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Reproducing kernels, Bergman kernels, Poisson kernels

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The identity map 1_V on a finite-dimensional inner-product vector space V is expressible in terms of any orthonormal basis $\{e_i\}$ as

$$v \;=\; 1_V(v) \;=\; \sum_i \langle v, e_i\rangle \cdot e_i \qquad\qquad (\text{for } v \in V)$$

That is, the matrix of the identity map with respect to any basis $\{e_i\}$ is simply the identity matrix:

$$1_V \;=\; \sum_i \langle -, e_i\rangle \otimes e_i \;\in\; V^* \otimes V$$

One infinite-dimensional analogue involves a Hilbert space V of functions on a measure space Z. The identity *matrix* is replaced by the *reproducing kernel* $K(z,w)$: this should be a function of some kind on $Z \times Z$ giving the identity map on V:

$$f(z) \;=\; \int_Z K(z,w)\cdot f(w)\, dw \qquad\qquad (\text{for } f \in V)$$

With orthonormal basis $\{f_n\}$ for V, apparently

$$K(z,w) \;=\; \sum_n f_n(z)\cdot \overline{f_n(w)}$$

It is completely unclear in what sense this converges, if any. For example, square integrable functions $L^2(\mathbb{T})$ on the circle $\mathbb{T}$ have orthonormal basis $e^{i\theta} \to e^{in\theta}/\sqrt{2\pi}$, but the corresponding expression

$$K(\theta,\theta') \;=\; \sum_i e^{in\theta}\cdot e^{-in\theta'}$$

does not converge *pointwise* for any $\theta, \theta' \in \mathbb{R}$. It does converge *distributionally*, in fact in a Sobolev space, but that is another story, about *Schwartz kernel functions* for a great variety of operators. Here, we are interested in situations where a reproducing kernel does converge pointwise in a more elementary sense.

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1. *Bergman kernel on the unit disk*

Let D be the unit disk in $\mathbb{C}$, and V the closure in $L^2(D)$ of the vector space of polynomial functions. This is a Hilbert subspace of $L^2(D)$. The inner product of two holomorphic monomials z^m and z^n is

$$\langle z^m, z^n\rangle \;=\; \int_D z^m\cdot \overline{z}^m\, dx\, dy \;=\; \int_0^1\int_0^{2\pi} (re^{i\theta})^m\cdot(re^{-i\theta})^n\, d\theta\, r\, dr \;=\; \int_0^1\int_0^{2\pi} r^{m+n+1}\cdot e^{i\theta(m-n)}\, d\theta\, dr$$

$$= \begin{cases} 0 & (\text{for } m \neq n) \\ \dfrac{\pi}{n+1} & (\text{for } m = n)\end{cases}$$

Thus, the reproducing kernel here, the *Bergman kernel*, should be

$$K(z,w) \;=\; \frac{1}{\pi}\sum_{n\geq 0}(n+1)\cdot z^n\cdot \overline{w}^n \qquad\qquad (\text{for } |z|<1 \text{ and } |w|<1)$$

We can sum the series:

$$\sum_{n\geq 0}(n+1)\cdot z^n\cdot \overline{w}^n \;=\; \frac{1}{\overline{w}}\frac{\partial}{\partial z}\Big(\sum_{n\geq 0}(z\overline{w})^n\Big) \;=\; \frac{1}{\overline{w}}\frac{\partial}{\partial z}\frac{1}{1-z\overline{w}} \;=\; \frac{1}{\overline{w}}\frac{\overline{w}}{(1-z\overline{w})^2} \;=\; \frac{1}{(1-z\overline{w})^2}$$

and the convergence is uniform on compacts. Thus, the Bergman kernel for the disk is

$$K(z,w) \;=\; \frac{1}{\pi}\frac{1}{(1-z\overline{w})^2} \qquad\qquad (\text{for } |z|<1 \text{ and } |w|<1)$$

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2. *Poisson kernel on the unit disk*

