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Iwasawa-Tate on ζ-functions and L-functions

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After a too-brief introduction to *adeles* and *ideles*, we sketch proof of *analytic continuation* and *functional equation* of Riemann's zeta, in the modern form due independently to Iwasawa and Tate about 1950. The sketch is repeated for Dedekind zeta functions of number fields, noting some additional complications. The sketch is repeated again for Hecke's (größencharakter) L-functions, noting further complications.

These sketches are as terse as possible. Thus, many details are suppressed. The goal is to make the largest points clear, and discover details needed to convert sketches to proofs. More sensible descriptions of adeles and ideles, for example, are postponed to another occasion, despite the opacity of the quick-and-easy definitions given here.

One virtue of the Iwasawa-Tate viewpoint is that concern about *units* and *class numbers* evaporates completely, and all number fields are treated in a fashion scarcely different from Riemann's treatment of zeta. Fujisaki's compactness lemma (below) supplants discussion of units and class numbers, in fact, *proving* both Dirichlet's Units Theorem and the finiteness of generalized class numbers as corollaries.

This discussion is the theory of L-functions attached to GL_1. The classical theory of modular forms or automorphic forms is that attached to GL_2. The adelic reformulation of the GL_2 case in [Gelfand-Graev-PS 1969] and [Jacquet-Langlands 1971] demonstrated the technical benefits of the the adelic viewpoint, especially in facilitating the application of *representation theory*. The advantages are even greater for larger groups, as illustrated in [Jacquet-PS-Shalika 1979] for GL_3.

[Tamagawa 1963] extended the Iwasawa-Tate arguments to L-functions associated to finite-dimensional *division algebras* over number fields. [Godement-Jacquet 1972] refined the argument to treat cuspidal automorphic L-functions for GL_n. Adelic *Poisson summation* is a key analytic device in these extensions.

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1. *Simplest case: Riemann's zeta*

We prove *analytic continuation* and *functional equation* for Riemann's zeta from the Iwasawa-Tate viewpoint, to show that the modern argument is completely parallel to Riemann's. Many details are deliberately left for later. As a result, as presented, many aspects are unmotivated or obscure, but this will be remedied eventually.

In particular, the reader is asked to be temporarily indulgent of allusions to objects whose existence, construction, or supporting properties are unclear, and of causal connections whose obviousness or subtlety are apparently ignored.

[1.1] Quick version of p-adics, adeles, ideles The standard collection of *norms* or *absolute values* on $\mathbb{Q}$ consists of the *usual* one, now denoted $|\cdot|_\infty$, and the p-adic norms

$$\left|\frac{a}{b}\cdot p^\ell\right|_p \;=\; p^{-\ell} \qquad \text{(for a prime number } p\text{, with } \ell\in\mathbb{Z}\text{, and } a,b \text{ prime to } p)$$

and corresponding p-adic *metric* $d_p(x,y) = |x-y|_p$. These give the usual *archimedean* topology on $\mathbb{R}$, and p-adic topologies and corresponding completions, the p-adic rational numbers $\mathbb{Q}_p$. To have a uniform notation, write $\mathbb{Q}_\infty = \mathbb{R}$. Let v be an index ranging over $\{\infty, 2, 3, 5, 7, 11, 13, 17, \ldots\}$, so $\mathbb{Q}_v$ can be either $\mathbb{Q}_\infty = \mathbb{R}$ or $\mathbb{Q}_p$ for a prime number p. Say that v is *finite* or *non-archimedean* if it is a prime number; otherwise v is *infinite* or *archimedean*.

For finite v, there is the corresponding *ring of local integers*

$$\mathbb{Z}_v \;=\; \mathbb{Z}_p \;=\; \{x \in \mathbb{Q}_p \;:\; |x|_p \le 1\} \qquad\qquad \text{(for } v \text{ corresponding to prime number } p\text{)}$$

The *adeles* $\mathbb{A}$ of $\mathbb{Q}$ consist of elements $\alpha = \{\alpha_v\} \in \prod_v \mathbb{Q}_v$ such that $\alpha \in \mathbb{Z}_v$ for all but finitely-many places v. The *ideles* $\mathbb{J}$ of $\mathbb{Q}$ consist of elements $\alpha = \{\alpha_v\} \in \prod_v \mathbb{Q}_v^\times$ such that $\alpha \in \mathbb{Z}_v^\times$ for all but finitely-many places v. [1]

[1.2] Schwartz functions

The usual space $\mathscr{S}(\mathbb{R})$ of Schwartz functions on $\mathbb{R}$ consists of the infinitely differentiable functions of rapid decay, with all derivatives also of rapid decay.

