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# *Meromorphic continuation and functional equation of $GL_2$ Eisenstein series*

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Poisson summation suffices to give a general argument for meromorphic continuation and functional equation of Eisenstein series for $GL_2$ over number fields, by integral representations of renormalized forms of the Eisenstein series.

I saw the basic idea in [Godement 1966a], and later learned that a version already appeared in [Rankin 1939]. I seem to recall that Rankin commented somewhere that his advisor Ingham already knew this device. An adelic version for $GL_2$ appears in my old book [Garrett 1989], superficially for totally real fields, but obviously not using this feature.

Much as Iwasawa-Tate theory proves analytic continuation and functional equation for *all* zeta integrals, and delays concern about refinements and symmetry, this approach proves analytic continuation and functional equation for a large class of Eisenstein series, delaying concern for details.

Let $\chi$ be a character on $M_k\backslash M_{\mathbb{A}}$, where $M$ is the diagonal subgroup of $G = GL_2$ over $k$. Let $N$ be upper-triangular unipotent matrices. Extend $\chi$ trivially to $\widetilde{\chi}$ on $N_{\mathbb{A}}$, that is, $\widetilde{\chi}$ is given by

$$\widetilde{\chi}\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \;=\; \chi_1(a)\cdot\chi_2(d) \qquad\qquad \text{(for some characters } \chi_1,\chi_2 \text{ on } \mathbb{J}/k^\times)$$

Let $\Phi$ be a Schwartz function on $\mathbb{A}^2$. Let $e_2 = (0,1)$ be the usual second basis-vector for $\mathbb{A}^2$, and define a vector $\varphi_{\chi,\Phi}$ in the $\chi^{th}$ principal series by

$$\varphi_{\chi,\Phi}(g) \;=\; \chi_1(\det g)\int_{\mathbb{J}} \chi_1\chi_2^{-1}(t)\;\Phi(t\cdot e_2\cdot g)\,dt$$

The required left equivariance follows from changing variables. Note the normalization: on $P_{\mathbb{A}}$,

$$\varphi_{\chi,\Phi}(p) \;=\; \widetilde{\chi}(p)\cdot\int_{\mathbb{J}} \chi_1\chi_2^{-1}(t)\;\Phi(t\cdot e_2)\,dt \qquad\qquad \text{(for } p\in P_{\mathbb{A}})$$

The restriction of $\Phi$ to the second coordinate produces a Schwartz function on $\mathbb{A}$, and the latter integral is the Iwasawa-Tate zeta integral attached to that Schwartz function and to the idele class character $\chi_1\chi_2^{-1}$. Let $\mathbb{J}^1$ be the ideles of idele-norm 1.

Define a kind of Eisenstein series by

$$\mathscr{E}(\chi,\Phi)(g) \;=\; \sum_{\gamma\in P_k\backslash G_k} \varphi_{\chi,\Phi}(\gamma\cdot g)$$

Let $w_o = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be the long Weyl element. For a character $\chi$ on $M$,

$$(\chi\circ w_o)(m) \;=\; \chi(w_o^{-1} m w_o) \qquad\qquad \text{(for } m\in M_{\mathbb{A}})$$

Put

$$\widehat{\chi}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \;=\; |a|\chi_2(a)\cdot|d|^{-1}\chi_1(d) \;=\; |\frac{a}{d}|\cdot(\chi\circ w_o)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

**[0.0.1] Theorem:** The Eisenstein series $\mathscr{E}(\chi, \Phi)$ has a meromorphic continuation in the (complex parameter part of) $\chi$, with functional equation

$$\mathscr{E}(\chi, \Phi) \;=\; \mathscr{E}(\widehat{\chi}, \widehat{\Phi})$$

The Eisenstein series has poles only for $\chi_1/\chi_2 = |\cdot|^{2s}$ for some $s \in \mathbb{C}$, in which case the poles are at $s = 1, 0$, are simple, with respective residues

$$\begin{cases} \mathrm{Res}_{s=1}\, \mathscr{E}(\chi, \Phi) &=& \dfrac{\chi_1(\det g)}{|\det g|} \cdot \widehat{\Phi}(0) \cdot \dfrac{\kappa}{2} \\ \mathrm{Res}_{s=0}\, \mathscr{E}(\chi, \Phi) &=& -\chi_1(\det g) \cdot \Phi(0) \cdot \dfrac{\kappa}{2} \end{cases} \qquad (\text{when } \chi_1/\chi_2 = |\cdot|^{2s})$$

where $\kappa$ is the residue at 1 of the completed zeta function of the groundfield, that is, the natural volume of $\mathbb{J}^1/k^\times$.

*Proof:* Since $P_k$ is the isotropy subgroup in $G_k$ of the line generated by $e_2$,

$$P_k\backslash G_k \;\longleftrightarrow\; k^\times\backslash(k^2 - \{0\}) \qquad \text{by} \qquad P_k g \longrightarrow k^\times e_2 g$$

Thus, with

$$\Theta(\Phi, g) \;=\; \sum_{\xi \in k^2} \Phi(\xi \cdot g)$$

$$\mathscr{E}(\chi, \Phi)(g) \;=\; \sum_{\gamma \in P_k\backslash G_k} \varphi_{\chi,\Phi}(\gamma \cdot g) \;=\; \sum_{\gamma \in P_k\backslash G_k} \chi_1(\det g) \int_{\mathbb{J}} \chi_1\chi_2^{-1}(t)\; \Phi(t \cdot e_2 \cdot \gamma g)\, dt$$

$$= \sum_{\gamma \in P_k\backslash G_k} \chi_1(\det g) \int_{\mathbb{J}/k^\times} \chi_1\chi_2^{-1}(t) \sum_{\alpha \in k^\times} \Phi(t \cdot \alpha e_2 \cdot \gamma g)\, dt$$

$$= \chi_1(\det g) \int_{\mathbb{J}/k^\times} \chi_1\chi_2^{-1}(t) \sum_{\gamma \in P_k\backslash G_k} \sum_{\alpha \in k^\times} \Phi(t \cdot \alpha e_2 \cdot \gamma g)\, dt$$

$$= \chi_1(\det g) \int_{\mathbb{J}/k^\times} \chi_1\chi_2^{-1}(t) \Big(\Theta(\Phi, tg) - \Phi(0)\Big)\, dt$$

Just as Riemann did, break the integral into two parts, on over $\mathbb{J}^+ = \{y \in \mathbb{J} : |y| \geq 1\}$ and the other over $\mathbb{J}^- = \{y \in \mathbb{J} : |y| \leq 1\}$:

$$\int_{\mathbb{J}/k^\times} \chi_1\chi_2^{-1}(t) \Big(\Theta(\Phi, tg) - \Phi(0)\Big)\, dt$$

$$= \int_{\mathbb{J}^+/k^\times} \chi_1\chi_2^{-1}(t) \Big(\Theta(\Phi, tg) - \Phi(0)\Big)\, dt \;+\; \int_{\mathbb{J}^-/k^\times} \chi_1\chi_2^{-1}(t) \Big(\Theta(\Phi, tg) - \Phi(0)\Big)\, dt$$

Estimates identical to those in Iwasawa-Tate show that the integral over $\mathbb{J}^+/k^\times$ is *entire* in $\chi$.

Use Poisson summation to convert the integral over $\mathbb{J}^-/k^\times$ into an integral over $\mathbb{J}^+k^\times$, with two extra, elementary terms. First, by Poisson summation, letting $g^\theta = (g^\top)^{-1}$,

$$\Theta(\Phi, tg) \;=\; \sum_{\xi \in k^2} \Phi(t\xi g) \;=\; \frac{1}{|t|^2 \cdot |\det g|} \sum_{\xi \in k^2} \widehat{\Phi}(t^{-1}\, \xi\, (g^\top)^{-1}) \qquad (\text{with } g^\theta = (g^\top)^{-1})$$

Then

$$\Theta(\Phi, tg) - \Phi(0) \;=\; \frac{1}{|t|^2 \cdot |\det g|} \cdot \Big(\Theta(\widehat{\Phi}\, t^{-1} g^\theta) - \widehat{\Phi}(0)\Big) \;+\; \frac{\widehat{\Phi}(0)}{|t|^2 \cdot |\det g|} \;-\; \Phi(0)$$

Then

$$\int_{\mathbb{J}^-/k^\times} \chi_1\chi_2^{-1}(t)\Big(\Theta(\Phi,tg)-\Phi(0)\Big)\,dt \;=\; \int_{\mathbb{J}^-/k^\times} \frac{\chi_1\chi_2^{-1}(t)}{|t|^2\cdot|\det g|}\cdot\Big(\Theta(\widehat{\Phi},\,t^{-1}g^\theta)-\widehat{\Phi}(0)\Big)\,dt$$

$$+\;\frac{\widehat{\Phi}(0)}{|\det g|}\int_{\mathbb{J}^-/k^\times}|t|^{-2}\chi_1\chi_2^{-1}(t)\,dt \;-\; \Phi(0)\int_{\mathbb{J}^-/k^\times}\chi_1\chi_2^{-1}(t)\,dt$$

The latter two integrals are elementary, as in Iwasawa-Tate:

$$\int_{\mathbb{J}^-/k^\times}|t|^{-2}\chi_1\chi_2^{-1}(t)\,dt \;=\; \begin{cases}\dfrac{\kappa/2}{s-1} & (\text{for } \chi_1\chi_2^{-1}=|\cdot|^{2s})\\[1ex] 0 & (\text{for } \chi_1\chi_2^{-1} \text{ non-trivial on } \mathbb{J}^1)\end{cases}$$

$$\int_{\mathbb{J}^-/k^\times}\chi_1\chi_2^{-1}(t)\,dt \;=\; \begin{cases}\dfrac{\kappa/2}{s} & (\text{for } \chi_1\chi_2^{-1}=|\cdot|^{2s})\\[1ex] 0 & (\text{for } \chi_1\chi_2^{-1} \text{ non-trivial on } \mathbb{J}^1)\end{cases} \qquad (\text{with } s=s_\chi)$$

where $\kappa$ is the residue at 1 of the completed zeta function of the groundfield. Write $\delta_\chi$ for 1 when $\chi_1\chi_2^{-1}$ is of the form $|\cdot|^{2s}$, and 0 otherwise. Replacing $t$ by $t^{-1}$ in the second non-elementary integral,

$$\mathscr{E}(\chi,\Phi)(g) \;=\; \chi_1(\det g)\int_{\mathbb{J}^+/k^\times}\chi_1\chi_2^{-1}(t)\Big(\Theta(\Phi,tg)-\Phi(0)\Big)\,dt$$

$$+\;\frac{\chi_1(\det g)}{|\det g|}\int_{\mathbb{J}^+/k^\times}|t|^2\,\chi_1^{-1}\chi_2(t)\Big(\Theta(\widehat{\Phi},tg^\theta)-\widehat{\Phi}(0)\Big)\,dt \;+\; \delta_\chi\Big(\frac{\chi_1(\det g)}{|\det g|}\frac{\widehat{\Phi}(0)\kappa/2}{s_\chi-1}-\chi_1(\det g)\frac{\Phi(0)\kappa/2}{s_\chi}\Big)$$

Noting that

$$\frac{\chi_1(\det g)}{|\det g|} \;=\; (|\cdot|\chi_1^{-1})(\det\big(g^\theta\big)$$

we are motivated to regroup

$$|t|^2\,\chi_1^{-1}\chi_2(t) \;=\; (|\cdot|\chi_1^{-1})(t)\cdot(|\cdot|\chi_2)(t) \;=\; (|\cdot|\chi_1^{-1})(t)\cdot(|\cdot|^{-1}\chi_2^{-1})^{-1}(t)$$

to comply with the form of the integral representation of $\mathscr{E}$. For convenience, let $\chi'$ be the character on $M$ given by

$$\chi'\begin{pmatrix}a&0\\0&d\end{pmatrix} \;=\; (|\cdot|\chi_1^{-1})(a)\cdot(|\cdot|^{-1}\chi_2^{-1})(d)$$

with corresponding $\chi_1'$ and $\chi_2'$. Then $s_{\chi'}=1-s_\chi$. Thus, so far,

$$\mathscr{E}(\chi,\Phi)(g) \;=\; \chi_1(g)\int_{\mathbb{J}^+/k^\times}\chi_1\chi_2^{-1}\Big(\Theta(\Phi,tg)-\Phi(0)\Big)\,dt$$

$$+\;\chi_1'(g^\theta)\int_{\mathbb{J}^+/k^\times}\chi_1'\chi_2'^{-1}\Big(\Theta(\widehat{\Phi},tg^\theta)-\widehat{\Phi}(0)\Big)\,dt \;-\; \delta_\chi\Big(\chi_1'(\det g^\theta)\frac{\widehat{\Phi}(0)\kappa/2}{s_{\chi'}}+\chi_1(\det g)\frac{\Phi(0)\kappa/2}{s_\chi}\Big)$$

The visible symmetry under $g\to g^\theta$, $\chi\to\chi'$, $\Phi\to\widehat{\Phi}$ on the right-hand side gives

$$\mathscr{E}(\chi,\Phi)(g) \;=\; \mathscr{E}(\chi',\widehat{\Phi})(g^\theta)$$

This is nice, but for $GL_2$ we need not tolerate the transpose-inverse. Recall that

$$g^\theta \;=\; \frac{w_o^{-1}gw_o}{\det g}$$

Thus, in the region of convergence,

$$\mathscr{E}(\chi', \widehat{\Phi})(g^\theta) \;=\; \chi_1'(\det g^\theta) \int_{\mathbb{J}/k^\times} |t|^2 \chi_1^{-1}\chi_2(t) \sum_{0\neq\xi\in k^2} \widehat{\Phi}(t\xi g^\theta)\, dt$$

$$= \;|\det\big(\frac{g}{\det g}\big)| \cdot \chi_1^{-1}(\det\big(\frac{g}{\det g}\big)) \int_{\mathbb{J}/k^\times} |t|^2 \chi_1^{-1}\chi_2(t) \sum_{0\neq\xi\in k^2} \widehat{\Phi}(t\xi \frac{w_o^{-1} g w_o}{\det g})\, dt$$

$$= \;|\det g| \cdot \chi_2(\det g) \int_{\mathbb{J}/k^\times} |t|^2 \chi_1^{-1}\chi_2(t) \sum_{0\neq\xi\in k^2} \widehat{\Phi}(t\xi w_o^{-1} g w_o)\, dt$$

by replacing $t$ by $t \cdot \det g$. Since $k^2$ is stable under multiplication by $w_o$, this is

$$\mathscr{E}(\chi', \widehat{\Phi})(g^\theta) \;=\; |\det g| \cdot \chi_2(\det g) \int_{\mathbb{J}/k^\times} |t|^2 \chi_1^{-1}\chi_2(t) \sum_{0\neq\xi\in k^2} \widehat{\Phi}(t\xi g w_o)\, dt$$

Thus, with

$$\widehat{\chi}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \;=\; |a|\chi_2(a) \cdot |d|^{-1}\chi_1(d) \;=\; |\frac{a}{d}| \cdot (\chi \circ w_o)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

with corresponding

$$\widehat{\chi}_1(a) \;=\; |a|\chi_2(a) \qquad \text{and} \qquad \widehat{\chi}_2(d) \;=\; |d|^{-1}\chi_1(d)$$

we have

$$\mathscr{E}(\chi', \widehat{\Phi})(g^\theta) \;=\; \mathscr{E}(\widehat{\chi}, \widehat{\Phi} \circ w_o)(g)$$

That is, we have the functional equation

$$\mathscr{E}(\chi, \Phi) \;=\; \mathscr{E}(\widehat{\chi}, \widehat{\Phi} \circ w_o)$$

This completes the discussion. ///

[0.0.2] **Remark**: One should really prove that the half-zeta integrals are entire, as is similarly necessary in Iwasawa-Tate, and already in Riemann.

[0.0.3] **Remark**: More importantly, there remains some sorting-out of the Eisenstein series here in terms of Eisenstein series defined more directly and used in spectral theory. This sorting-out is a slightly more complicated version of the analogous refined choices of Schwartz data to get the best outcome in Iwasawa-Tate.

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