With the usual Euclidean Laplacian $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, a function f on the disk D is *harmonic* when $\Delta f = 0$. Granting standard properties of Gelfand-Pettis integrals, the integral

$$f_n(z) \;=\; \int_0^{2\pi} e^{-in\theta}\, f(e^{i\theta}z)\, d\theta$$

is still harmonic, because the differential operator passes inside the integral in the obviously plausible fashion:

$$\Delta f_n(z) \;=\; \Delta\int_0^{2\pi} e^{-in\theta}\, f(e^{i\theta}z)\, d\theta \;=\; \int_0^{2\pi} e^{-in\theta}\, \Delta_z f(e^{i\theta}z)\, d\theta \;=\; \int_0^{2\pi} e^{-in\theta}\cdot 0\, d\theta \;=\; 0$$

By design, this integral picks out the part of f equivariant by $e^{i\theta}\to e^{in\theta}$: by a change of variables, replacing θ' by $\theta'-\theta$,

$$f_n(e^{i\theta}z) \;=\; \int_0^{2\pi} e^{-in\theta'}\, f(e^{i\theta'}\cdot e^{i\theta}z)\, d\theta' \;=\; e^{in\theta}\cdot\int_0^{2\pi} e^{-in\theta'}\, f(e^{i\theta'}z)\, d\theta' \;=\; e^{in\theta}\cdot f_n(z)$$

Thus, $f_n(e^{i\theta}r) = e^{in\theta}\varphi_n(r)$ for a function φ_n of radius alone. Writing $f(e^{i\theta}r) = F(r,\theta)$, the Laplacian in radial coordinates is $F_{rr}+\frac{1}{r}F_r+\frac{1}{r^2}F_{\theta\theta}$. The differential equation satisfied by φ_n is obtained by separation of variables:

$$0 \;=\; \Delta\big(e^{in\theta}\varphi_n(r)\big) \;=\; \big(\varphi_n''+\frac{1}{r}\varphi_n'+\frac{1}{r^2}(in)^2\varphi_n\big)\cdot e^{in\theta}$$

so

$$\varphi_n''+\frac{1}{r}\varphi_n'-\frac{n^2}{r^2}\varphi_n \;=\; 0$$

This is an Euler-type equation, meaning of the form $u''+\frac{b}{r}u'+\frac{c}{r^2}u=0$. Solutions are of the form $u(r)=r^\lambda$ for λ a solution of the indicial equation $\lambda(\lambda-1)+b\lambda+c=0$, and also $r^\lambda\log r$ for a double root λ. For $n\neq 0$, the solutions are $r\to r^{\pm|n|}$, but $r^{-|n|}$ blows up at $r\to 0^+$, so this solution cannot appear in a component $f_n(re^{i\theta}) = e^{in\theta}\varphi_n(r)$ of f continuous on the disk. Likewise, the second solution $\log r$ for $n=0$ cannot occur. Thus,

$$f(re^{i\theta}) \;=\; \sum_{n\in\mathbb{Z}} c_n\, r^{|n|}\, e^{in\theta} \;=\; \sum_{n\geq 0} c_n\, z^n + \sum_{n<0} c_n\, \overline{z}^{|n|}$$

The boundary value of $\sum_{n\geq 0} c_n\, z^n + \sum_{n<0} c_n\, \overline{z}^{|n|}$ appears to be the Fourier series $\sum_n c_n\, e^{in\theta}$. The not-classically-convergent reproducing kernel on the circle

$$K_1(e^{i\theta},e^{i\theta'}) \;=\; \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{in\theta}\, e^{-in\theta'} \qquad\qquad (\text{convergent in a Sobolev space, not pointwise})$$

becomes classically convergent upon replacing $e^{in\theta}$ by $r^{|n|}e^{in\theta}$ with $0 \le r < 1$, and we obtain the Poisson kernel:

$$K(re^{i\theta}, e^{i\theta'}) \;=\; \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} r^{|n|}\, e^{in\theta}\, e^{-in\theta'} \;=\; \frac{1}{2\pi}\Big(\frac{1}{1-re^{i\theta}\cdot e^{-i\theta'}} + \frac{re^{-i\theta}\cdot e^{i\theta'}}{1-re^{-i\theta}\cdot e^{i\theta'}}\Big)$$

$$= \frac{1}{2\pi}\frac{(1-re^{-i\theta}\cdot e^{i\theta'}) + re^{-i\theta}\cdot e^{i\theta'}(1-re^{i\theta}\cdot e^{-i\theta'})}{(1-re^{i\theta}\cdot e^{-i\theta'})(1-re^{-i\theta}\cdot e^{i\theta'})} \;=\; \frac{1}{2\pi}\frac{1-r^2}{(1-re^{i\theta}\cdot e^{-i\theta'})(1-re^{-i\theta}\cdot e^{i\theta'})}$$

$$= \frac{1}{2\pi}\frac{1-|z|^2}{(1-z\cdot e^{-i\theta'})(1-\overline{z}\cdot e^{i\theta'})} \;=\; \frac{1}{2\pi}\frac{1-|z|^2}{|1-z\cdot e^{-i\theta'}|^2}$$

