Jensen’s formula

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1. Mean-value property of harmonic functions
2. Connections to holomorphic functions
3. Jensen’s formula

The Jensen formula usually appears as follows: for holomorphic $f$ on $|z| \leq r$, no zeros on $|z| = r$, and $f(0) \neq 0$,

$$
\log |f(0)| - \sum_{\rho} \log \left| \frac{\rho}{r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \quad \text{ (summed over zeros } |\rho| < r \text{ of } f)$$

This is a corollary of the mean-value property for values of a harmonic function $u$[1] in the interior of a disk in terms of its values on the boundary:

$$
u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \cdot \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\theta \quad \text{ (for } |z| < 1)$$

The Poisson formula in turn follows from the mean-value property, that $u(0)$ is the average of values of $u$ over the circle.

For non-vanishing holomorphic $f$, $\log |f|$ is harmonic, so the Poisson formula applies. When $f$ has (finitely-many) zeros $\rho$ in $|z| < 1$, an auxiliary function such as

$$F(z) = \frac{f(z)}{\prod_{\rho}(z - \rho)}$$

has no zeros there, and the Poisson formula applies to $\log |F|$.

1. Mean-value property

Among other features, in two dimensions harmonic functions form a useful, strictly larger class of functions including holomorphic functions. For example, harmonic functions still enjoy a mean-value property, as holomorphic functions do:

[1.1] Theorem: (Mean-value property) For harmonic $u$ on a neighborhood of the closed unit disk,

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \, d\theta$$

Proof: Consider the rotation-averaged function

$$v(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta} \cdot z) \, d\theta \quad \text{ (for } |z| \leq 1)$$

[1] A continuously twice_differentiable function on $\mathbb{R}^2$ is harmonic when it is annihilated by the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In some contexts, a harmonic function is understood to be real-valued.
Since the Laplacian $\Delta$ is rotation-invariant, $v$ is a rotation-invariant harmonic function. In polar coordinates, for rotation-invariant functions $v(z) = f(|z|)$, the Laplacian is

$$\Delta v = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(\sqrt{x^2 + y^2}) = \frac{\partial}{\partial x} \left( \frac{x}{|z|} f'(|z|) \right) + \frac{\partial}{\partial y} \left( \frac{y}{|z|} f'(|z|) \right)$$

$$= \frac{1}{|z|} f' - \frac{x^2}{|z|^3} f' + \frac{x^2}{|z|^3} f'' + \frac{1}{|z|} f' - \frac{y^2}{|z|^3} f' + \frac{y^2}{|z|^3} f'' = f'' + \frac{1}{|z|} f'$$

The ordinary differential equation $f'' + f'/r = 0$ on an interval $(0, R)$ is an equation of Euler type, meaning expressible in the form $r^2 f'' + Brf' + Cf = 0$ with constants $B, C$. In general, such equations are solved by letting $f(r) = r^\lambda$, substituting, dividing through by $r^\lambda$, and solving the resulting indicial equation for $\lambda$:

$$\lambda(\lambda - 1) + A\lambda + B = 0$$

Distinct roots $\lambda_1, \lambda_2$ of the indicial equation produce linearly independent solutions $r^{\lambda_1}$ and $r^{\lambda_2}$. However, as in the case at hand, a repeated root $\lambda$ produces a second solution $r^\lambda \cdot \log r$. Here, the indicial equation is $\lambda^2 = 0$, so the general solution is $a + b \log r$. When $b \neq 0$, the solution $a + b \log r$ blows up as $r \to 0^\pm$. Since $f(0) = v(0) = u(0)$ is finite, it must be that $b = 0$. Thus, a rotation-invariant harmonic function on the disk is constant. Thus, its average over a circle is its central value, proving the mean-value property for harmonic functions.

1.2 Remark: One might worry about commutation of the Laplacian with the integration above. In the first place, it is clear that we must have this commutativity. Second, the best and most final argument for such is in terms of Gelfand-Pettis (also called weak) integrals of function-valued functions, rather than temporary elementary arguments.

1.3 Remark: The solutions $a + b \log r$ do indeed exhaust the possible solutions: given $f'' + f'/r = 0$ on $(0, R)$, we see $r \cdot f'$ is constant because

$$\frac{\partial}{\partial r} (r \cdot f') = r \cdot f'' + f' = r \cdot (-f'/r) + f' = 0$$

2. Connections to holomorphic functions

With the notation

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

we have

$$\frac{\partial}{\partial z} \circ \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} \circ \frac{\partial}{\partial z} = \frac{1}{4} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

A holomorphic function $u$ satisfies the Cauchy-Riemann equation $\partial u / \partial \bar{z} = 0$, so every holomorphic function is harmonic. Similarly, every every conjugate-holomorphic function is harmonic. Thus, for holomorphic $f$, the real and imaginary parts

$$\text{Re}(f(z)) = \frac{1}{2} \left( f(z) + \overline{f(z)} \right) \quad \text{Im}(f(z)) = \frac{1}{2i} \left( f(z) - \overline{f(z)} \right)$$

are harmonic, and real-valued.

The class of harmonic functions includes useful non-holomorphic real-valued functions. For example, (real-valued) logarithms of absolute values of non-vanishing holomorphic functions are harmonic:

$$\log |f(z)| = \frac{1}{2} \cdot \left( \log f + \log f^* \right) = \frac{1}{2} \cdot \left( \text{holomorphic} + \text{anti-holomorphic} \right)$$
so is annihilated by $\Delta = 4 \frac{\partial^2}{\partial z^2} \circ \frac{\partial}{\partial \overline{z}}$.

## 3. Jensen's formula

**[3.1] Theorem:** For holomorphic $f$ on an open containing $|z| \leq r$, with no zeros on $|z| = r$, and with $f(0) \neq 0$,

$$\log |f(0)| - \sum_{\rho} \log |\frac{\rho}{r}| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta$$

(summed over zeros $|\rho| < r$ of $f$)

**Proof:** First, for clarity, take $r = 1$. Letting

$$F(z) = \frac{f(z)}{\prod_{\rho}(z - \rho)}$$

we have

$$\log |F(z)| = \log |f(z)| - \sum_{\rho} \log |\rho - z|$$

By the Poisson formula applied to $\log |F|$,

$$\log |f(0)| - \sum_{\rho} \log |\rho| = \log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(e^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| - \sum_{\rho} \log |\rho - e^{i\theta}| d\theta$$

A small trick:

$$\int_0^{2\pi} \log |\rho - e^{i\theta}| d\theta = \int_0^{2\pi} \log |1 - e^{-i\theta}\rho| d\theta = \text{Re} \int_0^{2\pi} \log(1 - e^{-i\theta}\rho) d\theta = \text{Re}(0) = 0$$

because $\log(1 - w\rho)$ is holomorphic on an open containint $|w| \leq 1$. For general $r > 0$, take $g(z) = f(z/r)$ and apply the above argument to $g$.

**[3.2] Remark:** Letting $\nu(t)$ be the number of zeros of size less than $t$, we can also rewrite

$$- \sum_{\rho} \log \left| \frac{\rho}{r} \right| = \sum_{\rho} (\log r - \log |\rho|) = \sum_{\rho} \int_{|\rho|}^{r} \frac{dt}{t} = \int_0^{r} \nu(t) \frac{dt}{t}$$

Bibliography: