First proof of \( L(1, \chi) \neq 0 \)

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[This document is http://www.math.umn.edu/~garrett/m/mfms/notes_2019-20/06b_ugly_nonvanishing.pdf]

1. First proof of non-vanishing on \( \text{Re}(s) = 1 \)
2. Landau’s lemma: Dirichlet series with positive coefficients

The subtle element in Dirichlet’s theorem about primes in arithmetic progressions is non-vanishing of \( L \)-functions \( L(s, \chi) \) at \( s = 1 \).

Here, we give a rather ugly and unexplanatory proof. However, the argument has few prerequisites. It uses Landau’s Lemma, which we prove here.

1. First proof of non-vanishing on \( \text{Re}(s) = 1 \)

We prove that \( L(1, \chi) \neq 0 \) for all \( \chi \mod N \), granting that these \( L \)-functions have meromorphic extensions to some neighborhood of \( s = 1 \). We also need to know that for the trivial character \( \chi_o \mod N \) the \( L \)-function \( L(s, \chi_o) \) has a simple pole at \( s = 1 \).

[1.1] Theorem: For a Dirichlet character \( \chi \mod N \) other than the trivial character \( \chi_o \mod N \),

\[
L(1, \chi) \neq 0
\]

Proof: To prove that the \( L \)-functions \( L(s, \chi) \) do not vanish at \( s = 1 \), and in fact do not vanish on the whole line \( \text{Re}(s) = 1 \), direct arguments involve tricks similar to what we do here.

First, for \( \chi \) whose square is not the trivial character \( \chi_o \mod N \), the standard trick is to consider

\[
\lambda(s) = L(s, \chi_o)^3 \cdot L(s, \chi)^4 \cdot L(s, \chi^2)
\]

Then, letting \( \sigma = \text{Re}(s) \), from the Euler product expressions for the \( L \)-functions noted earlier, in the region of convergence,

\[
|\lambda(s)| = |\exp \left( \sum_{m,p} \frac{3 + 4 \chi(p^m) + \chi^2(p^m)}{mp^{\sigma}} \right)| = \exp \left| \sum_{m,p} \frac{3 + 4 \cos \theta_{m,p} + \cos 2\theta_{m,p}}{mp^{\sigma}} \right|
\]

where for each \( m \) and \( p \) we let

\[
\theta_{m,p} = (\text{the argument of } \chi(p^m)) \in \mathbb{R}
\]

The trick, presumably found after considerable experimentation, is that for any real \( \theta \)

\[
3 + 4 \cos \theta + \cos 2\theta = 3 + 4 \cos \theta + 2 \cos^2 \theta - 1 = 2 + 4 \cos \theta + 2 \cos^2 \theta = 2(1 + \cos \theta)^2 \geq 0
\]

[1] Non-vanishing of \( \zeta(s) \) on the whole line \( \text{Re}(s) = 1 \) yields the Prime Number Theorem: let \( \pi(x) \) be the number of primes less than \( x \). Then \( \pi(x) \sim x/\ln x \), meaning that the limit of the ratio of the two sides as \( x \to \infty \) is 1. This was first proven in 1896, separately, by Hadamard and de la Vallée Poussin. The same sort of argument also gives an analogous asymptotic statement about primes in each congruence class modulo \( N \), namely that \( \pi_{a,N}(x) \sim x/\varphi(N) \cdot \ln x \), where \( \gcd(a, N) = 1 \) and \( \varphi \) is Euler’s totient function.
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Therefore, miraculously, all the terms inside the large sum being exponentiated are non-negative, and

$$|\lambda(s)| \geq e^0 = 1$$

In particular, if $L(1, \chi) = 0$ were to be 0, then, since $L(s, \chi_o)$ has a simple pole at $s = 1$ and since $L(s, \chi^2)$ does not have a pole (since $\chi^2 \neq \chi_o$), the multiplicity $\geq 4$ of the 0 in the product of $L$-functions would overwhelm the three-fold pole, and $\lambda(1) = 0$. This would contradict the inequality just obtained.