That is, certainly for smooth functions f on the circle, and even for *distributions* f on the circle,

$$z \longrightarrow \int_0^{2\pi} K(z, e^{i\theta'})\, f(e^{i\theta'})\, d\theta'$$

is a harmonic function on the open disk with boundary values (in a range of possible senses) f.

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3. *Bergman kernel on the unit ball in* $\mathbb{C}^n$

Let B be the unit ball in $\mathbb{C}^n$. Because B is stable under coordinate-wise rotations

$$(z_1,\ldots,z_n) \longrightarrow (e^{i\theta_1}z_1,\ldots,e^{i\theta_1}z_n)$$

the hermitian inner products of monomials (in multi-index notation)

$$\int_B z^\alpha\cdot\overline{z}^\beta \;=\; \int_B (z_1^{\alpha_1}\ldots z_n^{\alpha_n})\cdot\overline{(z_1^{\beta_1}\ldots z_n^{\beta_n})}$$

must vanish unless $\alpha=\beta$: changing variables,

$$\int_B z^\alpha\cdot\overline{z}^\beta \;=\; e^{i(\alpha_1-\beta_1)\theta_1}\ldots e^{i(\alpha_n-\beta_n)\theta_n}\cdot\int_B z^\alpha\cdot\overline{z}^\beta$$

That is, unless $\alpha_j=\beta_j$ for every j, some choice of angles θ_j will make the factor on the right-hand side 0, so the integral must be 0. With $\alpha=\beta$,

$$\int_B z^\alpha\cdot\overline{z}^\alpha \;=\; \int_B |z_1|^{2\alpha_1}\ldots|z_n|^{2\alpha_n} \;=\; \int_0^1 r^{2\alpha_1+\ldots+2\alpha_n+(2n-1)}\,dr\int_{S^{2n-1}}\Big|\frac{z_1}{r}\Big|^{2\alpha_1}\ldots\Big|\frac{z_n}{r}\Big|^{2\alpha_n}$$

$$= \frac{1}{2\alpha_1+\ldots+2\alpha_n+2n}\int_{S^{2n-1}}\Big|\frac{z_1}{r}\Big|^{2\alpha_1}\ldots\Big|\frac{z_n}{r}\Big|^{2\alpha_n}$$

To integrate a homogeneous degree-zero function f over a sphere, integrate $f(x/|x|)\cdot\varphi(|x|)$ over $\mathbb{C}^n$ for any non-negative φ on $(0,\infty)$, and divide by the integral of $\varphi(|x|)$ over $\mathbb{C}^n$. Uniqueness of the invariant integral up to scalars assures that this is independent of the auxiliary function φ. Given α, take $\varphi(r)=r^{2\alpha_1+\ldots+2\alpha_n}e^{-\pi r^2}$. Letting $|z|^2=|z_1|^2+\ldots+|z_n|^2$, the normalizing constant is

$$\int_{\mathbb{C}^n}|z|^{2\alpha_1+\ldots+2\alpha_n}\,e^{-\pi|z|^2} \;=\; |S^{2n-1}|\int_0^\infty r^{2\alpha_1+\ldots+2\alpha_n+2n}\,e^{-\pi r^2}\,\frac{dr}{r}$$

$$= \tfrac{1}{2}|S^{2n-1}|\int_0^\infty r^{\alpha_1+\ldots+\alpha_n+n}\,e^{-\pi r}\,\frac{dr}{r} \;=\; \pi^{-(\alpha_1+\ldots+\alpha_n+n)}\cdot\tfrac{1}{2}|S^{2n-1}|\cdot\Gamma(\alpha_1+\ldots+\alpha_n+n)$$

and

$$\int_{\mathbb{C}^n} \Big|\frac{z_1}{r}\Big|^{2\alpha_1} \ldots \Big|\frac{z_n}{r}\Big|^{2\alpha_n} \cdot r^{2\alpha_1+\ldots+2\alpha_n}\, e^{-\pi r^2} \;=\; \int_{\mathbb{C}^n} |z_1|^{2\alpha_1} \ldots |z_n|^{2\alpha_n}\, e^{-\pi r^2} \;=\; \prod_j \int_{\mathbb{C}} |z_j|^{2\alpha_j}\, e^{-\pi |z_j|^2}$$

