Reproducing kernels, Bergman kernels, Poisson kernels

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The identity map $1_V$ on a finite-dimensional inner-product vector space $V$ is expressible in terms of any orthonormal basis $\{e_i\}$ as

$$v = 1_V(v) = \sum_i \langle v, e_i \rangle \cdot e_i \quad \text{(for } v \in V)$$

That is, the matrix of the identity map with respect to any basis $\{e_i\}$ is simply the identity matrix:

$$1_V = \sum_i \langle - , e_i \rangle \otimes e_i \in V^* \otimes V$$

One infinite-dimensional analogue involves a Hilbert space $V$ of functions on a measure space $Z$. The identity matrix is replaced by the reproducing kernel $K(z, w)$: this should be a function of some kind on $Z \times Z$ giving the identity map on $V$:

$$f(z) = \int_Z K(z, w) \cdot f(w) \, dw \quad \text{(for } f \in V)$$

With orthonormal basis $\{f_n\}$ for $V$, apparently

$$K(z, w) = \sum_n f_n(z) \cdot \overline{f_n(w)}$$

It is completely unclear in what sense this converges, if any. For example, square integrable functions $L^2(\mathbb{T})$ on the circle $\mathbb{T}$ have orthonormal basis $e^{in\theta} \to e^{in\theta}/\sqrt{2\pi}$, but the corresponding expression

$$K(\theta, \theta') = \sum_i e^{in\theta} \cdot e^{-in\theta'}$$

does not converge pointwise for any $\theta, \theta' \in \mathbb{R}$. It does converge distributionally, in fact in a Sobolev space, but that is another story, about Schwartz kernel functions for a great variety of operators. Here, we are interested in situations where a reproducing kernel does converge pointwise in a more elementary sense.

1. Bergman kernel on the unit disk

Let $D$ be the unit disk in $\mathbb{C}$, and $V$ the closure in $L^2(D)$ of the vector space of polynomial functions. This is a Hilbert subspace of $L^2(D)$. The inner product of two holomorphic monomials $z^m$ and $z^n$ is

$$\langle z^m, z^n \rangle = \int_D z^m \cdot \overline{z^n} \, dx \, dy = \int_0^1 \int_0^{2\pi} (re^{i\theta})^m \cdot \overline{(re^{-i\theta})^n} \, d\theta \, r \, dr = \int_0^1 \int_0^{2\pi} r^{m+n+1} \cdot e^{i\theta(m-n)} \, d\theta \, dr$$

$$= \begin{cases} 0 & \text{for } m \neq n \\ \frac{\pi}{n+1} & \text{for } m = n \end{cases}$$
Thus, the reproducing kernel here, the **Bergman kernel**, should be

\[
K(z, w) = \frac{1}{\pi} \sum_{n \geq 0} (n + 1) \cdot z^n \cdot \bar{w}^n \quad \text{(for } |z| < 1 \text{ and } |w| < 1)\]

We can sum the series:

\[
\sum_{n \geq 0} (n + 1) \cdot z^n \cdot \bar{w}^n = \frac{1}{\bar{w}} \frac{\partial}{\partial z} \left( \sum_{n \geq 0} (z \bar{w})^n \right) = \frac{1}{\bar{w}} \frac{\partial}{\partial z} \frac{1}{1 - z \bar{w}} = \frac{1}{\bar{w}} \frac{\bar{w}}{(1 - z \bar{w})^2} = \frac{1}{(1 - z \bar{w})^2}
\]

and the convergence is uniform on compacts. Thus, the Bergman kernel for the disk is

\[
K(z, w) = \frac{1}{\pi} \frac{1}{(1 - z \bar{w})^2} \quad \text{(for } |z| < 1 \text{ and } |w| < 1)\]

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### 2. Poisson kernel on the unit disk

With the usual Euclidean Laplacian \(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\), a function \(f\) on the disk \(D\) is **harmonic** when \(\Delta f = 0\). Granting standard properties of Gelfand-Pettis integrals, the integral

\[
f_n(z) = \int_0^{2\pi} e^{-in\theta} f(e^{i\theta} z) \, d\theta
\]

is still harmonic, because the differential operator passes inside the integral in the obviously plausible fashion:

\[
\Delta f_n(z) = \Delta \int_0^{2\pi} e^{-in\theta} f(e^{i\theta} z) \, d\theta = \int_0^{2\pi} e^{-in\theta} \Delta f(e^{i\theta} z) \, d\theta = \int_0^{2\pi} e^{-in\theta} \cdot 0 \, d\theta = 0
\]

By design, this integral picks out the part of \(f\) equivariant by \(e^{i\theta} \rightarrow e^{in\theta}\): by a change of variables, replacing \(\theta'\) by \(\theta' - \theta\),

\[
f_n(e^{i\theta} z) = \int_0^{2\pi} e^{-in\theta'} f(e^{i\theta'} \cdot e^{i\theta} z) \, d\theta' = e^{in\theta} \cdot \int_0^{2\pi} e^{-in\theta'} f(e^{i\theta'} z) \, d\theta' = e^{in\theta} \cdot f_n(z)
\]

Thus, \(f_n(e^{i\theta} r) = e^{in\theta} \varphi_n(r)\) for a function \(\varphi_n\) of radius alone. Writing \(f(e^{i\theta} r) = F(r, \theta)\), the Laplacian in radial coordinates is \(F_{rr} + \frac{1}{r} F_r + \frac{1}{r^2} F_{\theta\theta}\). The differential equation satisfied by \(\varphi_n\) is obtained by separation of variables:

\[
0 = \Delta (e^{in\theta} \varphi_n(r)) = (\varphi_n'' + \frac{1}{r} \varphi_n' + \frac{1}{r^2} (in)^2 \varphi_n) \cdot e^{in\theta}
\]

so

\[
\varphi_n'' + \frac{1}{r} \varphi_n' + \frac{n^2}{r^2} \varphi_n = 0
\]

This is an Euler-type equation, meaning of the form \(u'' + \frac{1}{r} u' + \frac{\lambda}{r^2} u = 0\). Solutions are of the form \(u(r) = r^\lambda\) for \(\lambda\) a solution of the indicial equation \(\lambda(\lambda - 1) + b\lambda + c = 0\), and also \(r^\lambda \log r\) for a double root \(\lambda\). For \(n \neq 0\), the solutions are \(r \rightarrow r^{\pm|n|}\), but \(r^{-|n|}\) blows up at \(r \rightarrow 0^+\), so this solution cannot appear in a component \(f_n(r e^{i\theta}) = e^{in\theta} \varphi_n(r)\) of \(f\) continuous on the disk. Likewise, the second solution \(\log r\) for \(n = 0\) cannot occur. Thus,

\[
f(r e^{i\theta}) = \sum_{n \in \mathbb{Z}} c_n r^{|n|} e^{in\theta} = \sum_{n \geq 0} c_n z^n + \sum_{n < 0} c_n \bar{z}^{|n|}
\]

The boundary value of \(\sum_{n \geq 0} c_n z^n + \sum_{n < 0} c_n \bar{z}^{|n|}\) appears to be the Fourier series \(\sum_n c_n e^{in\theta}\). The not-classically-convergent reproducing kernel on the circle

\[
K_1(e^{i\theta}, e^{i\theta'}) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\theta} e^{-in\theta'}
\]

(convergent in a Sobolev space, not pointwise)
Letting $f : \mathbb{C} \to \mathbb{C}$, the Poisson kernel $P(z) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} r^n e^{in\theta}$ with $0 < r < 1$, and we obtain the Poisson kernel:

$$K(re^{i\theta}, e^{i\theta'}) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} r^n e^{in\theta} e^{-in\theta'} = \frac{1}{2\pi} \left( \frac{1}{1 - re^{i\theta} \cdot e^{-i\theta'}} + \frac{re^{-i\theta} \cdot e^{i\theta'}}{1 - re^{-i\theta} \cdot e^{i\theta'}} \right)$$

$$= \frac{1}{2\pi} \left( \frac{1 - re^{-i\theta}}{1 - re^{i\theta}} \cdot e^{i\theta'} + re^{-i\theta} \cdot e^{i\theta'} \right) \frac{1 - r^2}{(1 - re^{i\theta}) (1 - re^{-i\theta})} = \frac{1}{2\pi} \frac{1 - |z|^2}{(1 - z \cdot e^{-i\theta})(1 - z \cdot e^{i\theta})}$$

That is, certainly for smooth functions $f$ on the circle, and even for distributions $f$ on the circle,

$$z \mapsto \int_0^{2\pi} K(z, e^{i\theta}) f(e^{i\theta'}) d\theta'$$

is a harmonic function on the open disk with boundary values (in a range of possible senses) $f$.

