Continuous spectrum for \( SL_2(\mathbb{Z}) \backslash \mathfrak{H} \)

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We prove that the orthogonal complement in \( L^2(\Gamma \backslash \mathfrak{H}) \) to cuspforms is spanned by pseudo-Eisenstein series, in turn expressible as integrals or wave packets of Eisenstein series \( E_s \). We prove a Plancherel theorem for this space. The essential harmonic analysis is Fourier transform on the real line, in coordinates on which it is called a Mellin transform. That is, the non-cuspidal part of harmonic analysis on \( SL_2(\mathbb{Z}) \backslash \mathfrak{H} \) reduces to harmonic analysis on a related, lower-dimensional group.

1. Pseudo-Eisenstein series, adjunction to constant term

As usual, let \( N \) be the upper-triangular unipotent matrices in \( G = SL_2(\mathbb{R}) \), \( A \) the diagonal matrices, \( A^+ \) the diagonal matrices with positive diagonal entries, and \( P = NA \) the parabolic subgroup of upper-triangular matrices, \( P^+ = NA^+ \), \( \Gamma = SL_2(\mathbb{Z}) \), \( \Gamma_\infty = P^+ \cap \Gamma = N \cap \Gamma \), and \( \mathfrak{H} \) the upper half-plane.

[1.1] Constant terms \( c_P f \)  In most-elementary terms, the constant term \( c_P f \) along \( P \) of a reasonable function \( f \) on \( \Gamma \backslash \mathfrak{H} \) is

\[
\text{(constant term of } f \text{)}(z) = c_P f(z) = \int_{\Gamma_\infty \backslash N} f(n \cdot z) \ dn
\]

Changing variables in the integral shows that \( c_P f \) is left \( N \)-invariant. Identify left \( N \)-invariant functions on \( \mathfrak{H} \) as functions of \( y = \text{im}(z) \) alone, thus as functions on \( (0, \infty) \).

[1.2] Cuspforms  A reasonable function \( f \) on \( \Gamma \backslash \mathfrak{H} \) is a cuspform when its constant term essentially vanishes:

\[
\text{f cuspform } \iff c_P f = 0
\]

The cuspform condition is better recast as

\[
f \text{ cuspform } \iff \int_{N \backslash \mathfrak{H}} c_P f(z) \cdot \varphi(z) \ dz = 0 \quad \text{(for all } \varphi \in C^\infty_c(N \backslash \mathfrak{H}) \approx C^\infty_c(0, \infty))
\]

That is, the cuspform condition is that the constant term vanishes as a distribution on \( N \backslash \mathfrak{H} \).

[1.3] Pseudo-Eisenstein series  Pseudo-Eisenstein series are the solutions to an adjunction problem, as follows. Identify \( \mathfrak{H} \approx G/K \), where \( K = SO(2, \mathbb{R}) \), as usual. We need asymmetrical pairings of the form

\[
\langle f, F \rangle_{H \backslash \mathfrak{H}} = \int_{H \backslash \mathfrak{H}} f \cdot F \quad \text{(C-bilinear, not hermitian)}
\]

even when \( f, F \) may not be in the same space of functions, and use the same notation for extensions of integral pairings to distributions. The problem is, given \( \varphi \) in \( C^\infty_c(N \backslash \mathfrak{H}) \), find \( \Psi_\varphi \in C^\infty_c(\Gamma \backslash \mathfrak{H}) \) such that

\[
\langle c_P f, \varphi \rangle_{N \backslash \mathfrak{H}} = \langle f, \Psi_\varphi \rangle_{\Gamma \backslash \mathfrak{H}} \quad \text{(for } f \text{ on } \Gamma \backslash \mathfrak{H})
\]
Exhibition of $\Psi_\varphi$ will legitimize treatment of $c_P$ as a map from distributions on $\Gamma \backslash \mathfrak{H}$ to distributions on $N \backslash \mathfrak{H}$, even though this generality is not needed.

[1.3.1] **Remark:** The measure $dx \, dy/y^2$ on $\mathfrak{H} \approx G/K$ is the usual left $G$-invariant measure $dx \, dy/y^2$ inherited from Haar measure on $G = NA^+ K$. This descends to $dy/y^2$ on $N \backslash \mathfrak{H} \approx N \backslash G/K$, not to the measure $dy/y$ from the Haar measure on $A^+$.

Direct computation yields a canonical expression for the desired $\Psi_\varphi$, using the left $N$-invariance of $\varphi$ and the left $\Gamma$-invariance of $f$, as follows. Note that $P \cap \Gamma$ differs from $N \cap \Gamma$ only by $\pm 1_2$, which act trivially on $\mathfrak{H} \approx G/K$. Abuse notation by writing $\varphi(z) = \varphi(\text{Im} \, z)$:

$$\langle c_P f, \varphi \rangle_{N \backslash \mathfrak{H}} = \int_{N \backslash \mathfrak{H}} c_P f(z) \varphi(z) \, \frac{dy}{y^2} = \int_{\Gamma \backslash \mathfrak{H}} f(z) \varphi(z) \, \frac{dx \, dy}{y^2}$$

Winding up,

$$\int_{\Gamma \backslash \mathfrak{H}} f(z) \varphi(z) \, \frac{dx \, dy}{y^2} = \int_{\Gamma \backslash \mathfrak{H}} \sum_{\gamma \in P \cap \Gamma} f(\gamma z) \varphi(\gamma z) \, \frac{dx \, dy}{y^2} = \int_{\Gamma \backslash \mathfrak{H}} f(z) \left( \sum_{\gamma \in P \cap \Gamma} \varphi(\gamma z) \right) \, \frac{dx \, dy}{y^2}$$

The inner sum in the last integral is the pseudo-Eisenstein series\[^{[1]}\] attached to $\varphi$:

$$\Psi_\varphi(z) = \sum_{\gamma \in P \cap \Gamma} \varphi(\gamma z)$$

designed to fit into the adjunction

$$\langle c_P f, \varphi \rangle_{N \backslash \mathfrak{H}} = \langle f, \Psi_\varphi \rangle_{\Gamma \backslash \mathfrak{H}}$$

While cuspforms are mysterious, these pseudo-Eisenstein series are completely explicit.

[1.3.2] **Claim:** The series for a pseudo-Eisenstein series $\Psi_\varphi$ is locally finite, meaning that for $z$ in a fixed compact in $\mathfrak{H}$, there are only finitely-many non-zero summands in $\Psi_\varphi(z) = \sum_{\gamma} \varphi(\gamma z)$. Thus, $\Psi_\varphi \in C^\infty_c(\Gamma \backslash \mathfrak{H})$.

**Proof:** Identify $N \backslash \mathfrak{H} \approx N \backslash G/K$, so $C^\infty_c(N \backslash \mathfrak{H})$ can be identified with right $K$-invariant functions in $C^\infty_c(N \backslash G)$ since $K$ is compact. Given $\varphi \in C^\infty_c(N \backslash \mathfrak{H})$, let $C \subset G$ be compact so that $N \cdot C$ contains the support of $\varphi$. Fix compact $C_o \subset G$ in which $g \in G$ is constrained to lie. Then a summand $\varphi(\gamma g)$ is non-zero only if $\gamma \in N \cdot C$, which holds only if

$$\gamma \in NC \cdot g^{-1}$$

so

$$\gamma \in \Gamma \cap NC \cdot C_o^{-1}$$

In the quotient $G \to (P \cap \Gamma) \backslash G$, the image of $\Gamma$ is discrete. The image of the compact set $N \cdot C \cdot C_o^{-1}$ under the continuous quotient map is compact, since $(N \cap \Gamma) \backslash N$ is compact, and continuous images of compacts are compact. Thus, left modulo $P \cap \Gamma$, that intersection is the intersection of a (closed) discrete set and a compact set, so finite. Therefore, the series is locally finite, and defines a smooth function on $\Gamma \backslash G$. Summing over left translates certainly retains right $K$-invariance.