For $\chi^2 = \chi_o$, instead consider

$$\lambda(s) = L(s, \chi) \cdot L(s, \chi_o) = \exp \left( \sum_{p, m} \frac{1 + \chi(p^m)}{mp^{ms}} \right)$$

If $L(1, \chi) = 0$, then this would cancel the simple pole of $L(s, \chi_o)$ at 1, giving a non-zero finite value at $s = 1$

The series inside the exponentiation is a **Dirichlet series with non-negative coefficients**, and for real $s$

$$\sum_{p, m} \frac{1 + \chi(p^m)}{mp^{ms}} \geq \sum_{p, m \text{ even}} \frac{1 + 1}{mp^{ms}} = \sum_{p, m} \frac{1 + 1}{2mp^{2ms}} = \sum_{p, m} \frac{1}{mp^{2ms}} = \log \zeta(2s)$$

Since $\zeta(2s)$ has a simple pole at $s = \frac{1}{2}$ the series

$$\log (L(s, \chi) \cdot L(s, \chi_o)) = \sum_{p, m} \frac{1 + \chi(p^m)}{mp^{ms}} \geq \log \zeta(2s)$$

necessarily blows up as $s \to \frac{1}{2}^+$. But by **Landau’s Lemma** below, a Dirichlet series with non-negative coefficients cannot blow up as $s \to s_0$ along the real line unless the function represented by the series fails to be holomorphic at $s_0$. Since the function given by $\lambda(s)$ is holomorphic at $s = 1/2$, this gives a contradiction to the supposition that $\lambda(s)$ is holomorphic at $s = 1$ (which had allowed this discussion at $s = 1/2$). That is, $L(1, \chi) \neq 0$.

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[1.2] **Remark**: Again, the above argument is quick, but unilluminating. We will give better proofs later.

2. **Landau’s Lemma on Dirichlet series with positive coefficients**

[2.1] **Theorem**: (Landau) Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series with real coefficients $a_n \geq 0$. Suppose that the series defining $f(s)$ converges for $\text{Re}(s) > \sigma_o$. Suppose further that the function $f$ extends to a function holomorphic in a neighborhood of $s = \sigma_o$. Then, in fact, the series defining $f(s)$ converges for $\text{Re}(s) > \sigma_o - \varepsilon$ for some $\varepsilon > 0$.

**Proof**: First, by replacing $s$ by $s - \sigma_o$ reduce to the case that $\sigma_o = 0$. Since the function $f(s)$ given by the series is holomorphic on $\text{Re}(s) > 0$ and on a neighborhood of 0, there is $\varepsilon > 0$ such that $f(s)$ is holomorphic on $|s - 1| < 1 + 2\varepsilon$, and the power series for the function converges nicely on this open disk. Differentiating the original series termwise (Abel’s theorem), evaluate the derivatives of $f(s)$ at $s = 1$ as

$$f^{(i)}(1) = \sum_n \frac{(-\log n)^i a_n}{n} = (-1)^i \sum_n \frac{(\log n)^i a_n}{n}$$

and Cauchy’s formulas yield, for $|s - 1| < 1 + 2\varepsilon$,

$$f(s) = \sum_{i \geq 0} \frac{f^{(i)}(1)}{i!} (s - 1)^i$$

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In particular, for $s = -\varepsilon$, we are assured of the convergence to $f(-\varepsilon)$ of

$$f(-\varepsilon) = \sum_{i \geq 0} \frac{f^{(i)}(1)}{i!} (-\varepsilon - 1)^i$$

Since $(-1)^if^{(i)}(1)$ is a positive Dirichlet series, move the powers of $-1$ a little to obtain

$$f(-\varepsilon) = \sum_{i \geq 0} \frac{(-1)^if^{(i)}(1)}{i!} (\varepsilon + 1)^i$$

The series

$$(-1)^i f^{(i)}(1) = \sum_n (\log n)^i \frac{a_n}{n}$$

has positive terms, so the double series, convergent, with positive terms,

$$f(-\varepsilon) = \sum_{n,i} \frac{a_n (\log n)^i}{i!} (1 + \varepsilon)^i \frac{1}{n}$$

can be rearranged to

$$f(-\varepsilon) = \sum_n \frac{a_n}{n} \left( \sum_i \frac{(\log n)^i (1 + \varepsilon)^i}{i!} \right) = \sum_n \frac{a_n}{n} n^{1+\varepsilon} = \sum_n \frac{a_n}{n^{1-\varepsilon}}$$

That is, the latter series converges (absolutely). ///

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