### 3. Bergman kernel on the unit ball in $\mathbb{C}^n$

Let $B$ be the unit ball in $\mathbb{C}^n$. Because $B$ is stable under coordinate-wise rotations

$$(z_1, \ldots, z_n) \mapsto (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$$

the hermitian inner products of monomials (in multi-index notation)

$$\int_B z^\alpha \cdot \overline{z}^\beta = \int_B \left( (z_1^{\alpha_1} \cdots z_n^{\alpha_n}) \cdot (\overline{z}_1^{\beta_1} \cdots \overline{z}_n^{\beta_n}) \right)$$

must vanish unless $\alpha = \beta$: changing variables,

$$\int_B z^\alpha \cdot \overline{z}^\beta = e^{i(\alpha_1 - \beta_1)\theta_1} \cdots e^{i(\alpha_n - \beta_n)\theta_n} \int_B z^\alpha \cdot \overline{z}^\beta$$

That is, unless $\alpha_j = \beta_j$ for every $j$, some choice of angles $\theta_j$ will make the factor on the right-hand side 0, so the integral must be 0. With $\alpha = \beta$,

$$\int_B z^\alpha \cdot \overline{z}^\alpha = \int_B |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} = \int_0^1 r^{2\alpha_1 + \cdots + 2\alpha_n + (2n - 1)} dr \int_{S^{2n-1}} \frac{|z_1|^{2\alpha_1}}{r} \cdots \frac{|z_n|^{2\alpha_n}}{r}$$

$$= \frac{1}{2\alpha_1 + \cdots + 2\alpha_n + 2n} \int_{S^{2n-1}} \frac{|z_1|^{2\alpha_1}}{r} \cdots \frac{|z_n|^{2\alpha_n}}{r}$$

To integrate a homogeneous degree-zero function $f$ over a sphere, integrate $f(x/|x|) \cdot \varphi(|x|)$ over $\mathbb{C}^n$ for any non-negative $\varphi$ on $(0, \infty)$, and divide by the integral of $\varphi(|x|)$ over $\mathbb{C}^n$. Uniqueness of the invariant integral up to scalars assures that this is independent of the auxiliary function $\varphi$. Given $\alpha$, take $\varphi(r) = r^{2\alpha_1 + \cdots + 2\alpha_n} e^{-\pi r^2}$. Letting $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$, the normalizing constant is

$$\int_{\mathbb{C}^n} |z|^{2\alpha_1 + \cdots + 2\alpha_n} e^{-\pi |z|^2} = |S^{2n-1}| \int_0^\infty r^{2\alpha_1 + \cdots + 2\alpha_n + 2n} e^{-\pi r^2} dr$$

$$= \frac{1}{2} |S^{2n-1}| \int_0^\infty r^{\alpha_1 + \cdots + \alpha_n + n} e^{-\pi r^2} dr = \pi^{-(\alpha_1 + \cdots + \alpha_n + n)} \cdot \frac{1}{2} |S^{2n-1}| \cdot \Gamma(\alpha_1 + \cdots + \alpha_n + n)$$
and
\[
\int_{\mathbb{C}} |z_j|^{2m} e^{-\pi |z|^2} = 2\pi \int_0^\infty r^{2\alpha_j+2} e^{-\pi r^2} \frac{dr}{r} = \pi \int_0^\infty r^{\alpha_j+1} e^{-\pi r} \frac{dr}{r} = \alpha_j \cdot \Gamma(\alpha_j + 1)
\]

Thus,
\[
\int_{\mathbb{C}} |z_j|^{2m} e^{-\pi |z|^2} = \frac{\alpha_1 \cdot \Gamma(\alpha_1 + 1) \ldots \alpha_n \cdot \Gamma(\alpha_n + 1)}{\pi^{-\alpha_1 + \ldots + \alpha_n + n} \cdot \left|S^{2n-1}\right| \cdot \Gamma(\alpha_1 + \ldots + \alpha_n + n)}
\]

and
\[
\int_B z^\alpha \cdot z^n = \frac{1}{2\alpha_1 + \ldots + 2\alpha_n + 2n} \int_{S^{2n-1}} |z_1|^{2\alpha_1} \ldots |z_n|^{2\alpha_n} = \frac{\pi^n \cdot \Gamma(\alpha_1 + 1) \ldots \Gamma(\alpha_n + 1)}{\left|S^{2n-1}\right| \cdot \Gamma(\alpha_1 + \ldots + \alpha_n + n + 1)}
\]

