(*September 14, 2015*)

01 Euler and $\zeta(s)$

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[This document is http://www.math.umn.edu/˜garrett/m/mfms/notes_2015-16/01_Euler_and_zeta.pdf]

Infinite sums that *telescope* can be understood in simpler terms. The archetype is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots \;=\; \Big(\frac{1}{1}-\frac{1}{2}\Big)+\Big(\frac{1}{2}-\frac{1}{3}\Big)+\Big(\frac{1}{3}-\frac{1}{4}\Big)+\ldots \;=\; 1$$

Other subtler examples analyze-able via trigonometric functions give more interesting answers, but these are still elementary, the archetype being [1]

$$\frac{1}{1}-\frac{1}{3}-\frac{1}{5}+\ldots \;=\; \Big(\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\Big)\Big|_{x=1} \;=\; \int_0^x \frac{dt}{1+t^2}\;\Big|_{x=1} \;=\; \arctan 1 \;=\; \frac{\pi}{4}$$

As late as 1735, no one knew how to express in simpler terms

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$$

This was the *Basel problem*, after the town in Switzerland home to the Bernoullis, collectively a dominant force in European mathematics at the time.

L. Euler solved the problem, winning him useful notoriety at an early age, but more notably introducing larger ideas, as follows.

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1. *Euler and the Basel Problem*

Given non-zero numbers $a_1,\ldots,a_n$, a polynomial with constant term 1 having these numbers as roots is

$$\Big(1-\frac{x}{a_1}\Big)\Big(1-\frac{x}{a_2}\Big)\ldots\Big(1-\frac{x}{a_n}\Big) \;=\; 1-(\frac{1}{a_1}\ldots+\frac{1}{a_n})x+(\frac{1}{a_1a_2}+\frac{1}{a_1a_3}+\frac{1}{a_2a_3}+\ldots)x^2+\ldots+(-1)^n\frac{x^n}{a_1\ldots a_n}$$

Imagine that $\frac{\sin \pi x}{\pi x}$ has an analogous product expansion, in terms of its zeros at $\pm1,\pm2,\pm3,\ldots$, up to a normalizing constant needing determination: using the power series expansion of $\sin x$,

$$\frac{(\pi x)-\frac{(\pi x)^3}{3!}+\ldots}{\pi x} \;=\; \frac{\sin \pi x}{\pi x} \;=\; C\cdot\prod_{n=1}^{\infty}\big(1-\frac{x^2}{n^2}\big) \;=\; C\cdot\Big(x-x^3\sum_n\frac{1}{n^2}+\ldots\Big)$$

Assuming this works, equating constant terms gives $C=1$, and equating coefficients of x^2 gives

$$\frac{\pi^2}{6} \;=\; \sum\frac{1}{n^2}$$

Slightly messier manipulations yield all values $\sum\frac{1}{n^{2k}}$.

Euler did not manage to prove that the sine function had such a product expansion for some years. Nevertheless, *just this heuristic*, without the eventual proof, was what won him considerable notoriety, in part because it suggested an *underlying mechanism*. Further, everyone believed the heuristic because, the *numerical* plausibility was easy to check, once observed.

[1] There is some delicacy about convergence, but this is secondary.

2. *Euler product expansion of* $\zeta(s)$

In effect, the Basel problem was about explaining the values $\zeta(2), \zeta(4), \ldots$, of the simplest *zeta function*

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad\qquad \text{(real } s\text{?, complex } s\text{?)}$$

Euler made another striking observation about $\zeta(s)$: it has an *Euler product* factorization/expansion, coming from *unique factorization* in $\mathbb{Z}$:

$$\sum_n \frac{1}{n^s} \;=\; \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{10^s} + \frac{1}{11^s} + \frac{1}{12^s} + \ldots\ldots$$

$$= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{2^{2s}} + \frac{1}{5^s} + \frac{1}{2^s \cdot 3^s} + \frac{1}{7^s} + \frac{1}{2^{3s}} + \frac{1}{3^{2s}} + \frac{1}{2^s \cdot 5^s} + \frac{1}{11^s} + \frac{1}{2^{2s} \cdot 3^s} + \ldots$$

$$= \Big(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + \ldots\Big)\Big(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + \ldots\Big)\Big(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \frac{1}{5^{3s}} + \ldots\Big) \ldots \;=\; \prod_{\text{primes } p} \frac{1}{1 - \frac{1}{p^s}}$$

from factoring *uniquely* into prime powers and massively regrouping in terms of geometric series for each prime:

$$1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots \quad = \quad \frac{1}{1 - \frac{1}{p^s}}$$

In short,

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{\text{primes } p} \frac{1}{1 - \frac{1}{p^s}}$$

That is, we have an expression involving *just primes* equated to an expression *not* overtly involving primes, suggesting the relevance of the zeta function to prime numbers. Whether or not we care greatly about prime numbers, the *connection* between primes and behavior of $\zeta(s)$ is striking.