To show that $\Psi_\varphi$ has compact support in $\Gamma \backslash G$, proceed similarly. That is, for a summand $\varphi(\gamma g)$ to be non-zero, it must be that $g \in \Gamma \cdot C$. The image $\Gamma \backslash (\Gamma \cdot C)$ is compact, being the continuous image of the compact set $C$ under the continuous map $G \to \Gamma \backslash G$, proving the compact support. \[///\]

\[^{[1]}\] In 1966 Godement called these incomplete theta series, but more recently Moeglin-Waldspurger strengthened the precedent of calling them pseudo-Eisenstein series.
[1.3.3] Corollary: The square-integrable cuspforms are the orthogonal complement of the (closed) space subspace of \( L^2(\Gamma \backslash \mathfrak{H}) \) spanned by the pseudo-Eisenstein series \( \Psi_\varphi \) with \( \varphi \in C_0^\infty(N \backslash \mathfrak{H}) \approx C_0^\infty(0, \infty) \).

2. Decomposition of pseudo-Eisenstein series: beginning

Spectral decomposition of the data \( \varphi \) in \( \Psi_\varphi \) induces a spectral decomposition of \( \Psi_\varphi \) itself.

[2.1] Fourier transform  Fourier transform \( \hat{f} \) of \( f \) on the real line is

\[
\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx
\]

Fourier inversion for Schwartz functions \( f \) is

\[
f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} \, d\xi
\]

Replacing \( \xi \) by \( \xi/(2\pi) \) gives the variant

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(t) e^{-i t \xi} \, dt \right) e^{i \xi x} \, d\xi
\]

[2.2] Multiplicative coordinates, Mellin transforms  Fourier transforms on \( \mathbb{R} \) put into multiplicative coordinates are Mellin transforms: for \( \varphi \in C_0^\infty(0, +\infty) \), take

\[
f(x) = \varphi(e^x)
\]

Let \( y = e^x \) (and \( r \) the exponentiated variable in the implied inner integral) and rewrite Fourier inversion as

\[
\varphi(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{0}^{\infty} \varphi(r) r^{-i \xi} \, dr \right) y^{i \xi} \, d\xi
\]

The Fourier transform in these coordinates is called a Mellin or Laplace transform \( \mathcal{M}F \),

\[
\mathcal{M}\varphi(i\xi) = \int_{0}^{\infty} \varphi(r) r^{-i \xi} \, dr
\]

For compactly-supported \( \varphi \), the integral definition extends to complex \( s \):

\[
\mathcal{M}\varphi(s) = \int_{0}^{\infty} \varphi(r) r^{-s} \, dr
\]

The variant Fourier inversion identity gives Mellin inversion

\[
\varphi(y) = \frac{1}{2\pi i} \int_{-i\infty}^{0+i\infty} \mathcal{M}\varphi(i\xi) y^{i \xi} \, d\xi
\]

With \( \xi \) the imaginary part of a complex variable \( s \), rewrite the latter integral as a complex path integral, noting \( d\xi = -i \, ds \),

\[
\varphi(y) = \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} \mathcal{M}\varphi(s) y^s \, ds
\]
For $f \in C_c^\infty(\mathbb{R})$ the integral giving the Fourier transform $\hat{f}(\xi)$ converges nicely for all complex values of $\xi$, so extends to an entire function in $\xi$, of rapid decay on horizontal lines.\[2\] This extension property applies to $\varphi$, thus allowing movement of the contour: for compactly-supported $\varphi$, Mellin inversion is

$$\varphi(y) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}_c(s) \, y^s \, ds \quad \text{(for any real } \sigma)$$

[2.3] Spectral decomposition of pseudo-Eisenstein series

Identifying $N \backslash \mathcal{H} \approx N \backslash G/K \approx \mathbb{A}^+$, Mellin inversion is

$$\varphi(z) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}_c(s) \, (\text{Im}(z))^s \, ds \quad \text{(for any real } \sigma)$$

Thus, the pseudo-Eisenstein series is

$$\Psi_\varphi(z) = \sum_{\gamma \in \Gamma \backslash \Gamma} \varphi(\gamma z) = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma \backslash \Gamma} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}_c(s) \cdot (\text{Im}(\gamma z))^s \, ds$$

Taking $\sigma = 0$ would be natural, but, with $\sigma = 0$, the double integral (sum and integral) is not absolutely convergent, and the two integrals cannot be interchanged. We will eventually see that the best line is $\sigma = 1/2$, but this is not in the region of convergence, either. For $\sigma > 1$, elementary estimates show that the double integral is absolutely convergent, and by Fubini the two integrals can be interchanged:

$$\Psi_\varphi(z) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}_c(s) \sum_{\gamma \in \Gamma \backslash \Gamma} (\text{Im}(\gamma z))^s \, ds \quad \text{(with } \sigma > 1)$$

The inner sum is the Eisenstein series $E_s(z)$, so

$$\Psi_\varphi(z) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}_c(s) \cdot E_s(z) \, ds \quad \text{(for } \sigma > 1)$$

[2.3.1] Remark: This decomposition is partly unsatisfactory, because it should refer to $\mathcal{M}_cP(\Psi_\varphi)$, not $\mathcal{M}_c\varphi$, to have whatever integral formulas expressed in terms of the automorphic forms $\Psi_\varphi$ themselves, not in terms of the auxiliary functions $\varphi$ from which they’re made.

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\[2\] This is an easy part of the Paley-Wiener theorem, proven directly: for complex $w = \xi + i\eta$,

$$\hat{f}(w) = \int_{\mathbb{R}} e^{-2\pi i \omega x} f(x) \, dx = \int_{\mathbb{R}} e^{-2\pi i \xi x} \left( e^{2\pi \eta x} \cdot f(x) \right) \, dx$$

Since $f$ has compact support, multiplying by an exponential still gives a compact-support function, hence Schwartz, so the Fourier transform is Schwartz.
3. Eisenstein series

[3.1] Adjunction  As do the pseudo-Eisenstein series, the Eisenstein series $E_s$ fits into an adjunction relation

$$\langle E_s, f \rangle_{\Gamma \backslash \mathcal{H}} = \langle y^s \cdot c_P f \rangle_{N \backslash \mathcal{H}} \quad \text{ (for } f \text{ on } \Gamma \backslash \mathcal{H})$$

whenever the implied integrals converge. The Eisenstein series is determined by this relation:

$$\langle y^s \cdot c_P f \rangle_{A^+} = \int_{N \backslash \mathcal{H}} c_P f(z) \cdot y^s \frac{dy}{y^2} = \int_{N \backslash \mathcal{H}} \left( \int_{\Gamma_{\infty} \backslash \mathcal{H}} f(nz) \, dn \right) \cdot y^s \frac{dy}{y^2} = \int_{\Gamma_{\infty} \backslash \mathcal{H}} f(z) \cdot y^s \frac{dx \, dy}{y^2}$$

Winding up,

$$\int_{P \subset \Gamma \backslash \mathcal{H}} f(z) \cdot y^s \frac{dx \, dy}{y^2} = \int_{\Gamma \backslash \mathcal{H}} \sum_{\gamma \in P \cap \Gamma} f(\gamma z) \cdot \Im(\gamma z) \frac{dy}{y^2} = \int_{\Gamma \backslash \mathcal{H}} f(z) \sum_{\gamma \in P \cap \Gamma} \Im(\gamma z) \frac{dy}{y^2}$$

Thus, the adjunction produces the expression for $E_s(z)$ in the region of convergence $\Re(s) > 1$. By the analytic continuation of $E_s$, the adjunction characterizing $E_s$ asserts that integrals against Eisenstein series are Mellin transforms of constant terms:

$$\langle E_s, f \rangle_{\Gamma \backslash \mathcal{H}} = \int_0^\infty c_P f(iy) y^s \frac{dy}{y^2} = \int_0^\infty c_P f(iy) y^{-(1-s)} \frac{dy}{y} = \mathcal{M}(c_P f)(1-s)$$

[3.2] Meromorphic continuation and functional equation  We grant the meromorphic continuation. The constant term of the Eisenstein series is

$$c_P E_s = y^s + c_s y^{1-s} \quad \text{(with } c_s = \xi(2s-1)/\xi(2s))$$

where $\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ is the completed zeta function. The Eisenstein series $E_s$ can be characterized as the (unique) moderate-growth automorphic form on $\Gamma \backslash \mathcal{H}$ satisfying

$$y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) w = s(s-1) \cdot w \quad \text{ and } \quad \left( y \frac{\partial}{\partial y} - (1-s) \right) c_P w = (2s-1) \cdot y^s$$

noting that the latter differential operator annihilates $y^{1-s}$ and has $y^s$ as an eigenfunction. This characterization determines the functional equation from the constant term. Namely, $E_{1-s}$ satisfies the first of the two equations, and

$$\left( y \frac{\partial}{\partial y} - (1-s) \right) c_P E_{1-s} = \left( y \frac{\partial}{\partial y} - (1-s) \right) (y^{1-s} + c_1 y^s) = 0 + c_{1-s} \cdot (2s-1) \cdot y^s$$

Thus, we obtain the functional equation $E_{1-s} = c_{1-s} E_s$. Applying the functional equation twice gives $c_s c_{1-s} = 1$. Since $E_s = \overline{E_s}$, also $c_s = c_{\overline{s}}$ and $|c_{1-t}|^2 = 1$ for $t \in \mathbb{R}$. In particular, $c_s$ does not vanish on the line $\Re(s) = \frac{1}{2}$.

[3.2.1] Remark: Just below we will see that, from the expression of pseudo-Eisenstein series in terms of Eisenstein series, poles of $E_s$ to the right of $\Re(s) = \frac{1}{2}$ play a role in the decomposition of $L^2(\Gamma \backslash \mathcal{H})$.

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[3] This characterization is the simplest example of treatments of more general Eisenstein series by Selberg, Bernstein, Colin de Verdière, and Jacquet.
4. Decomposition of pseudo-Eisenstein series: conclusion

So far, we have a spectral decomposition of $\Psi_\varphi$ in terms of $\varphi$:

$$
\Psi_\varphi = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) \cdot E_s \, ds \quad (\sigma > 1)
$$

We proceed to rewrite this to refer only to $\Psi_\varphi$, not $\varphi$.

[4.1] Final re-arrangements, invocation of adjunction

Move the line of integration to the left, to $\sigma = 1/2$, stabilized by the functional equation of $E_s$:

$$
\Psi_\varphi = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s) \cdot E_s \, ds + \sum_{s_\sigma} \text{res}_{s=s_\sigma} (E_s \cdot \mathcal{M}\varphi(s))
$$

The $1/2\pi i$ from inversion cancels the $2\pi i$ in the residue formula. To rewrite this in terms of $\Psi_\varphi$, on one hand, from above,

$$
\langle E_s, \Psi_\varphi \rangle_{\Gamma \setminus \mathfrak{H}} = \mathcal{M}(c_\varphi \Psi_\varphi)(1-s)
$$

On the other hand, by the adjunction/unwinding property of $\Psi_\varphi$,

$$
\langle E_s, \Psi_\varphi \rangle = \langle c_\varphi E_s, \varphi \rangle_{N \setminus \mathfrak{H}} = \langle y^s + c_s y^{1-s}, \varphi \rangle_{N \setminus \mathfrak{H}} = \int_0^\infty (y^s + c_s y^{1-s}) \cdot \varphi(y) \frac{dy}{y^2}
$$

$$
= \int_0^\infty (y^{-(1-s)} + c_s y^{-s}) \cdot \varphi(y) \frac{dy}{y} = \mathcal{M}\varphi(1-s) + c_s \mathcal{M}\varphi(s)
$$

Thus,

$$
\mathcal{M}(c_\varphi \Psi_\varphi)(1-s) = \langle E_s, \Psi_\varphi \rangle_{\Gamma \setminus \mathfrak{H}} = \mathcal{M}\varphi(1-s) + c_s \mathcal{M}\varphi(s)
$$

or, replacing $s$ by $1-s$,

$$
\mathcal{M}(c_\varphi \Psi_\varphi)(s) = \langle E_{1-s}, \Psi_\varphi \rangle_{\Gamma \setminus \mathfrak{H}} = \mathcal{M}\varphi(s) + c_{1-s} \mathcal{M}\varphi(1-s)
$$

At the same time, the integral part of the expression of $\Psi_\varphi$ in terms of Eisenstein series can be folded in half, integrating from $\frac{1}{2} + i0$ to $\frac{1}{2} + i\infty$ rather than from $\frac{1}{2} - i\infty$ to $\frac{1}{2} + i\infty$:

$$
\Psi_\varphi - \text{(residual part)} = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s) \cdot E_s \, ds = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s) \cdot E_s + \mathcal{M}\varphi(1-s) \cdot E_{1-s} \, ds
$$