The j^{th} integral over $\mathbb{C}$ is

$$\int_{\mathbb{C}} |z_j|^{2m}\, e^{-\pi|z_j|^2} \;=\; 2\pi \int_0^\infty r^{2\alpha_j+2}\, e^{-\pi r^2}\, \frac{dr}{r} \;=\; \pi \int_0^\infty r^{\alpha_j+1}\, e^{-\pi r}\, \frac{dr}{r} \;=\; \pi^{\alpha_j} \cdot \Gamma(\alpha_j+1)$$

Thus,

$$\int_{S^{2n-1}} \Big|\frac{z_1}{r}\Big|^{2\alpha_1} \ldots \Big|\frac{z_n}{r}\Big|^{2\alpha_n} \;=\; \frac{\pi^{\alpha_1}\,\Gamma(\alpha_1+1)\ldots\pi^{\alpha_n}\,\Gamma(\alpha_1+n)}{\pi^{-(\alpha_1+\ldots+\alpha_n+n)}\cdot\frac{1}{2}|S^{2n-1}|\cdot\Gamma(\alpha_1+\ldots+\alpha_n+n)}$$
$$=\; \frac{\pi^n\,\Gamma(\alpha_1+1)\ldots\Gamma(\alpha_1+n)}{\frac{1}{2}|S^{2n-1}|\cdot\Gamma(\alpha_1+\ldots+\alpha_n+n)}$$

and

$$\int_B z^\alpha \cdot \overline{z}^\alpha \;=\; \frac{1}{2\alpha_1+\ldots+2\alpha_n+2n}\int_{S^{2n-1}} \Big|\frac{z_1}{r}\Big|^{2\alpha_1} \ldots \Big|\frac{z_n}{r}\Big|^{2\alpha_n} \;=\; \frac{\pi^n\,\Gamma(\alpha_1+1)\ldots\Gamma(\alpha_1+n)}{|S^{2n-1}|\cdot\Gamma(\alpha_1+\ldots+\alpha_n+n+1)}$$

The ratio of Gamma values is almost the inverse of a multinomial coefficient: letting $|\alpha| = \alpha_1+\ldots+\alpha_n$,

$$\frac{\Gamma(|\alpha|+n+1)}{\Gamma(\alpha_1+\ldots+\alpha_n+n+1)} \;=\; \frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!} \;=\; \begin{pmatrix} |\alpha| \\ \alpha_1\ \ldots\ \alpha_n\end{pmatrix}\cdot(|\alpha|+1)\ldots(|\alpha|+n)$$

and

$$(|\alpha|+1)\ldots(|\alpha|+n) \;=\; \Big(\frac{\partial}{\partial t}\Big)^n\Big|_{t=1} t^{|\alpha|+n}$$

Thus, up to a constant, the reproducing kernel $K(z,w)$ is

$$\Big(\frac{\partial}{\partial t}\Big)^n\Big|_{t=1} t^n \sum_\alpha \begin{pmatrix}\alpha_1+\ldots+\alpha_n+n \\ \alpha_1\ \ldots\ \alpha_n\end{pmatrix} z^\alpha\,\overline{w}^\alpha\, t^{|\alpha|} \;=\; \Big(\frac{\partial}{\partial t}\Big)^n\Big|_{t=1} t^n \sum_{N\ge 0} t^N \sum_{|\alpha|=N}\begin{pmatrix} N \\ \alpha_1\ \ldots\ \alpha_n\end{pmatrix} z^\alpha\,\overline{w}^\alpha$$

$$=\; \Big(\frac{\partial}{\partial t}\Big)^n\Big|_{t=1} t^n \sum_{N\ge 0} t^N (z_1\overline{w}_1+\ldots z_n\overline{w}_n)^N \;=\; (z\cdot\overline{w})^{-n}\Big(\frac{\partial}{\partial t}\Big)^n\Big|_{t=1}\sum_{\ell\ge 0}(t\,z\cdot\overline{w})^\ell$$

$$=\; (z\cdot\overline{w})^{-n}\Big(\frac{\partial}{\partial t}\Big)^n\Big|_{t=1}\frac{1}{1-tz\cdot\overline{w}} \;=\; (z\cdot\overline{w})^{-n}\cdot\frac{(n-1)!\cdot(z\cdot\overline{w})^n}{(1-z\cdot\overline{w})^{n+1}} \;=\; \frac{(n-1)!}{(1-z\cdot\overline{w})^{n+1}}$$

