(*November 30, 2015*)

A rationality principle

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/
[This document is http://www.math.umn.edu/˜garrett/m/mfms/notes_2015-16/rationality_principle.pdf]

The following rationality principle was used by H. Klingen, *Über die Werte der Dedekindschen Zetafunktionen*, Math. Ann. **145** (1961/2), 265-272, to study special values of L-function on totally real number fields.

[0.0.1] Theorem: The constant term c_o in the Fourier expansion $f(z) = \sum_{n=0}^\infty c_n\, e^{2\pi i n z}$ of a holomorphic elliptic modular form $f(z)$ lies in the field $\mathbb{Q}(c_1, c_2, \ldots)$ generated over $\mathbb{Q}$ by the higher Fourier coefficients.

Proof: In fact, we prove that the map $\sum_n c_n\, e^{2\pi i n z} \to c_o$ is a finite $\mathbb{Q}$-linear combination of maps $\sum_n c_n\, e^{2\pi i n z} \to c_n$ with $n > 0$, depending on the weight $2k$.

[0.0.2] Lemma: Let $K \subset L$ be fields, V a finite-dimensional L-vectorspace, and $\Lambda \subset \mathrm{Hom}_L(V, L)$. Suppose that Λ spans $\mathrm{Hom}_L(V, L)$ over L, and that $V_o = \{v \in V : \lambda v \in K,\ \forall \lambda \in \Lambda\}$ spans V over L. For any L-basis $B \subset \Lambda$ of $\mathrm{Hom}_L(V, L)$, every $\lambda \in \Lambda$ is a K-linear (not merely L-linear) combination of elements of B.

Proof: Let $\{v_\beta : \beta \in B\}$ be the dual L-basis to B, namely, $\beta(v_\beta) = 1$ and $\beta'(v_\beta) = 0$ for $\beta' \neq \beta$. Since $\beta(V_o) \subset K$,

$$V_o \ \subset\ K\text{-span of } \{v_\beta : \beta \in B\}$$

If V_o failed to contain some v_β, then the K-dimension of V_o would be less than the L-dimension of V, so V_o could not span V over L. Thus, V_o is exactly the K-span of $\{v_\beta : \beta \in B\}$. Write $\lambda \in \Lambda$ as a L-linear combination of elements of B: $\lambda = \sum_{\beta \in B} c_\beta\, \beta$, with $c_\beta \in L$. Since $v_{\beta'} \in V_o$,

$$c_{\beta'} \ =\ \sum_{\beta \in B} c_\beta\ \beta(v_{\beta'}) \ =\ \Big(\sum_{\beta \in B} c_\beta\ \beta\Big)(v_{\beta'}) \ =\ \lambda(v_{\beta'}) \ \in\ K$$

That is, λ is a K-linear combination of the functionals in B, as claimed. ///

[0.0.3] Claim: The space of holomorphic modular forms of level one, of a fixed weight $2k$, have a $\mathbb{C}$-basis consisting of modular forms with *rational* Fourier coefficients.

Proof: First, we prove that monomials $E_4^a E_6^b$ with $4a + 6b = 2k$ (and non-negative integers a, b) *span* the space of weight $2k$ modular forms of level one. Every $2k \geq 4$ has at least one expression $2k = 4a + 6b$, and there are no non-zero modular forms of weight 2, and only constants of weight 0. For weights $4, 6, 8, 10$, the divisor relation shows that $E_4, E_6, E_4^2, E4E_6$ span these spaces.

Suppose $2k \geq 12$. Note that $\Delta = E_4^3 - E_6^2$ is not identically 0, and is a nowhere-vanishing (except at $i\infty$) cuspform. Given $f(z) = \sum_n c_n\, e^{2\pi i n z}$, and any a, b with $2k = 4a + 6b$, $f - c_o \cdot E_4^a E_6^b$ is a cuspform, and

$$\frac{f - c_o \cdot E_4^a E_6^b}{\Delta}$$

is of weight $2k - 12$, so induction gives the result. The Eisenstein series E_4 and E_6 have rational Fourier coefficients, so $E_4^a E_6^b$ has rational Fourier coefficients, for all non-negative integers a, b. Thus, each space of weight $2k$ modular forms is spanned by modular forms with rational Fourier coefficients. ///

Now we prove the theorem. In the lemma, let V be the space of modular forms of weight $2k > 0$, and Λ the collection of all functionals $\lambda_n : \sum_n c_n\, e^{2\pi i n z} \to c_n$. Certainly Λ spans the dual space $\mathrm{Hom}_\mathbb{C}(V, \mathbb{C})$, and the claim shows that the space V_o of modular forms of weight $2k$ with rational coefficients spans the whole space over $\mathbb{C}$. For weight $2k > 0$, the simultaneous kernel of all λ_n with $n > 0$ consists of constants, and the only constant modular form of positive weight is 0. Thus, we can choose a basis $B \subset \Lambda$ for $\mathrm{Hom}_\mathbb{C}(V, \mathbb{C})$ from among functionals λ_n with $n > 0$. Thus, λ_0 is a $\mathbb{Q}$-linear combination of the λ_n's with $n > 0$. ///