The ratio of Gamma values is almost the inverse of a multinomial coefficient: letting $|\alpha| = \alpha_1 + \ldots + \alpha_n$,
\[
\frac{\Gamma(|\alpha| + n + 1)}{\Gamma(\alpha_1 + \ldots + \alpha_n + n + 1)} = \frac{|\alpha| + n)!}{\alpha_1! \ldots \alpha_n!} = \left(\frac{|\alpha|}{\alpha_1 \ldots \alpha_n}\right) \cdot (|\alpha| + 1) \ldots (|\alpha| + n)
\]

and
\[
(|\alpha| + 1) \ldots (|\alpha| + n) = \left(\frac{\partial}{\partial t}\right)^{|\alpha|+n} t^{|\alpha|+n}
\]

Thus, up to a constant, the reproducing kernel $K(z, w)$ is
\[
\left(\frac{\partial}{\partial t}\right)^n t^n \sum_{\alpha} \binom{\alpha_1 + \ldots + \alpha_n + n}{\alpha_1 \ldots \alpha_n} z^\alpha \overline{w}^\alpha t^{\alpha} = \left(\frac{\partial}{\partial t}\right)^n t^n \sum_{N=0}^{N} \sum_{|\alpha|=N} \binom{N}{\alpha_1 \ldots \alpha_n} z^\alpha \overline{w}^\alpha
\]

\[
= \left(\frac{\partial}{\partial t}\right)^n t^n \sum_{N=0}^{N} \sum_{|\alpha|=N} \binom{N}{\alpha_1 \ldots \alpha_n} z^\alpha \overline{w}^\alpha = (z \cdot \overline{w})^{-n} \sum_{t=0}^{N} (t \cdot \overline{w})^t = (z \cdot \overline{w})^{-n} \cdot \frac{(n-1)! \cdot (z \cdot \overline{w})^n}{(1 - z \cdot \overline{w})^{n+1}} = \frac{(n-1)!}{(1 - z \cdot \overline{w})^{n+1}}
\]

Rather than organizing all the constants above, granting that $K(z, w) = c/(1 - z \cdot \overline{w})^{n+1}$ for some constant $c$, the constant can be determined by integrating against the function $1$ and evaluating at $0$:
\[
1 = \int_B K(0, w) \cdot 1 = \int_B c \cdot 1 = \text{vol}(B)
\]

so $c = 1 / \text{vol}(B)$, and the Bergman kernel for the complex $n$-ball $B$ is
\[
K(z, w) = \frac{1}{\text{vol}(B)} \cdot \frac{1}{(1 - z \cdot \overline{w})^{n+1}}
\]
4. Reproducing kernel for holomorphic cuspforms

Recall that a holomorphic cuspform $f$ on the upper half-plane $\mathcal{H}$ for $\Gamma = SL_2(\mathbb{Z})$ of weight $2k$ is a holomorphic function with the automorphy property

$$f(\gamma z) = (cz+d)^{2k} \cdot f(z) \quad \text{(for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\text{)}$$

and a decay constraint at $i\infty$ describable by requiring that its Fourier expansion has no non-positive-index terms:

$$f(x+iy) = \sum_{n \geq 1} c_n e^{2\pi in(x+iy)}$$

and that the coefficients are of polynomial growth. The Petersson inner product on holomorphic cuspforms of weight $2k$ for $\Gamma$ is

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathcal{H}} f(z) \cdot \overline{g(z)} \cdot y^{2k} \frac{dx \, dy}{y^2}$$

It is completely unreasonable to expect to explicitly identify an orthonormal basis for all these spaces of cuspforms. Instead, we anticipate that the reproducing kernel is obtained by winding up a suitable holomorphic function $\varphi(z, w)$ on $\mathcal{H} \times \mathcal{H}$:

$$K(z, w) = \sum_\gamma \varphi(\gamma z, w) \cdot \frac{1}{(cz+d)^{2k}} \quad \text{(summed over } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\text{)}$$

Presumably $k(z, w)$ is holomorphic in $z$ and anti-holomorphic in $w$. More importantly, we want $k(z, w)$ so that the winding-up creates $K(z, w)$ a holomorphic cuspform in $z$ and anti-holomorphic cuspform in $w$. Thus, for $\gamma' \in \Gamma$, we would want

$$K(z, \gamma' w) = (c'w + d')^{2k} \cdot K(z, w) \quad \text{(with } \gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma\text{)}$$

