\infty)$. ///

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2. *Decomposition of pseudo-Eisenstein series: beginning*

Spectral decomposition of the data φ in Ψ_φ induces a spectral decomposition of Ψ_φ itself.

[2.1] Fourier transform Fourier transform $\widehat{f}$ of f on the real line is

$$\widehat{f}(\xi) \;=\; \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx$$

Fourier inversion for Schwartz functions f is

$$f(x) \;=\; \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

Replacing ξ by $\xi/(2\pi)$ gives the variant

$$f(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} f(t)\, e^{-it\xi}\, dt \Big)\, e^{i\xi x}\, d\xi$$

[2.2] Multiplicative coordinates, Mellin transforms Fourier transforms on $\mathbb{R}$ put into multiplicative coordinates are *Mellin* transforms: for $\varphi \in C_c^\infty(0,+\infty)$, take

$$f(x) \;=\; \varphi(e^x)$$

Let $y = e^x$ (and r the exponentiated variable in the implied inner integral) and rewrite Fourier inversion as

$$\varphi(y) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big(\int_0^{\infty} \varphi(r)\, r^{-i\xi}\, \frac{dr}{r} \Big)\, y^{i\xi}\, d\xi$$

The Fourier transform in these coordinates is called a *Mellin* or *Laplace* transform $\mathcal{M}F$,

$$\mathcal{M}\varphi(i\xi) \;=\; \int_0^{\infty} \varphi(r)\, r^{-i\xi}\, \frac{dr}{r}$$

For *compactly-supported* φ, the integral definition extends to complex s:

$$\mathcal{M}\varphi(s) \;=\; \int_0^{\infty} \varphi(r)\, r^{-s}\, \frac{dr}{r}$$

The variant Fourier inversion identity gives *Mellin inversion*

$$\varphi(y) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}\varphi(i\xi)\; y^{i\xi}\;\, d\xi$$

With ξ the imaginary part of a complex variable s, rewrite the latter integral as a complex path integral, noting $d\xi = -i\, ds$,

$$\varphi(y) \;=\; \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} \mathcal{M}\varphi(s)\; y^s\, ds$$

For $f \in C_c^\infty(\mathbb{R})$ the integral giving the Fourier transform $\widehat{f}(\xi)$ converges nicely for all *complex* values of ξ, so extends to an *entire* function in ξ, of rapid decay on horizontal lines. [2] This extension property applies to φ, thus allowing movement of the contour: for compactly-supported φ, Mellin inversion is

$$\varphi(y) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\, y^s\, ds \qquad\qquad (\text{for } \textit{any} \text{ real } \sigma)$$

[2.3] Spectral decomposition of pseudo-Eisenstein series

Identifying $N\backslash\mathfrak{H} \approx N\backslash G/K \approx A^+$, Mellin inversion is

$$\varphi(z) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\, \big(\mathrm{Im}\, z)\big)^s\, ds \qquad\qquad (\text{for any real } \sigma)$$

Thus, the pseudo-Eisenstein series is

$$\Psi_\varphi(z) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(\gamma z) \;=\; \frac{1}{2\pi i}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\cdot \big(\mathrm{Im}\,(\gamma z)\big)^s\, ds$$

Taking $\sigma = 0$ would be natural, but, with $\sigma = 0$, the double integral (sum and integral) is not absolutely convergent, and *the two integrals cannot be interchanged.* We will eventually see that the best line is $\sigma = 1/2$, but this is not in the region of convergence, either. For $\sigma > 1$, elementary estimates show that the double integral *is* absolutely convergent, and by Fubini the two integrals can be interchanged:

$$\Psi_\varphi(z) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \big(\mathrm{Im}\,(\gamma z)\big)^s\, ds \qquad\qquad (\text{with } \sigma > 1)$$

The inner sum is the *Eisenstein series* $E_s(z)$, so

$$\Psi_\varphi(z) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\cdot E_s(z)\, ds \qquad\qquad (\text{for } \sigma > 1)$$

[2.3.1] Remark: This decomposition is partly unsatisfactory, because it should refer to $\mathcal{M}c_P(\Psi_\varphi)$, not $\mathcal{M}\varphi$, to have whatever integral formulas expressed in terms of the automorphic forms Ψ_φ themselves, not in terms of the auxiliary functions φ from which they're made.

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[2] This is an easy part of the *Paley-Wiener* theorem, proven directly: for complex $w = \xi + i\eta$,

$$\widehat{f}(w) \;=\; \int_{\mathbb{R}} e^{-2\pi i w x}\, f(x)\, dx \;=\; \int_{\mathbb{R}} e^{-2\pi i \xi x}\left(e^{2\pi\eta x}\cdot f(x)\right) dx$$

Since f has compact support, multiplying by an exponential still gives a compact-support function, hence Schwartz, so the Fourier transform is Schwartz.

3. *Eisenstein series*

[3.1] Adjunction As do the pseudo-Eisenstein series, the Eisenstein series E_s fits into an *adjunction relation*

$$\langle E_s, f\rangle_{\Gamma\backslash\mathfrak{H}} \;=\; \langle y^s, c_P f\rangle_{N\backslash\mathfrak{H}} \qquad\qquad (\text{for } f \text{ on } \Gamma\backslash\mathfrak{H})$$

whenever the implied integrals converge. The Eisenstein series is *determined* by this relation:

$$\langle y^s, c_P f\rangle_{A^+} \;=\; \int_{N\backslash\mathfrak{H}} c_P f(z)\cdot y^s\,\frac{dy}{y^2} \;=\; \int_{N\backslash\mathfrak{H}} \Big(\int_{\Gamma_\infty\backslash N} f(nz)\,dn\Big)\cdot y^s\,\frac{dy}{y^2} \;=\; \int_{\Gamma_\infty\backslash\mathfrak{H}} f(z)\cdot y^s\,\frac{dx\,dy}{y^2}$$

Winding up,

$$\int_{P\cap\Gamma\backslash\mathfrak{H}} f(z)\cdot y^s\,\frac{dx\,dy}{y^2} \;=\; \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} f(\gamma z)\cdot \mathrm{Im}\,(\gamma z)^s\,\frac{dx\,dy}{y^2} \;=\; \int_{\Gamma\backslash\mathfrak{H}} f(z) \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \mathrm{Im}\,(\gamma z)^s\,\frac{dx\,dy}{y^2}$$

Thus, the adjunction produces the expression for $E_s(z)$ in the region of convergence $\mathrm{Re}\,(s) > 1$. By the analytic continuation of E_s, the adjunction characterizing E_s asserts that integrals against Eisenstein series are Mellin transforms of constant terms:

$$\langle E_s, f\rangle_{\Gamma\backslash\mathfrak{H}} \;=\; \int_0^\infty c_P f(iy)\,y^s\,\frac{dy}{y^2} \;=\; \int_0^\infty c_P f(iy)\,y^{-(1-s)}\,\frac{dy}{y} \;=\; \mathcal{M}(c_P f)\big(1-s\big)$$

[3.2] Meromorphic continuation and functional equation We grant the meromorphic continuation. The constant term of the Eisenstein series is

$$c_P E_s \;=\; y^s + c_s y^{1-s} \qquad\qquad (\text{with } c_s = \xi(2s-1)/\xi(2s))$$

where $\xi(s) = \pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})\,\zeta(s)$ is the completed zeta function. The Eisenstein series E_s can be characterized as the (unique) moderate-growth automorphic form on $\Gamma\backslash\mathfrak{H}$ satisfying [3]

$$y^2\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\big)w \;=\; s(s-1)\cdot w \qquad\qquad \text{and} \qquad\qquad \Big(y\frac{\partial}{\partial y} - (1-s)\Big)c_P w \;=\; (2s-1)\cdot y^s$$

noting that the latter differential operator annihilates y^{1-s} and has y^s as an eigenfunction. This characterization determines the functional equation from the constant term. Namely, E_{1-s} satisfies the first of the two equations, and

$$\Big(y\frac{\partial}{\partial y} - (1-s)\Big)c_P E_{1-s} \;=\; \Big(y\frac{\partial}{\partial y} - (1-s)\Big)(y^{1-s} + c_{1-s}y^s) \;=\; 0 + c_{1-s}\cdot(2s-1)\cdot y^s$$

Thus, we obtain the functional equation $E_{1-s} = c_{1-s}\,E_s$. Applying the functional equation twice gives $c_s c_{1-s} = 1$. Since $E_{\overline{s}} = \overline{E_s}$, also $\overline{c_s} = c_{\overline{s}}$ and $|c_{\frac{1}{2}+it}|^2 = 1$ for $t\in\mathbb{R}$. In particular, c_s does not vanish on the line $\mathrm{Re}\,(s) = \frac{1}{2}$.

[3.2.1] Remark: Just below we will see that, from the expression of pseudo-Eisenstein series in terms of Eisenstein series, poles of E_s to the right of $\mathrm{Re}\,(s) = \frac{1}{2}$ play a role in the decomposition of $L^2(\Gamma\backslash\mathfrak{H})$.

[3] This characterization is the simplest example of treatments of more general Eisenstein series by Selberg, Bernstein, Colin de Verdière, and Jacquet.

4. *Decomposition of pseudo-Eisenstein series: conclusion*

So far, we have a spectral decomposition of Ψ_φ in terms of φ:

$$\Psi_\varphi \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\cdot E_s\,ds \qquad\qquad (\sigma > 1)$$

We proceed to rewrite this to refer only to Ψ_φ, not φ.

[4.1] Final re-arrangments, invocation of adjunction Move the line of integration to the left, to $\sigma = 1/2$, stabilized by the functional equation of E_s:

$$\Psi_\varphi \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s)\,E_s\,ds \;+\; \sum_{s_o}\mathrm{res}_{s=s_o}(E_s\cdot\mathcal{M}\varphi(s))$$

The $1/2\pi i$ from inversion cancels the $2\pi i$ in the residue formula. To rewrite this in terms of Ψ_φ, on one hand, from above,

$$\langle E_s,\Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}} \;=\; \mathcal{M}(c_P\Psi_\varphi)(1-s)$$

On the other hand, by the adjunction/unwinding property of Ψ_φ,

$$\langle E_s,\Psi_\varphi\rangle \;=\; \langle c_PE_s,\varphi\rangle_{N\backslash\mathfrak{H}} \;=\; \langle y^s+c_sy^{1-s},\varphi\rangle_{N\backslash\mathfrak{H}} \;=\; \int_0^\infty (y^s+c_sy^{1-s})\cdot\varphi(y)\,\frac{dy}{y^2}$$

$$=\; \int_0^\infty (y^{-(1-s)}+c_sy^{-s})\cdot\varphi(y)\,\frac{dy}{y} \;=\; \mathcal{M}\varphi(1-s)+c_s\mathcal{M}\varphi(s)$$

Thus,

$$\mathcal{M}(c_P\Psi_\varphi)(1-s) \;=\; \langle E_s,\Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}} \;=\; \mathcal{M}\varphi(1-s)+c_s\mathcal{M}\varphi(s)$$

or, replacing s by $1-s$,

$$\mathcal{M}(c_P\Psi_\varphi)(s) \;=\; \langle E_{1-s},\Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}} \;=\; \mathcal{M}\varphi(s)+c_{1-s}\mathcal{M}\varphi(1-s)$$

At the same time, the integral part of the expression of Ψ_φ in terms of Eisenstein series can be *folded in half*, integrating from $\frac{1}{2}+i0$ to $\frac{1}{2}+i\infty$ rather than from $\frac{1}{2}-i\infty$ to $\frac{1}{2}+i\infty$:

$$\Psi_\varphi - (\text{residual part}) \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s)\cdot E_s\,ds \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s)\,E_s+\mathcal{M}\varphi(1-s)\,E_{1-s}\,ds$$

Using the functional equation $E_{1-s}=c_{1-s}E_s$, and recognizing $\mathcal{M}(c_P\Psi_\varphi)$ as expressed above in terms of $\mathcal{M}\varphi$, this becomes

$$\Psi_\varphi - (\text{residual part}) \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} (\mathcal{M}\varphi(s)+c_{1-s}\mathcal{M}\varphi(1-s))\cdot E_s\,ds \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}c_P\Psi_\varphi(s)\,E_s\,ds$$

by the expression for the Mellin transform of the constant term of Ψ_φ. That is, a pseudo-Eisenstein series is expressible as an integral of Eisenstein series E_s on the line $\mathrm{Re}(s)=1/2$, plus a sum of residues:

$$\Psi_\varphi - (\text{residual part}) \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}c_P\Psi_\varphi(s)\cdot E_s\,ds \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \langle E_{1-s},\Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}}\cdot E_s\,ds$$

When desired, the integral can be written as an integral over the whole line $\mathrm{Re}(s) = \frac{1}{2}$, by the functional equation of E_s and dividing by 2:

$$\Psi_\varphi - (\text{residual part}) \;=\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle E_{1-s}, \Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}} \cdot E_s\, ds \qquad \text{(complex-}\textit{bilinear}\text{ pairing)}$$

$$= \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle \Psi_\varphi, E_s\rangle_{\Gamma\backslash\mathfrak{H}} \cdot E_s\, ds \qquad \text{(complex-}\textit{hermitian}\text{ pairing)}$$

[4.2] The residual part

For $\Gamma = SL_2(\mathbb{Z})$, the only pole of E_s in the half-plane $\mathrm{Re}(s) \geq 1/2$ is at $s_o = 1$, is simple, and the residue is a *constant* function. Thus,

$$\Psi_\varphi \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \langle \Psi_\varphi, E_s\rangle \cdot E_s\, ds \;+\; \mathcal{M}\varphi(1)\cdot \mathrm{res}_{s=1} E_s \qquad \text{(complex-}\textit{hermitian}\text{ pairing)}$$

Continuing with the abuse of notation $\varphi(z) = \varphi(\mathrm{Im}\, z)$, the coefficient $\mathcal{M}\varphi(1)$ is

$$\mathcal{M}\varphi(1) \;=\; \int_0^{+\infty} \varphi(y)\, y^{-1}\, \frac{dy}{y} \;=\; \int_0^{+\infty} \varphi(y)\, \frac{dy}{y^2} \;=\; \int_{N\backslash\mathfrak{H}} \varphi(z)\, \frac{dy}{y^2}$$

Rearranging, since the natural volume of $\Gamma_\infty\backslash N$ is 1 and φ is left N-invariant,

$$\mathcal{M}\varphi(1) \;=\; \int_{N\backslash\mathfrak{H}}\int_{\Gamma_\infty\backslash N} \varphi(nz)\, dn\, \frac{dy}{y^2} \;=\; \int_{N\backslash\mathfrak{H}} \varphi(nz)\Big(\int_{\Gamma_\infty\backslash N} 1\, dn\Big)\frac{dy}{y^2} \;=\; \int_{\Gamma_\infty\backslash\mathfrak{H}} \varphi(z)\, \frac{dx\, dy}{y^2}$$

Winding up,

$$\mathcal{M}\varphi(1) \;=\; \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(z)\, \frac{dx\, dy}{y^2} \;=\; \int_{\Gamma\backslash\mathfrak{H}} \Psi_\varphi(z)\, \frac{dx\, dy}{y^2} \;=\; \int_{\Gamma\backslash\mathfrak{H}} \Psi_\varphi(z)\cdot 1\, \frac{dx\, dy}{y^2} \;=\; \langle \Psi_\varphi, 1\rangle$$

That is, $\mathcal{M}\varphi(1)$ is the inner product of Ψ_φ with the constant function 1.

[4.2.1] Remark: The above discussion, and the fact that *constants* are residues of Eisenstein series E_s proves that constants are orthogonal to cuspforms: $c_P f = 0$ for cuspform f, so

$$0 \;=\; \mathcal{M}(c_P f)(1-s) \;=\; \langle f, E_s\rangle \qquad \text{(complex-}\textit{bilinear}\text{ pairing)}$$

The complex-bilinear pairing value $\langle f, E_s\rangle$ is *meromorphic* in s, so

$$\langle f, \mathrm{res}_{s=1} E_s\rangle \;=\; \mathrm{res}_{s=1}\langle f, E_s\rangle \;=\; \mathrm{res}_{s=1} 0 \;=\; 0$$

5. *Plancherel theorem*

For $\varphi \in C_c^\infty(N\backslash G/N) \approx C_c^\infty(0,+\infty)$, the pseudo-Eisenstein series Ψ_φ is expressed in terms of Eisenstein series, with complex-*hermitian* pairing $\langle,\rangle$,

$$\Psi_\varphi \;=\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle \Psi_\varphi, E_s\rangle\, E_s\, ds + \langle \Psi_\varphi, 1\rangle\, \mathrm{res}_{s=1} E_s$$

For $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$ [4]

$$\langle \Psi_\varphi, f\rangle \;=\; \Big\langle \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle \Psi_\varphi, E_s\rangle\, E_s\, ds + \langle \Psi_\varphi, 1\rangle\, \mathrm{res}_{s=1} E_s,\; f\Big\rangle$$

$$=\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle \Psi_\varphi, E_s\rangle\, \langle E_s, f\rangle\, ds \;+\; \langle \Psi_\varphi, 1\rangle\, \langle \mathrm{res}_{s=1} E_s,\; f\rangle$$

With $f = \Psi_\psi$ for $\psi \in C_c^\infty(N\backslash\mathfrak{H})$, this gives an *isometry*:

[5.0.1] Theorem: Using the *hermitian* pairing on $L^2(\Gamma\backslash\mathfrak{H})$, for $\varphi, \psi \in C_c^\infty(0,+\infty)$,

$$\langle \Psi_\varphi, \Psi_\psi\rangle \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \langle \Psi_\varphi, E_s\rangle\, \overline{\langle \Psi_\psi, E_s\rangle}\, ds \;+\; \langle \Psi_\varphi, 1\rangle\, \overline{\langle \Psi_\psi, 1\rangle} \cdot \mathrm{res}_{s=1} E_s$$

Incidentally, the Plancherel isometry determines the constant $\mathrm{res}_{s=1}E_s$: from $\langle 1,1\rangle = \langle 1,1\rangle\cdot\langle 1,1\rangle\cdot \mathrm{res}_{s=1}E_s$ we find $\mathrm{res}_{s=1}E_s = \frac{1}{\langle 1,1\rangle} = \frac{1}{\mathrm{vol}\,(\Gamma\backslash\mathfrak{H})}$. Compare [Langlands 1966].

Since the collection of pseudo-Eisenstein series Ψ_φ is *dense* in the orthogonal complement in $L^2(\Gamma\backslash\mathfrak{H})$ to *cuspforms*, this isometry *extends by continuity* to give an isometry

$$\Big\{\text{orthogonal complement in } L^2(\Gamma\backslash\mathfrak{H}) \text{ to cuspforms}\Big\} \;\approx\; L^2(\tfrac{1}{2}+i\mathbb{R}) \oplus \mathbb{C}$$

by (the isometric extension of) $f \longrightarrow \langle f, E_s\rangle \oplus \langle f, 1\rangle / \langle 1,1\rangle^{\frac{1}{2}}$. This is the *Plancherel theorem* on the orthogonal complement to cuspforms. That is, *in an L^2-sense*, for f in the orthogonal complement to cuspforms,

$$f \;=_{L^2}\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle f, E_s\rangle \cdot E_s\, ds \;+\; \frac{\langle f, 1\rangle\cdot 1}{\langle 1,1\rangle} \qquad \text{(complex-}\textit{hermitian}\text{ pairing)}$$

That is, the indicated integrals are not *literal* integrals, but are the extension-by-continuity of the corresponding integrals.

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[4] Although $\Gamma\backslash\mathfrak{H}$ is not quite *smooth*, via $\mathfrak{H} \approx G/K$ we identify $C_c^\infty(\Gamma\backslash\mathfrak{H})$ with right K-invariant smooth functions on $\Gamma\backslash G$, denoted $C_c^\infty(\Gamma\backslash G)^K$.

Bibliography

I first saw a form of this argument in [Godement 1966]. [Selberg 1956] and [Roelcke 1956] addressed somewhat more general versions, with [Langlands 1967/76] treating a very general case, partly sketched in [Harish-Chandra 1968].

[Godement 1966] R. Godement, *Decomposition of* $L^2(\Gamma\backslash G)$ *for* $\Gamma = SL(2, Z)$, in Proc. Symp. Pure Math. *9* (1966), AMS, 211-24.

[Harish-Chandra 1968] Harish-Chandra, *Automorphic forms on semi-simple Lie groups*, notes by G.J.M. Mars, SLN **62**, Springer-Verlag, 1968.

[Langlands 1966] R. Langlands, *Eisenstein series*, in Proc. Symp. Pure Math. IX, *Algebraic Groups and Discontinuous Subgroups, Boulder 1965* (1966), AMS, 1966.

[Langlands 1967/1976] R.P.Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, vol. 544, Springer-Verlag, Berlin and New York, 1976.

[Roelcke 1956] W. Roelcke, *Über die Wellengleichung bei Grenzkreisgruppen erster Art*, S.-B. Heidelberger Akad. Wiss. Math.Nat.Kl. 1953/1955 (1956), 159-267.

[Selberg 1956] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric spaces, with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47-87