Rather than organizing all the constants above, granting that $K(z,w) = c/(1-z\cdot\overline{w})^{n+1}$ for some constant c, the constant can be determined by integrating against the function 1 and evaluating at 0:

$$1 \;=\; \int_B K(0,w)\cdot 1 \;=\; \int_B c\cdot 1 \;=\; \mathrm{vol}\,(B)$$

so $c = 1/\mathrm{vol}\,(B)$, and the Bergman kernel for the complex n-ball B is

$$K(z,w) \;=\; \frac{1}{\mathrm{vol}\,(B)}\cdot\frac{1}{(1-z\cdot\overline{w})^{n+1}}$$

4. *Reproducing kernel for holomorphic cuspforms*

Recall that a holomorphic cuspform f on the upper half-plane $\mathfrak{H}$ for $\Gamma = SL_2(\mathbb{Z})$ of weight $2k$ is a holomorphic function with the *automorphy* property

$$f(\gamma z) \;=\; (cz+d)^{2k} \cdot f(z) \qquad\qquad (\text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma)$$

and a decay constraint at $i\infty$ describable by requiring that its Fourier expansion has no non-positive-index terms:

$$f(x+iy) \;=\; \sum_{n\geq 1} c_n\, e^{2\pi i n(x+iy)}$$

and that the coefficients are of *polynomial growth*. The Petersson inner product on holomorphic cuspforms of weight $2k$ for Γ is

$$\langle f, g\rangle \;=\; \int_{\Gamma\backslash\mathfrak{H}} f(z)\cdot \overline{g(z)}\; y^{2k}\; \frac{dx\,dy}{y^2}$$

It is completely unreasonable to expect to explicitly identify an orthonormal basis for all these spaces of cuspforms. Instead, we anticipate that the reproducing kernel is obtained by *winding up* a suitable holomorphic function $\varphi(z,w)$ on $\mathfrak{H}\times\mathfrak{H}$:

$$K(z,w) \;=\; \sum_\gamma \varphi(\gamma z,\, w)\cdot \frac{1}{(cz+d)^{2k}} \qquad\qquad (\text{summed over } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma)$$

Presumably $k(z,w)$ is holomorphic in z and anti-holomorphic in w. More importantly, we want $k(z,w)$ so that the winding-up creates $K(z,w)$ a holomorphic cuspform in z and anti-holomorphic cuspform in w. Thus, for $\gamma' \in \Gamma$, we would want

$$K(z,\gamma' w) \;=\; (c'\overline{w}+d')^{2k}\cdot K(z,w) \qquad\qquad (\text{with } \gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma)$$

After some experimentation, a plausible choice is

$$\varphi(z,w) \;=\; \frac{1}{(z-\overline{w})^{2k}}$$

and then

$$K(z,w) \;=\; \sum_\gamma \frac{1}{(\gamma z - \overline{w})^{2k}}\,\frac{1}{(cz+d)^{2k}} \;=\; \sum_\gamma \frac{1}{(\frac{az+b}{cz+d}-\overline{w})^{2k}}\,\frac{1}{(cz+d)^{2k}}$$

$$=\; \sum_\gamma \Big(\frac{1}{az+b-cz\overline{w}-d\overline{w}}\Big)^{2k} \;=\; \sum_\gamma \Big(\frac{1}{(1\;\; -\overline{w})\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix}}\Big)^{2k}$$

demonstrating the symmetry of $K(z,w)$ in z and $\overline{w}$. For a holomorphic cuspform f, with $w = u+iv$, *unwinding* gives

