1. Fourier analysis on finite abelian groups

Let $G$ be a finite abelian group, and $L^2(G)$ the space of complex-valued functions on $G$, with inner product

$$\langle f, \varphi \rangle = \sum_{x \in G} f(x) \overline{\varphi(x)}$$

Let $G$ act on $L^2(G)$ by the regular representation

$$(g \cdot f)(x) = f(xg) \quad (\text{for } x, g \in G \text{ and } f \in L^2(G))$$

Let

$$\widehat{G} = \{ \text{group homomorphisms } \sigma : G \rightarrow \mathbb{C}^\times \}$$

be the collection of characters on $G$. Since $G$ is finite, it is torsion, so $|\sigma(x)| = 1$ for all $\sigma \in \widehat{G}$ and $x \in G$. By a standard abuse of notation, $\sigma$ also denotes a one-dimensional complex vector space on which $G$ acts by the character $\sigma$, namely

$$g \cdot v = \sigma(g) \cdot v \quad (\text{for } v \text{ in the vector space, } g \in G)$$

[1.0.1] Theorem: The regular representation of $G$ decomposes $L^2(G)$ as a direct sum of (scalar multiples of) characters of $G$, each occurring exactly once. That is,

$$L^2(G) = \bigoplus_{\sigma \in \widehat{G}} \mathbb{C} \cdot \sigma \quad (\text{as } G \text{ representations})$$

Proof: On one hand, each $\sigma \in \widehat{G}$ is itself a complex-valued function on $G$, so it is in $L^2(G)$. Thus, all scalar multiples of the function $\sigma$ give a copy of the representation $\sigma$ inside $L^2(G)$. Further, for $\sigma \neq \tau$ in $\widehat{G},$

---

[1] Finite groups are both discrete and compact. One canonical measure on discrete groups is counting measure. Even without talking about invariant measures on groups, counting measure $\mu$ is translation-invariant in the sense that $\mu(X) = \mu(X \cdot g)$ for finite subsets $X$ of the group and for $g$ in the group. But there is another canonical invariant measure on compact groups, namely, that which gives the whole group measure 1. Thus, there are at least two canonical normalizations of invariant measure on a finite group.

[2] For abelian $G$, the left and right actions of $G$ on itself by multiplication are the same, so there is scant point in distinguishing them. For non-abelian groups these two actions are distinct, and in that case we would need to keep track of the action of $G \times G$ on $G$ and on functions on $G$.

[3] In the context of abelian groups, all irreducible complex representations are one-dimensional, and it is not unreasonable to identify any such thing with a group homomorphism of $G$ to $\mathbb{C}^\times$. With such an identification, the trace is essentially the same thing as the representation itself. For non-abelian groups and higher-dimensional irreducibles, the characters in the sense of traces are not group homomorphisms themselves.

[4] One should observe that the function $\sigma$ really does behave as required under right translation, namely, $\sigma(xg) = \sigma(x)\sigma(g) = \sigma(g)\sigma(x)$.
$\sigma \tau^{-1}$ is not the trivial character in $\hat{G}$, so by the cancellation lemma\[3\]

$$\langle \sigma, \tau \rangle = \sum_{x \in G} \sigma(x) \overline{\tau(x)} = \sum_{x \in G} (\sigma \tau^{-1})(x) = 0$$

That is, the one-dimensional subspaces $\sigma$ and $\tau$ inside $L^2(G)$ are orthogonal, since the functions $\sigma$ and $\tau$ are orthogonal. Thus, each $\sigma$ in $\hat{G}$ occurs in $L^2(G)$ at least once.

On the other hand, $G$ acts\[4\] on $L^2(G)$ by commuting unitary operators:

$$\langle g \cdot f, g \cdot \varphi \rangle = \sum_{x \in G} f(xg) \overline{\varphi(x)} = \sum_{x \in G} f(x) \overline{\varphi(x)} = \langle f, \varphi \rangle$$

by replacing $x$ by $xg^{-1}$. Thus, by the finite-dimensional spectral theorem\[5\] $L^2(G)$ has an orthogonal basis consisting of simultaneous eigenvectors\[6\] for $G$. That is, a simultaneous eigenvector $f$ has the behavior

$$f(xg) = \sigma(g) \cdot f(x) \quad \text{(for some } \sigma \in \hat{G}, \text{ for all } x, g \in G)$$

In fact, this gives

$$f(x) = f(1 \cdot x) = (x \cdot f)(1) = \sigma(x) \cdot f(1) = f(1) \cdot \sigma(x)$$

That is, such $f$ is a scalar multiple of the function $\sigma$. That is, it is already in the subspace of $L^2(G)$ spanned by the function $\sigma$. Thus, each representation $\sigma \in \hat{G}$ occurs just once in $L^2(G)$, in the sense that the space of functions in $L^2(G)$ that are eigenvectors for $G$ with eigenvalues given by $\sigma$ is one-dimensional. \[7\] ///

---

\[3\] The basic cancellation lemma that shows that the sum of a non-trivial character $\chi$ over a finite (not necessarily non-abelian) group $G$ is 0, by changing variables, as follows. For any $h \in G$, we change variables to obtain $\sum_{g \in G} \chi(g) = \sum_{g \in G} \chi(gh) = \chi(h) \sum_{g \in G} \chi(g)$. Since $\chi(h) \neq 1$ for some $h \in G$, the sum must be 0.

\[4\] More broadly, the action of a torsion group on a finite-dimensional vector space over an algebraically closed field of characteristic zero is simultaneously diagonalizable.

\[5\] The spectral theorem for single normal operators, that is, operators $T$ commuting with their adjoints, gives a spectral theorem for individual unitary operators $T$, since these are normal. Then note that the $\lambda$ eigenspace $V_\lambda$ for $T$ is stabilized by any operator $S$ that commutes with $T$, since $T(Sv) = S(Tv) = S(\lambda v) = \lambda \cdot Sv$. Restrictions of unitary operators $S$ to stable subspaces are still unitary, so $V_\lambda$ has an orthogonal basis of eigenvectors for $S$. These are necessarily still $\lambda$-eigenvectors for $T$. A downward induction gives the result.

\[6\] For a simultaneous eigenvector $v \neq 0$ for a group $G$, the set of eigenvalues $\lambda_g$ of $v$ for $g \in G$ is not an unstructured set of numbers. Rather, the map $g \mapsto \lambda_g$ is a character of $G$, since, by associativity, $\lambda_g \lambda_h v = g(hv) = (gh) \cdot v = \lambda_{gh} v$.

\[7\] Indeed, we just showed that the space of functions in $L^2(G)$ that are simultaneous eigenfunctions for $G$ with eigenvalues given by $\sigma$ consists exactly of scalar multiples of the function $\sigma$. 

2