3. *Quantitative infinitude of primes*

The ancient Greek mathematicians proved the infinitude of primes *qualitatively*, but experimentation quickly suggests that the estimates on asymptotic density arising from that classical argument are laughably bad.

The first serious *qualitative* result about primes was Euler's, using the Euler product expansion of $\zeta(s)$, proving

$$\sum_{\text{all primes } p} \frac{1}{p} \;=\; \infty$$

Comparison with $\int_1^\infty x^{-s}\,dx$ for real $s > 1$ proves that $\zeta(s) \to +\infty$ as $s \to 1^+$:

$$\frac{1}{s-1} \;=\; \int_1^\infty x^{-s}\,dx \;\le\; \zeta(s) \qquad\qquad \text{(for real } s > 1)$$

On the other hand, from $-\log(1-x) = x + x^2/2 + x^3/3 + \ldots$,

$$\log \zeta(s) \;=\; \sum_p -\log\big(1 - \frac{1}{p^s}\big) \;=\; \sum_p \Big(\frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \ldots\Big)$$

The terms after the initial $1/p^s$ do not contribute to any blow-up as $s \to 1^+$, by crudely estimating primes by positive integers:

$$\sum_p \sum_{\ell \geq 2} \frac{1}{\ell p^{\ell s}} \;\leq\; \sum_{n \geq 2} \sum_{\ell \geq 2} \frac{1}{\ell n^{\ell s}} \;\leq\; \sum_{\ell \geq 2} \frac{1}{\ell} \int_1^\infty \frac{dx}{x^{\ell s}}$$

$$= \sum_{\ell \geq 2} \frac{1}{\ell} \cdot \frac{1}{\ell s - 1} \;\leq\; \sum_{\ell \geq 2} \frac{1}{\ell} \cdot \frac{1}{\ell - 1} \;<\; \infty \qquad \text{(uniformly for } s \geq 1\text{)}$$

Thus, for some finite constant C,

$$\log \frac{1}{s-1} \;<\; \log \zeta(s) \;\leq\; C \;+\; \sum_p \frac{1}{p^s} \qquad \text{(for } s > 1\text{)}$$

Thus, the blow-up of $\log \zeta(s)$ as $s \to 1^+$ forces the divergence [2] of $\sum_p 1/p$.

This argument by itself did not attempt quantify *how fast* $\sum_{p<x} \frac{1}{p}$ blows up as a function of $x \to \infty$, although a similar idea can make progress there.

A related question of at least whimsical interest is about the prime-counting function [3]

$$\pi(x) \;=\; \sum_{p\prime < x} 1$$

That is, the *weights* in the counting are not $1/p^s$, not $1/p$, but just 1. About 160 years after Euler's first work on zeta, in 1896 Hadamard and de la Vallée-Poussin (independently) proved

$$\pi(x) \;\sim\; \frac{x}{\log x} \qquad \text{(as } x \to +\infty\text{)}$$

meaning that $\pi(x)/\big(x/\log x\big) \to 1$ as $x \to +\infty$. The question of refining the *error term* in this asymptotic is very much open currently.

Riemann's 1859 paper showed that the connection between prime numbers and the zeta function is far deeper than the above discussion suggests.

4. *Proof of Euler product*

Although the heuristic may be clear, we could be concerned about *convergence* of the infinite product of these geometric series to the sum expression for $\zeta(s)$, in $\mathrm{Re}(s) > 1$. All the worse if the Euler product seems intuitive, it is surprisingly non-trivial to carry out a detailed verification of the Euler factorization.

Ideas about *convergence* were different in 1750 than now. It is not accurate to glibly claim that the Cauchy-Weierstraß $\varepsilon - \delta$ viewpoint gives the only possible correct notion of *convergence*, since A. Robinson's 1966 *non-standard analysis* offers a rigorous modernization of Leibniz', Euler's, and others' use of *infinitesimals* and *unlimited natural numbers* to reach similar goals. [4] Thus, although an $\varepsilon - \delta$ discussion is alien to Euler's viewpoint, it is more familiar to contemporary readers, and we conduct the discussion in such terms.

[2] For real $s > 1$, certainly $\sum_p p^{-s} \leq \sum_p 1/p$, so the blow-up of the left-hand side as $s \to 1^+$ implies divergence of the right.

[3] Slightly unfortunate notation $\pi(x)$, but it is traditional.

[4] A. Robinson's *Non-standard Analysis*, North-Holland, 1966 was epoch-making, and really did justify some of the profoundly intuitive ideas in L. Euler's *Introductio in Analysin Infinitorum*, Opera Omnia, Tomi Primi, Lausanne,

[4.1] The main issue One central point is the discrepancy between *finite* products of *finite* geometric series involving primes, and *finite* sums of natural numbers. For example, for $T > 1$, because every positive integer $n < T$ is a product of prime powers $p^m < T$ in a unique manner,

$$\Big| \prod_{\text{prime}\, p<T} \Big(\sum_{m\,:\,p^m<T} \frac{1}{p^{ms}} \Big) \;-\; \sum_{n<T} \frac{1}{n^s} \Big| \;<\; \sum_{n\ge T} \frac{1}{|n^s|}$$

since the finitely-many leftovers from the product produce integers $n \ge T$ at most once each. The latter sum goes to 0 as $T \to \infty$, for fixed $\mathrm{Re}(s) > 1$, by comparison with an integral.

The finite sums $\sum_{n<T} 1/n^s$ are the usual partial sums of the infinite sum, and (simple) convergence is the assertion that this sequence converges to $\sum_n 1/n^s$.

Thus, this already proves that

$$\lim_{T\to\infty} \prod_{\text{prime}\, p<T} \Big(\sum_{m\,:\,p^m<T} \frac{1}{p^{ms}} \Big) \;=\; \sum_{n\ge 1} \frac{1}{n^s}$$

[4.2] Limits of varying products In contrast, the auxiliary question about infinite products is more complicated. We have products *whose factors themselves vary*: we would like to prove that because

$$1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots + \frac{1}{p^{ms}} \;\longrightarrow\; \frac{1}{1 - \frac{1}{p^s}} \qquad \text{(for } \mathrm{Re}(s) > 1\text{)}$$

the limit of *changing* products converges:

$$\prod_{p<T} \Big(\sum_{m:p^m<T} \frac{1}{p^{ms}} \Big) \;\longrightarrow\; \prod_p \frac{1}{1 - \frac{1}{p^s}} \qquad \text{(for } \mathrm{Re}(s) > 1\text{)}$$

where the infinite product on the right is the limit of its finite partial products. *That the individual factors on the left approach the individual factors on the right is not necessary (for fixed s), and in any case is not sufficient.*

Taking logarithms is convenient, since error estimates on *sums* is easier than error estimates on *products*. That is, we claim that

$$\log \prod_{p<T} \Big(\sum_{m:p^m<T} \frac{1}{p^{ms}} \Big) \;\longrightarrow\; \log \prod_p \frac{1}{1 - \frac{1}{p^s}} \qquad \text{(for } \mathrm{Re}(s) > 1\text{)}$$

The infinite product on the right is easily verified to converge to a non-zero limit, so continuity of *log* away from 0 allows us to move the logarithm inside the infinite product. Moving *log* inside a *finite* product is not an issue. Thus, it suffices to prove that

$$\sum_{p<T} \log \Big(\sum_{m:p^m<T} \frac{1}{p^{ms}} \Big) \;\longrightarrow\; \sum_p \log \frac{1}{1 - \frac{1}{p^s}} \qquad \text{(for } \mathrm{Re}(s) > 1\text{)}$$

1748. E. Nelson's reformulation *Internal set theory, a new approach to NSA*, Bull. AMS **83** (1977), 1165-1198, significantly improved the usability of these ideas. A. Robert's *Non-standard Analysis*, Dover, 2003 (original French version 1985, Presses polytechniques romandes, Lausanne) is a very clear exposition of non-standard analysis in Nelson's modified form.