Using the functional equation $E_{1-s} = c_{1-s} E_s$, and recognizing $\mathcal{M}(c_\varphi \Psi_\varphi)$ as expressed above in terms of $\mathcal{M}\varphi$, this becomes

$$
\Psi_\varphi - \text{(residual part)} = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} (\mathcal{M}\varphi(s) + c_{1-s} \mathcal{M}\varphi(1-s)) \cdot E_s \, ds = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}_\varphi(s) \cdot E_s \, ds
$$

by the expression for the Mellin transform of the constant term of $\Psi_\varphi$. That is, a pseudo-Eisenstein series is expressible as an integral of Eisenstein series $E_s$ on the line $\text{Re}(s) = 1/2$, plus a sum of residues:

$$
\Psi_\varphi - \text{(residual part)} = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}_\varphi(s) \cdot E_s \, ds = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} (E_{1-s}, \Psi_\varphi)_{\Gamma \setminus \mathfrak{H}} \cdot E_s \, ds
When desired, the integral can be written as an integral over the whole line \( \text{Re}(s) = \frac{1}{2} \), by the functional equation of \( E_s \) and dividing by 2:

\[
\Psi - \text{(residual part)} = \frac{1}{4\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} (E_{1-s}, \Psi_{\Gamma \backslash \mathcal{H}} \cdot E_s) ds \quad \text{(complex-bilinear pairing)}
\]

\[
= \frac{1}{4\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \left< \Psi_{\Gamma \backslash \mathcal{H}}, E_s \right> \cdot E_s ds \quad \text{(complex-hermitian pairing)}
\]

**[4.2] The residual part**  For \( \Gamma = \text{SL}_2(\mathbb{Z}) \), the only pole of \( E_s \) in the half-plane \( \text{Re}(s) \geq 1/2 \) is at \( s_o = 1 \), is simple, and the residue is a constant function. Thus,

\[
\Psi_{\varphi} = \frac{1}{2\pi i} \int_{\frac{1}{2} + i0}^{\frac{1}{2} + i\infty} \left< \Psi_{\varphi}, E_s \right> \cdot E_s ds + \mathcal{M}(1) \cdot \text{res}_{s=1} E_s \quad \text{(complex-hermitian pairing)}
\]

Continuing with the abuse of notation \( \varphi(z) = \varphi(\text{Im} z) \), the coefficient \( \mathcal{M}(1) \) is

\[
\mathcal{M}(1) = \int_0^\infty \varphi(y) y^{-1} \frac{dy}{y} = \int_0^\infty \varphi(y) \frac{dy}{y^2} = \int_{N \backslash \mathcal{H}} \varphi(z) \frac{dy}{y^2}
\]

Rearranging, since the natural volume of \( \Gamma_\infty \backslash N \) is 1 and \( \varphi \) is left \( N \)-invariant,

\[
\mathcal{M}(1) = \int_{N \backslash \mathcal{H}} \int_{\Gamma_\infty \backslash N} \varphi(nz) dn \frac{dy}{y^2} = \int_{N \backslash \mathcal{H}} \varphi(nz) \left( \int_{\Gamma_\infty \backslash N} 1 dn \right) \frac{dy}{y^2} = \int_{\Gamma_\infty \backslash N} \varphi(z) \frac{dy}{y^2}
\]

Winding up,

\[
\mathcal{M}(1) = \int_{\Gamma \backslash \mathcal{H}} \sum_{\gamma \in \Gamma \backslash \Gamma} \varphi(z) \frac{dx dy}{y^2} = \int_{\Gamma \backslash \mathcal{H}} \Psi_{\varphi}(z) \frac{dx dy}{y^2} = \int_{\Gamma \backslash \mathcal{H}} \Psi_{\varphi}(z) \cdot 1 \frac{dx dy}{y^2} = \left< \Psi_{\varphi}, 1 \right>
\]

That is, \( \mathcal{M}(1) \) is the inner product of \( \Psi_{\varphi} \) with the constant function 1.