$$\int_{\Gamma\backslash\mathfrak{H}} K(z,w)\, f(w)\; v^{2k}\; \frac{du\,dv}{v^2} \;=\; \int_{\Gamma\backslash\mathfrak{H}} \sum_\gamma \varphi(\gamma z, w)\, f(\gamma w)\; \mathrm{Im}\,(\gamma w)^{2k}\; \frac{du\,dv}{v^2} \;=\; \int_{\mathfrak{H}} \varphi(z,w)\, f(w)\; v^{2k}\; \frac{du\,dv}{v^2}$$

$$=\; \int_{\mathfrak{H}} \frac{1}{(z-\overline{w})^{2k}}\,\Big(\sum_n c_n\, e^{2\pi i n w}\Big)\; v^{2k}\; \frac{du\,dv}{v^2} \;=\; \sum_n c_n \int_{\mathfrak{H}} \frac{1}{(z-\overline{w})^{2k}}\; e^{2\pi i n w}\; v^{2k}\; \frac{du\,dv}{v^2}$$

The integral is

$$\int_{\mathfrak{H}} \frac{1}{(z-\overline{w})^{2k}}\, e^{2\pi i n w}\ v^{2k}\ \frac{du\,dv}{v^2} \;=\; \int_0^\infty \int_{\mathbb{R}} \frac{1}{((x+iy)-(u-iv))^{2k}}\, e^{2\pi i n(u+iv)}\ v^{2k}\ \frac{du\,dv}{v^2}$$

Replacing u by $u+v$ makes this into

$$e^{2\pi i n x} \int_0^\infty \int_{\mathbb{R}} \frac{1}{(iy-(u-iv))^{2k}}\, e^{2\pi i n(u+iv)}\ v^{2k}\ \frac{du\,dv}{v^2}$$

$$= \; e^{2\pi i n x} \int_0^\infty e^{-2\pi n v} \Big(\int_{\mathbb{R}} \frac{1}{(u-i(v+y))^{2k}}\, e^{2\pi i n u}\ du \Big) v^{2k}\ \frac{dv}{v^2}$$

The inner integral is evaluated by residues:

$$\int_{\mathbb{R}} \frac{1}{(u-i(v+y))^{2k}}\, e^{2\pi i n u}\ du \;=\; 2\pi i \cdot \frac{(2\pi i n)^{2k-1}}{(2k-1)!} \cdot e^{2\pi i n \cdot i(v+y)} \;=\; \frac{(2\pi i)^{2k}\cdot n^{2k-1}}{\Gamma(2k)} \cdot e^{-2\pi n(v+y)}$$

so the whole integral is

$$\int_0^\infty e^{-2\pi n v}\ v^{2k-1} \cdot \frac{(2\pi i)^{2k}\cdot n^{2k-1}}{\Gamma(2k)} \cdot e^{-2\pi n\cdot(v+y)}\ \frac{dv}{v} \;=\; \frac{(2\pi i)^{2k}}{\Gamma(2k)}\, n^{2k-1}\, e^{-2\pi n y} \int_0^\infty e^{-4\pi n v}\ v^{2k-1}\ \frac{dv}{v}$$

$$= \; \frac{(2\pi i)^{2k}}{\Gamma(2k)}\, n^{2k-1}\, e^{-2\pi n y} \cdot (4\pi n)^{1-2k} \cdot \Gamma(2k-1) \;=\; (2i)^{2k}\, 4^{1-2k}\, \pi \cdot e^{-2\pi n y} \cdot (2k-1)$$

$$= \; (-1)^k\, 2^{2-2k}\, e^{-2\pi n y} \cdot (2k-1)$$

Thus,

$$\int_{\mathfrak{H}} \frac{1}{(z-\overline{w})^{2k}}\, e^{2\pi i n w}\ v^{2k}\ \frac{du\,dv}{v^2} \;=\; (-1)^k\, 2^{2-2k}\, e^{-2\pi n y} \cdot (2k-1) \cdot e^{2\pi i n z}$$

and we have the desired reproducing property

$$\int_{\Gamma\backslash\mathfrak{H}} K(z,w)\, f(w)\ v^{2k}\ \frac{du\,dv}{v^2} \;=\; (-1)^k\, 2^{2-2k} \cdot (2k-1) \cdot \sum_n c_n\, e^{2\pi i n z} \;=\; (-1)^k\, 2^{2-2k} \cdot (2k-1) \cdot f(z)$$