[4.2.1] Claim: For fixed $0 < \delta < 1$, there is a constant $C > 0$ such that, for any $|x| < \delta$ and $|y| < \delta$,

$$\Big|\log(1+x) - \log(1+y)\Big| \;<\; C \cdot |x-y|$$

(This is the mean value theorem.) ///

Now the approximations of the factors by geometric series can be used: for fixed p, letting $\sigma = \mathrm{Re}(s) > 1$,

$$\Big|\sum_{m:p^m<T} \frac{1}{p^{ms}} \;-\; \frac{1}{1-\frac{1}{p^s}}\Big| \;\le\; \sum_{m:p^m\ge T} \frac{1}{p^{m\sigma}} \;\le\; \frac{1}{T^\sigma}\cdot\frac{1}{1-2^{-\sigma}}$$

Thus, for fixed $\sigma > 1$, for every p

$$\Big|\log\sum_{m:p^m<T} \frac{1}{p^{ms}} \;-\; \log\frac{1}{1-\frac{1}{p^s}}\Big| \;\le\; C\cdot\frac{1}{T^\sigma}\cdot\frac{1}{1-2^{-\sigma}}$$

and then

$$\sum_{p<T}\Big|\log\sum_{m:p^m<T} \frac{1}{p^{ms}} \;-\; \log\frac{1}{1-\frac{1}{p^s}}\Big| \;<\; \frac{1}{T^\sigma}\sum_{p<T}\frac{1}{1-2^{-\sigma}} \;\le\; \frac{1}{T^{\sigma-1}}\cdot\frac{1}{1-2^{-\sigma}}$$

Given $\varepsilon > 0$, take T_o large enough so that for $T > T_o$

$$\Big|\sum_{p<T}\log\frac{1}{1-\dfrac{1}{p^s}} \;-\; \sum_{p}\log\frac{1}{1-\dfrac{1}{p^s}}\Big| \;<\; \varepsilon$$

Then

$$\Big|\sum_{p<T}\log\Big(\sum_{m:p^m<T}\frac{1}{p^{ms}}\Big) \;-\; \sum_{p}\log\frac{1}{1-\dfrac{1}{p^s}}\Big| \;\le\; \Big|\sum_{p<T}\log\Big(\sum_{m:p^m<T}\frac{1}{p^{ms}}\Big) \;-\; \sum_{p<T}\log\frac{1}{1-\dfrac{1}{p^s}}\Big| \;+\; \varepsilon$$

$$<\; \frac{C}{1-2^\sigma}\cdot\frac{1}{T^{\sigma-1}} + \varepsilon$$

Since $\sigma > 1$, this can be made small by increasing T. ///

[4.2.2] Remark: Since everything turned out nicely, one might think that the above discussion made too much of a fuss about small issues. Indeed, nothing counter-intuitive transpired. However, at the least, it is worthwhile to understand exactly what the assertion of an Euler product factorization entails, in terms of *finite* expressions.

[4.3] Discussion One issue is a sensible meaning for *convergence* of an infinite *product* $\prod_{i=1}^\infty a_i$. Our general understanding of *infinite* processes rests on essentially a single notion, that of taking a *limit* of finite subprocesses. An *ordering* may be further specified, to distinguish a special class of finite subprocesses. For example, an infinite sum $\sum_{i=1}^\infty a_i$ has value the limit, if that limit exists, of the *special* finite subsums $s_N = \sum_{i=1}^N a_i$. We could make the stronger requirement of convergence of the *net*[5] of *all* finite subsums, indexed by the *directed poset* of *all* finite subsets of $\{1, 2, \ldots\}$. Convergence of such a more complicated net

[5] A *net* is a useful generalization of *sequence*: while a sequence is a set indexed by the ordered set $\{1, 2, \ldots\}$, a *net* is a set indexed by a *directed poset*. The word *poset* is a common abbreviation for *partially ordered set*, which is a set with a partial order $x < y$. A partial order is *transitive*, meaning that $x < y$ and $y < z$ implies $x < z$, and anti-symmetric, meaning that $x \not< x$. The *directed* condition on a poset S is that, given $x, y \in S$, there is $z \in S$ with both $x < z$ and $y < z$.

would mean that, given $\varepsilon > 0$, there is a finite subset $F \subset \{1, 2, \ldots\}$ such that, for any finite subsets X, Y of $\{1, 2, \ldots\}$ containing F,

$$\Big|\sum_{i\in X} a_i \;-\; \sum_{i\in Y} a_i\Big| \;<\; \varepsilon$$

One can prove that this stronger notion of convergence is equivalent to *absolute* convergence. For subsequent manipulations of infinite sums, usually we want and need absolute convergence.

Similarly, for infinite products $\prod_{i=1}^{\infty} a_i$, the weakest reasonable convergence requirement is convergence of the *sequence* of finite sub-products $\prod_{i=1}^{N} a_i$. *Absolute* convergence is equivalent to the stronger requirement of convergence of net of *all* finite sub-products. The stronger requirement is necessary to legitimize non-trivial subsequent manipulations.

The behavior of 0 in multiplication has an effect on infinite products with no counterpart in infinite sums, namely, that a single factor of 0 makes the whole product 0, regardless of the behavior of other factors. This might seem silly or undesirable, so some sources declare this behavior unacceptable, or given an impression of compromise by allowing only *finitely-many* factors of 0.

A more serious issue is *convergence to* 0. For example, the sequence of finite partial products

$$p_T \;=\; \prod_{n\le T} \big(1 - \frac{1}{n}\big)$$

converges to 0. Thus, it makes sense to say that the *infinite* product converges to 0. However, many sources disallow this, as part of their definition. On the face of it, there is no reason to object to convergence to 0, since it certainly fits with general principles about infinite processes being limits of finite sub-processes.

However, in the present situation as well as in many other applications, one immediately takes a *logarithm* of a product. Thus, we want infinite products which converge in a sense that makes the infinite sum of logarithms converge. The discrepancy between this goal and the general principles about infinite processes being limits of finite sub-processes is genuine, since logarithm is not *continuous* at 0. *It would be unreasonable to expect maps such as log to preserve limits at points where they are not continuous.*

For example, taking logarithms in the product displayed above,

$$\sum_n \log\big(1 - \frac{1}{n}\big) \;=\; \sum_n -\big(\frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\big) \;\le\; -\sum_n \frac{1}{n} \;=\; -\infty$$

That is, as expected, convergence of an infinite product to 0 becomes divergence to $-\infty$ under logarithm.

A detail: logarithms of infinite products of *complex* numbers require conventions to avoid meaningless divergence due to ambiguities in the imaginary part of logarithms. This issue is secondary, so we ignore it in the present discussion.

In any context in which logarithms of products matter, we might *define* convergence of a product $\prod_j a_j$ of positive reals a_j to be convergence of the sum $\sum_j \log a_j$, and expect to prove

[4.3.1] Claim: For positive real numbers a_j, if the infinite sum $\sum_j \log a_j$ converges, then $\prod_j a_j$ converges, in the sense that the sequence of partial products $p_N = \prod_{j\le N} a_j$ converges. ///

In terms of logarithms, from

$$-\log(1-x) \;=\; x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \qquad (\text{for } |x| < 1)$$

we have approximations that simplify sums of logarithms. For example, for $\delta > 0$, there are $A, B > 0$ such that

$$Ax \;<\; -\log(1-x) \;<\; Bx \qquad (\text{for } |x| < 1-\delta, \text{ constants } A, B \text{ depending on } \delta > 0)$$

Thus, an infinite product $\prod_n (1+a_n)$ with all the a_n's in the range $|a_n| < 1-\delta$ has partial products with logarithms satisfying

$$A \sum_{n<N} a_n \;<\; \log \prod_{n<N} (1+a_n) \;=\; \sum_{n<N} \log(1+a_n) \;<\; B \sum_{n<N} a_n \qquad (A, B \text{ depending on } \delta > 0)$$

Thus, in this situation, convergence of the sum of logarithms $\log(1+a_n)$ is equivalent to convergence of the sum of a_n. There are obvious variations.

Again, infinite products may converge to 0 in the sense that the sequence of partial subproducts converges to 0, but the sequence of sums of logarithms of partial subproducts diverges to $-\infty$. Equivalently, the comparison of $\log(1+x)$ and x fails as $x \to -1$. Equivalently, log is not continuous at 0.

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5. *Appendix: product expansion of* $\sin x$

Euler did eventually *prove* the product expansion

$$\sin \pi x \;=\; \pi x \cdot \prod_{n\geq 1} \Big(1 - \frac{x^2}{n^2}\Big)$$

The question does not mention complex numbers at all, but the simplest verification of this product expansion is a standard application of basic *complex analysis*, at the level of *Liouville's theorem* and *Laurent expansions* near poles, providing yet another powerful motivation for understanding basic complex analysis. [6]

[5.1] Partial fraction expansion of $\frac{\pi^2}{\sin^2 \pi x}$

We claim that there is a *partial fraction expansion*

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

To see this, first note that both sides have double poles exactly at integers. The Laurent expansion of the right-hand side near $n \in \mathbb{Z}$ begins

$$\frac{1}{(z-n)^2} + \text{holomorphic}$$

The left-hand side is periodic, so it suffices to see the Laurent expansion near 0:

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \frac{\pi^2}{\big(\pi z + \frac{(\pi z)^3}{3!} + \ldots\big)^2} \;=\; \frac{1}{z^2} \cdot \frac{1}{\big(1 - \frac{\pi^2 z^2}{3!} + \ldots\big)^2} \;=\; \frac{1}{z^2} \cdot \Big(1 + \frac{\pi^2 z^2}{3!} + \ldots\Big)^2 \;=\; \frac{1}{z^2} + \text{holomorphic}$$

This Laurent expansion matches that of the partial fraction expansion. Thus,

$$f(z) \;=\; \frac{\pi^2}{\sin^2 \pi z} \;-\; \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

has no poles in $\mathbb{C}$, so is *entire*. On the real line, after cancellation of poles, f *continuous*. The periodicity $f(z+1) = f(z)$ is visible, so f is *bounded* on the real line. In fact, since f is bounded on any region

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[6] Weierstraß and Hadamard product expansions apply to general *entire* functions, but with more overhead than needed here.

$\{x+iy : 0 \le x \le 1,\ |y| \le N\}$, the periodicity gives the boundedness of $f(z)$ on every band $|y| \le N$ containing $\mathbb{R}$.

Both parts of $f(z)$ go to 0 as $|y| \to \infty$. Thus, f is bounded and entire, so constant, by *Liouville's theorem.* Since $f(z) \to 0$ as $|y| \to \infty$, this constant is 0, giving the desired equality.

[5.2] Partial fraction expansion of $\pi \cot \pi x$

Next, regroup slightly, to improve convergence:

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \frac{1}{z^2} + \sum_{n \ge 1} \Big(\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} \Big)$$

The left-hand side is the derivative of $-\pi \cot \pi z$, and with the improved convergence the right-hand side is the obvious termwise derivative, so up to a constant C,

$$\pi \cot \pi z \;=\; C + \frac{1}{z} + \sum_{n \ge 1} \Big(\frac{1}{z-n} + \frac{1}{z+n} \Big)$$

The identity

$$\frac{1}{z-n} + \frac{1}{z+n} \;=\; \frac{(z+n)+(z-n)}{z^2-n^2} \;=\; \frac{2z}{z^2-n^2}$$

certifies that convergence is uniform and absolute, *and* that the summands are *odd* functions of z. Everything but the constant C is *odd* as a function of z, so $C = 0$. Thus,

$$\pi \cot \pi z \;=\; \frac{1}{z} + \sum_{n \ge 1} \Big(\frac{1}{z-n} + \frac{1}{z+n} \Big)$$

[5.3] Product expansion of $\sin \pi x$

Also, $\pi \cot \pi z$ is the logarithmic derivative of $\sin \pi z$:

$$\frac{d}{dz} \log(\sin \pi z) \;=\; \frac{(\sin \pi z)'}{\sin \pi z} \;=\; \frac{\pi \cos \pi z}{\sin \pi z} \;=\; \pi \cot \pi z$$

Thus,

$$\frac{d}{dz} \log(\sin \pi z) \;=\; \frac{(\sin \pi z)'}{\sin \pi z} \;=\; \frac{1}{z} + \sum_{n \ge 1} \Big(\frac{1}{z-n} + \frac{1}{z+n} \Big)$$

We intend to integrate. First, anticipating our goal, note that

$$\frac{d}{dz} \log\big(1 - \frac{z}{n}\big) \;=\; \frac{-\frac{1}{n}}{1-\frac{z}{n}} \;=\; \frac{1}{z-n}$$

Thus, integrating, for some constant C,

$$\log(\sin \pi z) \;=\; C + \log z + \sum_{n \ge 1} \Big(\log\big(1 - \frac{z}{n}\big) + \log\big(1 - \frac{z}{n}\big) \Big) \;=\; C + \log z + \sum_{n \ge 1} \log\Big(1 - \frac{z^2}{n^2}\Big)$$

Exponentiating,

$$\sin \pi z \;=\; e^C \cdot z \cdot \prod_{n \ge 1} \Big(1 - \frac{z^2}{n^2}\Big)$$

Looking at the power series at $z = 0$, we see that $e^C = \pi$, so

$$\sin \pi z \;=\; \pi z \cdot \prod_{n \ge 1} \Big(1 - \frac{z^2}{n^2}\Big)$$