Phragmén-Lindelöf Theorems

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The maximum modulus principle in complex analysis does not apply to unbounded regions. That is, holomorphic functions on an unbounded region may be bounded by 1 on the edges but be severely unbounded in the interior.

A simple example is \( f(z) = e^{e^z} \): for \( z \) real and going to \(+\infty\) this function blows up. Indeed,

\[
\left| e^{e^{x+iy}} \right| = e^{\Re(e^{x+iy})} = e^{e^x \cos y}
\]

Thus, for fixed \( y = \Im z \) with \( \cos y > 0 \), the function blows up as \( x = \Re z \to +\infty \). On the other hand, for \( \cos y = 0 \) the function is bounded. Thus, on the strip \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\), the function \( e^{e^z} \) is bounded on the edges but blows up as \( x \to +\infty \).

This example suggests growth conditions under which a bound of 1 on the edges implies the same bound throughout the strip.

[1.1] Theorem: Let \( f \) be a holomorphic function on the horizontal half-strip

\[
\{ z : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \text{ and } 0 \leq x \}
\]

If

\[
|f(z)| \ll e^{C \Re z} \quad \text{(for some constant } 0 \leq C < 1)\]

then \( |f(z)| \leq 1 \) on the edges of the half-strip implies \( |f(z)| \leq 1 \) in the interior, as well.

Proof: Unsurprisingly, the proof is a reduction to the usual maximum modulus principle. Take any fixed \( D \) in the range

\[
C < D < 1
\]

The function

\[
F_\varepsilon(z) = f(z) / e^{\varepsilon D z} \quad \text{(for } \varepsilon > 0)\]

is bounded by 1 on the edges of the half-strip, and in the interior goes to 0 uniformly in \( y \) as \( x \to +\infty \), for fixed \( \varepsilon > 0 \). This uniform decay in the interior uses the modification with \( D \). Thus, on a rectangle

\[
R_T = \{ z : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \text{ and } 0 \leq x \leq T \}
\]

for sufficiently large \( T > 0 \) depending upon \( \varepsilon \), the function \( F_\varepsilon \) is bounded by 1 on the edge. The usual maximum modulus principle implies that \( F_\varepsilon \) is bounded by 1 throughout. That is, for each fixed \( z_0 \) in the half-strip,

\[
|f(z_0)| \leq e^{\varepsilon - e^{D\Re z_0}} \quad \text{(for all } \varepsilon > 0)\]

We can let \( \varepsilon \to 0^+ \), giving \( |f(z_0)| \leq 1 \).

[1] The maximum modulus principle in complex analysis is that a holomorphic function \( f \) on a bounded region in \( \mathbb{C} \) with \( |f(z)| \leq 1 \) on the edges is bounded by 1 in the interior, as well.
[1.2] **Remark:** Analogous theorems on strips of other widths follow by using $e^{c \cdot e^z}$ with suitable constants $c$.

An analogous theorem on a full strip, rather than half-strip, follows by using a function like $e^{c \cosh z}$ in place of $e^z$, as follows.

[1.3] **Theorem:** Let $f$ be a holomorphic function on the full horizontal strip

$$\{z : -\frac{\pi}{2} \leq \text{Im} z \leq \frac{\pi}{2}\}$$

If

$$|f(z)| \ll e^{c \cosh C \cdot \text{Re} z} \quad \text{for some constant } 0 \leq C < 1$$

then $|f(z)| \leq 1$ on the edges of the strip implies $|f(z)| \leq 1$ in the interior, as well.

**Proof:** Again, reduce to the maximum modulus principle. Fix $D$ in the range $C < D < 1$. The function

$$F_\varepsilon(z) = \frac{f(z)}{e^{\varepsilon \cosh D z}} \quad \text{for } \varepsilon > 0$$

is bounded by 1 on the edges of the strip, and in the interior goes to 0 uniformly in $y$ as $x \to \pm \infty$, for fixed $\varepsilon > 0$. Thus, on a rectangle

$$R_T = \{z : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \text{ and } -T \leq x \leq T\} \quad \text{for large } T > 0, \text{ depending upon } \varepsilon$$

the function $F_\varepsilon$ is bounded by 1 on the edge. The usual maximum modulus principle implies that $F_\varepsilon$ is bounded by 1 throughout. That is, for each fixed $z_0$ in the half-strip,

$$|f(z_0)| \leq e^{\varepsilon \cosh D \text{Re } z_0} \quad \text{for all } \varepsilon > 0$$

We can let $\varepsilon \to 0^+$, giving $|f(z_0)| \leq 1$.

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The details of various adjustments can be made to disappear by strengthening the hypotheses:

[1.4] **Corollary:** Let $f$ be a holomorphic function on a strip or half-strip, with a bound

$$|f(z)| \ll e^{|z|^A} \quad \text{for some } A > 0$$

If $|f(z)| \leq 1$ on the edges of the (half-)strip, then $|f(z)| \leq 1$ in the interior, as well.

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[1.5] **Remark:** Further variations are easily possible, by additional adjustments of functions. For example, polynomial growth of a function $f$ on the edges of a strip or half-strip can be accommodated by considering $f(z)/(z - z_0)^M$ for $z_0$ outside the strip, and large $M$. 

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Paul Garrett: Phragmén-Lindelöf Theorem (September 15, 2019)