Guinand’s explicit formula

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We take for granted the basic analytical properties of Riemann’s zeta function and the Gamma function. In particular, with $\Gamma_R(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$ the Gamma factor for $\zeta(s)$, and $\xi(s) = \Gamma_R(s) \cdot \zeta(s)$, the functional equation is $\xi(1-s) = \xi(s)$. The completed zeta function $\xi$ has poles only at $s = 0, 1$, and these simple. We use the normalization of Fourier transform\[1\]

$$\mathcal{F}(x) = \int_{\mathbb{R}} e^{ix\xi} g(x) \, dx$$

[0.1] Theorem: (Guinand) For $g \in C^\infty_c(\mathbb{R})$, letting $\rho = \frac{1}{2} + i\gamma$ whether or not $\rho$ is on the cricial line,

$$\sum_{\rho=\frac{1}{2}+i\gamma} \hat{g}(\gamma) = \hat{g}(i/2) - \hat{g}(-i/2)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left( \frac{\Gamma_R'}{\Gamma_R} \left( \frac{1}{2} + it \right) + \frac{\Gamma_R'}{\Gamma_R} \left( \frac{1}{2} - it \right) \right) \hat{g}(t) \, dt - \sum_{p=\pm \sqrt{m}} \log p \left( g(m \log p) + g(-m \log p) \right)$$

Proof: The idea is in [Guinand 1947], and there is essentially only one route to take. [2] Consider

$$I = \frac{1}{2\pi i} \int_{\Re(s) = 1+\varepsilon} \frac{\xi'(s)}{\xi(s)} \hat{g}(t) \, ds$$

(with $s = \frac{1}{2} + it$ whether or not $\Re(s) = \frac{1}{2}$)

The integral makes sense because $\hat{g}(s)$ is entire. It converges because $\hat{g}$ is rapidly decreasing in $\Re(s)$, for every fixed $\Im(s)$. The logarithmic derivative has simple poles at the zeros and poles of $\xi(s)$ with residues the order of vanishing, namely, the two simple poles at $s = 0, 1$, trivial zeros at $s = -2, -4, -6, \ldots$, and non-trivial zeros $\rho$ in the critical strip. In terms of $t = (s - \frac{1}{2})/i$, these are at $\pm i/2$. Moving the contour\[3\] to $\Re(s) = -\varepsilon$ captures $t = \pm i/2$ and the non-trivial zeros $\rho$, written similarly as $\rho = \frac{1}{2} + i\gamma$, whether or not we know that $\Re(\gamma) = \frac{1}{2}$:

$$I = -\hat{g}(i/2) - \hat{g}(-i/2) + \sum_{\rho=\frac{1}{2}+i\gamma} \hat{g}(\gamma) + \frac{1}{2\pi i} \int_{\Re(s) = -\varepsilon} \frac{\xi'(s)}{\xi(s)} \hat{g}(t) \, ds$$

From the functional equation, $\xi'(s) = -\xi(1-s)$, and the integral on $\Re(s) = -\varepsilon$ is equal to an integral on $\Re(s) = 1+\varepsilon$:

$$I = -\hat{g}(i/2) - \hat{g}(-i/2) + \sum_{\rho=\frac{1}{2}+i\gamma} \hat{g}(\gamma) - \frac{1}{2\pi i} \int_{\Re(s) = 1+\varepsilon} \frac{\xi'(1-s)}{\xi(1-s)} \hat{g}(t) \, ds$$

[1] Often, there might be a $2\pi$ somewhere, or a different sign in the exponent, but those details are irrelevant. The present choice is for convenience in the situation at hand.


[3] To legitimize this contour move, we should identify a sequence $T_j \to +\infty$ such that the integrals along $[-\varepsilon + iT_j, 1 + \varepsilon + iT_j]$ and $[-\varepsilon - iT_j, 1 + \varepsilon - iT_j]$ go to 0. Since $\hat{g}(t)$ is rapidly decreasing, this is not delicate. However, it still does need an asymptotic estimate of the number of zeros of $\zeta(s)$ to height $T$, which comes from the functional equation, Stirling-Laplace asymptotics for $\Gamma(s)$, and Hadamard’s theorem on growth versus zeros of entire functions.
Rearranging,
\[ \sum_{\rho = \frac{1}{2} + i\gamma} \hat{g}(\gamma) - \hat{g}(i/2) - \hat{g}(-i/2) = \frac{1}{2\pi i} \int_{\text{Re}(s)=1+\varepsilon} \frac{\xi'(s)}{\xi(s)} \hat{g}(t) \, ds + \frac{1}{2\pi i} \int_{\text{Re}(s)=1+\varepsilon} \frac{\xi'(1-s)}{\xi(1-s)} \hat{g}(t) \, ds \]

Separating the archimedean factors,
\[ \frac{\xi'(s)}{\xi(s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{\Gamma'_R(s)}{\Gamma_R(s)} \]

The contour integrals of the archimedean part can be shifted to \( \text{Re}(s) = \frac{1}{2} \) without picking up any residues, giving
\[ \frac{1}{2\pi i} \int_{\text{Re}(s)=\frac{1}{2}} \frac{\Gamma'_R(s)}{\Gamma_R(s)} \hat{g}(t) \, ds + \frac{1}{2\pi i} \int_{\text{Re}(s)=\frac{1}{2}} \frac{\Gamma'_R(1-s)}{\Gamma_R(1-s)} \hat{g}(t) \, ds = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(t) \left( \frac{\Gamma'_R}{\Gamma_R} \left( \frac{1}{2} + it \right) + \frac{\Gamma'_R}{\Gamma_R} \left( \frac{1}{2} - it \right) \right) \, dt \]

The logarithmic derivative of the finite-prime part, in a right half-plane, is[4]
\[ \frac{\zeta'(s)}{\zeta(s)} = -\sum_{p,m \geq 1} \frac{\log p}{p^{ms}} \]

giving
\[ \frac{1}{2\pi i} \int_{\text{Re}(s)=1+\varepsilon} \frac{\zeta'(s)}{\zeta(s)} \hat{g}(t) \, ds = -\sum_{p,m} \log(p) \frac{1}{2\pi i} \int_{\text{Re}(s)=1+\varepsilon} \frac{1}{p^{ms}} \cdot \hat{g}(t) \, ds \]

Moving the contour to \( \text{Re}(s) = \frac{1}{2} \) gives \( p^{ms} = \sqrt{p^m} \cdot e^{itm\log p} \), and with \( ds = d(\frac{1}{2} + it) = i \, dt \), the integral becomes
\[ -\sum_{p,m} \log(p) \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itm\log p} \hat{g}(t) \, dt \]

Here, Fourier inversion is
\[ \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(t) e^{-itx} \, dt = g(x) \]

Thus, by Fourier inversion,
\[ -\sum_{p,m} \log(p) \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{p^m}} \cdot e^{-itm\log p} \hat{g}(t) \, dt = -\sum_{p,m} \frac{\log p}{\sqrt{p^m}} g(m \log p) \]

The \( 1 - s \) term is treated similarly, giving the assertion of the theorem. ///


[4] A different notational choice is to sum over all integers, rather than over powers of primes, and use the von Mangoldt function
\[ \Lambda(n) = \begin{cases} \log p & \text{for } n = p^k, \text{ prime } p, 1 \leq k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \]

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