(*September 27, 2015*)

# *The keyhole/Hankel contour and* $\zeta(-n)$

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[This document is http://www.math.umn.edu/˜garrett/m/mfms/notes_2015-16/05_keyhole_and_zeta.pdf]

The contour-integration device here is one of Riemann's proofs of analytic continuation of $\zeta(s)$. It immediately proves that values of $\zeta(s)$ at non-positive integers are *rational*, and shows the connection to the Laurent coefficients of $1/(e^t-1)$ at $t=0$.

## [1.1] An integral representation of $\Gamma(s)\cdot\zeta(s)$

Although the integral representation of $\zeta(s)$ using a theta function is perhaps better in the long run, there is a more elementary one:

**[1.1.1] Claim:** For $\mathrm{Re}(s) > 1$,

$$\Gamma(s)\cdot\zeta(s) \;=\; \int_0^\infty \frac{t^{s-1}\,dt}{e^t-1}$$

*Proof:* Expand a geometric series, exchange sum and integral, and change variables:

$$\int_0^\infty \frac{t^{s-1}\,dt}{e^t-1} \;=\; \int_0^\infty \frac{t^{s-1}\,e^{-t}\,dt}{1-e^{-t}} \;=\; \int_0^\infty t^{s-1}\sum_{n\ge1} e^{-nt}\,dt \;=\; \sum_{n\ge1}\int_0^\infty t^s\,e^{-nt}\,\frac{dt}{t}$$

$$=\; \sum_{n\ge1}\frac{1}{n^s}\int_0^\infty t^s\,e^{-t}\,\frac{dt}{t} \;=\; \Gamma(s)\cdot\sum_{n\ge1}\frac{1}{n^s} \;=\; \Gamma(s)\cdot\zeta(s)$$

as claimed. ///

## [1.2] Keyhole/Hankel contour

The *keyhole* or *Hankel* contour is a path from $+\infty$ inbound along the real line to $\varepsilon > 0$, counterclockwise around a circle of radius $\varepsilon$ at 0, back to $\varepsilon$ on the real line, and outbound back to $+\infty$ along the real line.

The usual elementary application is to evaluation of integrals similar to $\int_0^\infty \frac{t^s\,dt}{t^2+1}$, with $0 < \mathrm{Re}(s) < 1$. In such an example, analytically continuing counterclockwise around 0 has no impact on the denominator, but, significantly, the numerator changes by a factor $e^{2\pi is}$, since

$$t^s \;=\; (|t|\cdot e^{i\theta})^s \;=\; |t|^s\cdot e^{i\theta s} \qquad\qquad (\text{and } \theta \text{ goes from 0 to } 2\pi)$$

We want the out-bound value of $t^s$ to be real-valued for real $s$, so the inbound version of $t^s$ must be $t^s\cdot e^{2\pi is}$. The absolute value of the integrand goes to 0 as $|t|\to 0$, so the integral over the small circle goes to 0 as $\varepsilon\to 0$, as do the integrals to and from $0,\varepsilon$ along the real line.

Thus, letting $H_\varepsilon$ be the Hankel contour with circle of radius $\varepsilon > 0$,

$$\lim_{\varepsilon\to0}\int_{H_\varepsilon}\frac{t^s\,dt}{t^2+1} \;=\; \lim_{\varepsilon\to0}\Big(\int_{+\infty}^{\varepsilon}\frac{(t\cdot e^{2\pi i})^s\,dt}{t^2+1} + (\text{integral over little circle}) + \int_\varepsilon^{+\infty}\frac{t^s\,dt}{t^2+1}\Big)$$

$$=\; (1-e^{2\pi is})\int_0^\infty \frac{t^s\,dt}{t^2+1}$$

In this elementary example, the trick is to further modify $H_\varepsilon$ by *not* going all the way to $+\infty$ outbound, but stopping at $+R$ for large positive $R$, traversing clockwise a large circle of radius $R$ back to the positive real axis, and then inbound to $\varepsilon$. The integrals from $R$ to and from $+\infty$ go to 0 as $R\to+\infty$, as does the integral over the large circle, since

$$\Big|\text{integral over big circle}\Big| \le\; \text{length}\cdot\text{max value} \le\; 2\pi R\cdot\frac{R^{\mathrm{Re}(s)}}{R^2-1}$$

For each $R, \varepsilon$, this gives a path integral (counter-clockwise) over a *closed* path. By *residues*, this picks up $2\pi i$ times the sum of the residues inside the path. Thus, we discover that the integrals do not depend on the parameters $0 < \varepsilon < 1 < R$. Keeping track of the relevant versions of $t^s$,

$$(1 - e^{2\pi i s}) \int_0^\infty \frac{t^s\, dt}{t^2+1} \;=\; 2\pi i \cdot \Big( (\text{residue at } t = i) + (\text{residue at } t = -i) \Big)$$

$$=\; 2\pi i \cdot \Big( \frac{e^{\frac{1}{2}\pi i s}}{i+i} + \frac{e^{\frac{3}{2}\pi i s}}{-i-i} \Big) \;=\; \pi \cdot (e^{\frac{1}{2}\pi i s} - e^{\frac{3}{2}\pi i s})$$

That is,

$$\int_0^\infty \frac{t^s\, dt}{t^2+1} \;=\; \pi \cdot \frac{e^{\frac{1}{2}\pi i s} - e^{\frac{3}{2}\pi i s}}{1 - e^{2\pi i s}} \;=\; \pi \cdot \frac{e^{-\frac{1}{2}\pi i s} - e^{\frac{1}{2}\pi i s}}{e^{-\pi i s} - e^{\pi i s}} \;=\; \frac{\pi}{e^{\frac{1}{2}\pi i s} + e^{-\frac{1}{2}\pi i s}} \;=\; \frac{\pi/2}{\cos \frac{\pi s}{2}}$$

This is a charming and useful device, but a different secondary trick is applied to $\zeta(s)$:

## [1.3] Evaluation of $\zeta(-n)$

The first part of the Hankel contour discussion gives

$$\Gamma(s) \cdot \zeta(s) \;=\; \int_0^\infty \frac{t^{s-1}\, dt}{e^t - 1} \;=\; \frac{1}{1 - e^{2\pi i (s-1)}} \cdot \lim_{\varepsilon \to 0} \int_{H_\varepsilon} \frac{t^{s-1}\, dt}{e^t - 1} \;=\; \frac{1}{1 - e^{2\pi i s}} \cdot \lim_{\varepsilon \to 0} \int_{H_\varepsilon} \frac{t^{s-1}\, dt}{e^t - 1}$$

Rewrite this as

$$\zeta(s) \;=\; \frac{1}{\Gamma(s) \cdot (1 - e^{2\pi i s})} \cdot \lim_{\varepsilon \to 0} \int_{H_\varepsilon} \frac{t^{s-1}\, dt}{e^t - 1}$$

At $s = -n \in \{0, -1, -2, -3, -4, \ldots\}$ two fortunate things happen. First, the pole of $\Gamma(s)$ and the zero of $1 - e^{2\pi i s}$ cancel, giving a finite, computable value. Second, the function $t^{-n-1}$ is *single-valued*, so the inbound and outbound integrals of the Hankel contour simply *cancel* each other, *and* the integral over the small circle at 0 becomes $2\pi i$ times the residue of $\frac{t^{-n-1}}{e^t - 1}$ at 0.

The periodicity of $1 - e^{2\pi i s}$ assures that the leading (linear) term in the power series at any integer is the same as that at 0, namely,

$$1 - e^{2\pi i s} \;=\; 1 - \Big(1 + \frac{2\pi i s}{1!} + \frac{(2\pi i s)^2}{2!} + \ldots \Big) \;=\; -2\pi i s + \text{higher}$$

Grant for the moment that the residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$. Then

$$\zeta(-n) \;=\; \frac{1}{\frac{(-1)^n}{n!} \cdot (-2\pi i)} \cdot 2\pi i \cdot \text{Res}_{t=0} \frac{t^{-n-1}}{e^t - 1} \;=\; (-1)^{n+1} \cdot n! \cdot \text{Res}_{t=0} \frac{t^{-n-1}}{e^t - 1}$$

$$=\; (-1)^{n+1} \cdot n! \cdot \big(-1^{th} \text{ Laurent coefficient of } \tfrac{t^{-n-1}}{e^t - 1} \text{ at } t = 0\big)$$

$$=\; (-1)^{n+1} \cdot n! \cdot \big(n^{th} \text{ Laurent series coefficient of } \tfrac{1}{e^t - 1} \text{ at } t = 0\big)$$

The Laurent coefficients of $\frac{1}{e^t - 1}$ are more-or-less Bernoulli numbers. These are not completely elementary objects, but are certainly *rational*. Thus, $\zeta(-n) \in \mathbb{Q}$.

## [1.4] Vanishing $\zeta(-2) = \zeta(-4) = \ldots = 0$

A slightly finer analysis of the generating function $\frac{1}{e^t - 1}$ yields the vanishing of $\zeta(s)$ at negative even integers, as follows.

First, $\frac{1}{e^t - 1}$ is very close to being *odd* as a function of $t$:

$$\frac{1}{e^t - 1} + \frac{1}{e^{-t} - 1} \;=\; \frac{1}{e^t - 1} + \frac{e^t}{1 - e^t} \;=\; \frac{1}{e^t - 1} - \frac{e^t}{e^t - 1} \;=\; \frac{1 - e^t}{e^t - 1} \;=\; -1$$

Thus,

$$\Big(\frac{1}{e^t-1} + \tfrac{1}{2}\Big) + \Big(\frac{1}{e^{-t}-1} + \tfrac{1}{2}\Big) \;=\; 0$$

and $\frac{1}{e^t-1} + \frac{1}{2}$ is *odd*, so all its non-vanishing Laurent coefficients are odd-degree. Thus, for even $-2n < 0$,

$$\zeta(-2n) \;=\; (-1)^{2n+1}\,(2n)!\,(-2n^{th} \text{ Laurent coefficient of } \frac{1}{e^t-1}) \;=\; 0$$

## [1.5] Laurent expansion of $\frac{1}{e^t-1}$

We compute a few terms of the Laurent expansion near $t = 0$:

$$\frac{1}{e^t-1} \;=\; \frac{1}{(1+t+t^2/2+t^3/6+\ldots)-1} \;=\; \frac{1}{t+t^2/2+t^3/6+\ldots} \;=\; \frac{1}{t}\cdot\frac{1}{1+t/2+t^2/6+\ldots}$$

$$=\; \frac{1}{t}\cdot\Big(1-(t/2+t^2/6+\ldots)+(t/2+t^2/6+\ldots)^2-(t/2+t^2/6+\ldots)^3+\ldots\Big)$$

by expanding the geometric series for $\frac{1}{1+(t/2+\ldots)}$. Ignoring $t^4$ and higher-order terms,

$$\frac{t}{e^t-1} \;=\; 1-(\frac{t}{2}+\frac{t^2}{6}+\frac{t^3}{24})+\Big((\frac{t}{2})^2+2\cdot\frac{t}{2}\cdot\frac{t^2}{6}\Big)-(\frac{t}{2})^3+\ldots$$

$$=\; 1-\frac{1}{2}t+(-\frac{1}{6}+\frac{1}{4})t^2+(-\frac{1}{24}+\frac{1}{6}-\frac{1}{8})t^3+\ldots \;=\; 1-\frac{1}{2}t+\frac{1}{12}t^2+0\cdot t^3+\ldots$$

That is,

$$\frac{1}{e^t-1} \;=\; \frac{1}{t}-\frac{1}{2}+\frac{1}{12}t+0\cdot t^2+\ldots$$

## [1.6] Residues of $\Gamma(s)$

Finally, we determine the residues of $\Gamma(s)$. Certainly

$$\Gamma(1) \;=\; \int_0^\infty t^1\,e^{-t}\,\frac{dt}{t} \;=\; \int_0^\infty e^{-t}\,dt \;=\; 1$$

From the functional equation $s\Gamma(s) = \Gamma(s+1)$, near $s = 0$

$$\Gamma(s) \;=\; \frac{\Gamma(s+1)}{s} \;=\; \frac{1+\text{higher}}{s} \;=\; \frac{1}{s}+(\text{holomorphic at } s=0)$$

Thus, the residue at 0 is 1. Iterating the functional equation,

$$\Gamma(s) \;=\; \frac{\Gamma(s+1)}{s} \;=\; \frac{\Gamma(s+2)}{(s+1)s} \;=\; \frac{\Gamma(s+3)}{(s+2)(s+1)s} \;=\; \ldots \;=\; \frac{\Gamma(s+n+1)}{(s+n)(s+n-1)\ldots(s+2)(s+1)s}$$

Thus, the leading Laurent term at $s = -n$ is

$$\frac{1}{s+n}\;\cdot\;\frac{\Gamma(s+n+1)}{(s+n-1)\ldots(s+2)(s+1)s}\Big|_{s=-n} \;=\; \frac{1}{s+n}\;\cdot\;\frac{\Gamma(-n+n+1)}{(-n+n-1)\ldots(-n+2)(-n+1)(-n)}$$

$$=\; \frac{1}{s+n}\;\cdot\;\frac{1}{(-1)(-2)(-3)\ldots(-n+2)(-n+1)(-n)} \;=\; \frac{1}{s+n}\;\cdot\;\frac{(-1)^n}{n!}$$

That is, the residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$ as claimed.

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## Bibliography

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[Riemann 1859] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monats. Akad. Berlin (1859), 671-680.

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