**[4.2.1] Remark:** The above discussion, and the fact that constants are residues of Eisenstein series \( E_s \) proves that constants are orthogonal to cuspforms: \( c_P f = 0 \) for cuspform \( f \), so

\[
0 = \mathcal{M}(c_P f)(1-s) = \left< f, E_s \right> \quad \text{(complex-bilinear pairing)}
\]

The complex-bilinear pairing value \( \left< f, E_s \right> \) is meromorphic in \( s \), so

\[
\left< f, \text{res}_{s=1} E_s \right> = \text{res}_{s=1} \left< f, E_s \right> = \text{res}_{s=1} 0 = 0
\]
5. Plancherel theorem

For $\varphi \in C_c^\infty(N\backslash G/N) \approx C_c^\infty(0, +\infty)$, the pseudo-Eisenstein series $\Psi_\varphi$ is expressed in terms of Eisenstein series, with complex-hermitian pairing $\langle , \rangle$,

$$\Psi_\varphi = \frac{1}{4\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \langle \Psi_\varphi, E_s \rangle E_s \, ds + \langle \Psi_\varphi, 1 \rangle \text{res}_{s=1} E_s$$

For $f \in C_c^\infty(1\backslash \mathcal{H})$,

$$\langle \Psi_\varphi, f \rangle = \left\langle \frac{1}{4\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \langle \Psi_\varphi, E_s \rangle E_s \, ds + \langle \Psi_\varphi, 1 \rangle \text{res}_{s=1} E_s, f \right\rangle$$

$$= \frac{1}{4\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \langle \Psi_\varphi, E_s \rangle \langle E_s, f \rangle \, ds + \langle \Psi_\varphi, 1 \rangle \langle \text{res}_{s=1} E_s, f \rangle$$

With $f = \Psi_\psi$ for $\psi \in C_c^\infty(N\backslash \mathcal{H})$, this gives an isometry:

**[5.0.1] Theorem:** Using the hermitian pairing on $L^2(1\backslash \mathcal{H})$, for $\varphi, \psi \in C_c^\infty(0, +\infty)$,

$$\langle \Psi_\varphi, \Psi_\psi \rangle = \frac{1}{2\pi i} \int_{\frac{1}{2} - i0}^{\frac{1}{2} + i\infty} \langle \Psi_\varphi, E_s \rangle \langle \Psi_\psi, E_s \rangle \, ds + \langle \Psi_\varphi, 1 \rangle \langle \Psi_\psi, 1 \rangle \cdot \text{res}_{s=1} E_s$$

Incidentally, the Plancherel isometry determines the constant $\text{res}_{s=1} E_s$: from $\langle 1, 1 \rangle = \langle 1, 1 \rangle \cdot \text{res}_{s=1} E_s$ we find $\text{res}_{s=1} E_s = \frac{1}{\text{vol}(1\backslash \mathcal{H})}$. Compare [Langlands 1966].

Since the collection of pseudo-Eisenstein series $\Psi_\varphi$ is dense in the orthogonal complement in $L^2(1\backslash \mathcal{H})$ to cuspforms, this isometry extends by continuity to give an isometry

$$\{ \text{orthogonal complement in } L^2(1\backslash \mathcal{H}) \text{ to cuspforms} \} \approx L^2(1/2 + i\mathbb{R}) \oplus \mathbb{C}$$

by (the isometric extension of) $f \mapsto \langle f, E_s \rangle \oplus \langle f, 1 \rangle / \langle 1, 1 \rangle^{1/2}$. This is the Plancherel theorem on the orthogonal complement to cuspforms. That is, in an $L^2$-sense, for $f$ in the orthogonal complement to cuspforms,

$$f = L^2 \frac{1}{4\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \langle f, E_s \rangle \cdot E_s \, ds + \frac{\langle f, 1 \rangle \cdot 1}{\langle 1, 1 \rangle} \quad \text{(complex-hermitian pairing)}$$

That is, the indicated integrals are not literal integrals, but are the extension-by-continuity of the corresponding integrals.

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[4] Although $1\backslash \mathcal{H}$ is not quite smooth, via $\mathcal{H} \approx G/K$ we identify $C_c^\infty(1\backslash \mathcal{H})$ with right $K$-invariant smooth functions on $1\backslash G$, denoted $C_c^\infty(1\backslash G)^K$. 

8
Bibliography