After some experimentation, a plausible choice is

$$\varphi(z, w) = \frac{1}{(z-w)^{2k}}$$

and then

$$K(z, w) = \sum_\gamma \frac{1}{(\gamma z - \bar{w})^{2k}} \cdot \frac{1}{(cz+d)^{2k}} = \sum_\gamma \frac{1}{(\frac{az+b}{cz+d} - \bar{w})^{2k}} \cdot \frac{1}{(cz+d)^{2k}}$$

$$= \sum_\gamma \frac{1}{(az+b - c\bar{w} - dw)^{2k}} = \sum_\gamma \frac{1}{(1 - \bar{w})} \left( \frac{a}{c} \frac{b}{d} \frac{z}{\bar{w}} \right)^{2k}$$

demonstrating the symmetry of $K(z, w)$ in $z$ and $\bar{w}$. For a holomorphic cuspform $f$, with $w = u + iv$, unwinding gives

$$\int_{\Gamma \backslash \mathcal{H}} K(z, w) \ f(w) \ \frac{v^{2k}}{v^2} \ du \ dv = \int_{\Gamma \backslash \mathcal{H}} \sum_\gamma \varphi(\gamma z, w) \ f(\gamma w) \ \text{Im}(\gamma w)^{2k} \ \frac{du \ dv}{v^2} = \int_{\mathcal{H}} \varphi(z, w) \ f(w) \ \frac{v^{2k}}{v^2} \ du \ dv$$

$$= \int_{\mathcal{H}} \frac{1}{(z-w)^{2k}} \left( \sum_n c_n e^{2\pi inw} \right) \ \frac{v^{2k}}{v^2} \ du \ dv = \sum_n c_n \int_{\mathcal{H}} \frac{1}{(z-w)^{2k}} \ e^{2\pi inw} \ \frac{v^{2k}}{v^2} \ du \ dv$$
The integral is
\[ \int_B \frac{1}{(z - w)^{2k}} e^{2\pi i n w} v^{2k} \frac{du dv}{v^2} = \int_0^{\infty} \int_{\mathbb{R}} \frac{1}{((x + iy) - (u - iv))^{2k}} e^{2\pi i n(u + iv)} v^{2k} \frac{du dv}{v^2} \]

Replacing \( u \) by \( u + v \) makes this into
\[ e^{2\pi i n x} \int_0^{\infty} \int_{\mathbb{R}} \frac{1}{(iy - (u - iv))^{2k}} e^{2\pi i n(u + iv)} v^{2k} \frac{du dv}{v^2} \]

The inner integral is evaluated by residues:
\[ \int_{\mathbb{R}} \frac{1}{(u - i(v + y))^{2k}} e^{2\pi i n u} du = 2\pi i \cdot \frac{(2\pi i)^{2k-1}}{(2k - 1)!} \cdot e^{2\pi i n(i(v + y))} = \frac{(2\pi i)^{2k} \cdot n^{2k-1}}{\Gamma(2k)} \cdot e^{-2\pi n(v + y)} \]

so the whole integral is
\[ \int_0^{\infty} e^{-2\pi n v} v^{2k-1} \cdot \frac{(2\pi i)^{2k} \cdot n^{2k-1}}{\Gamma(2k)} \cdot e^{-2\pi n(v + y)} \frac{dv}{v} = \frac{(2\pi i)^{2k}}{\Gamma(2k)} n^{2k-1} e^{-2\pi n y} \int_0^{\infty} e^{-4\pi n v} v^{2k-1} \frac{dv}{v} \]
\[ = \frac{(2\pi i)^{2k}}{\Gamma(2k)} n^{2k-1} e^{-2\pi n y} \cdot (4\pi n)^{1-2k} \cdot \Gamma(2k - 1) = (2i)^{2k} 4^{1-2k} \pi \cdot e^{-2\pi n y} \cdot (2k - 1) \]
\[ = (-1)^k 2^{2-2k} e^{-2\pi n y} \cdot (2k - 1) \]

Thus,
\[ \int_B \frac{1}{(z - w)^{2k}} e^{2\pi i n w} v^{2k} \frac{du dv}{v^2} = (-1)^k 2^{2-2k} e^{-2\pi n y} \cdot (2k - 1) \cdot e^{2\pi i n z} \]

and we have the desired reproducing property
\[ \int_{\Gamma \setminus B} K(z, w) f(w) v^{2k} \frac{du dv}{v^2} = (-1)^k 2^{2-2k} \cdot (2k - 1) \cdot \sum_n c_n e^{2\pi i n z} = (-1)^k 2^{2-2k} \cdot (2k - 1) \cdot f(z) \]