For v finite, the space of Schwartz functions $\mathscr{S}(\mathbb{Q}_v)$ consists of compactly supported, locally constant functions on $\mathbb{Q}_v$. A *monomial* Schwartz function f on $\mathbb{A}$ is a function of the form $f(\alpha) = \prod_v f_v(\alpha_v)$ for adeles α, where f_v is the characteristic function of the v-adic integers $\mathbb{Z}_v$ for all but finitely-many places v. The space $\mathscr{S}(\mathbb{A})$ of complex-valued Schwartz functions on $\mathbb{A}$ is the vector space of *finite* linear combinations of monomial Schwartz functions.

[1.3] Global Fourier transform and Poisson summation

Let ψ be a non-trivial continuous *global* character $\psi : \mathbb{A} \to \mathbb{C}^\times$ trivial on the diagonal copy of $\mathbb{Q}$ in $\mathbb{A}$. Thus, ψ gives a non-trivial character on $\mathbb{A}/\mathbb{Q}$. As expected, for an addition-invariant measure dx on $\mathbb{A}$, the global Fourier transform is

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{A}} \overline{\psi}(\xi x)\, f(x)\, dx \qquad\qquad \text{(for } f \in \mathscr{S}(\mathbb{A})\text{)}$$

Normalize the measure so that the total measure of the compact quotient $\mathbb{A}/\mathbb{Q}$ is 1. With such ψ and dx, *Fourier inversion*

$$f(x) \;=\; \int_{\mathbb{A}} \psi(\xi x)\, \widehat{f}(\xi)\, d\xi$$

holds. Adelic *Poisson summation* is

$$\sum_{x \in \mathbb{Q}} f(x) \;=\; \sum_{\xi \in \mathbb{Q}} \widehat{f}(\xi) \qquad\qquad \text{(for } f \in \mathscr{S}(\mathbb{A})\text{)}$$

[1.4] Idele norm, product formula

The *idele norm* $|\cdot| : \mathbb{J} \to \mathbb{C}^\times$ is

$$|\alpha| \;=\; \prod_v |\alpha_v|_v \qquad\qquad \text{(for an idele } \alpha = \{\alpha_v\}\text{)}$$

[1] It is traditional to call the adeles the *restricted direct product* of $\mathbb{Q}_v$'s with respect to the subgroups $\mathbb{Z}_v$, and to call the ideles the *restricted direct product* of $\mathbb{Q}_v^\times$'s with respect to the subgroups $\mathbb{Z}_v^\times$. Obviously, this does not address the reasons for the formation of these objects. Further, most likely no other examples of anything deserving to be called *restricted direct products* are available to anyone learning about adeles for the first time. There are much better descriptions, but less immediate, so we give the easier ones as placeholders.

The *product formula* is the assertion that the idele norm is trivial on the diagonal copy of $\mathbb{Q}^\times$ inside $\mathbb{J}$, that is,

$$1 \;=\; |a| \;=\; \prod_v |a|_v \qquad\qquad (\text{for } a \in \mathbb{Q}^\times)$$

Thus, the idele norm descends to a character $\mathbb{J}/\mathbb{Q}^\times \to \mathbb{C}^\times$.

[1.5] Global zeta integrals

Let $d^\times x$ be a multiplication-invariant measure on $\mathbb{J}$. Define **global zeta integrals**

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\, f(x)\, d^\times x \qquad\qquad (\text{for } f \in \mathscr{S}(\mathbb{A}),\ s \in \mathbb{C},\ \mathrm{Re}\, s > 1)$$

We will see that, for suitable choice of f, the zeta integral is the zeta function with its gamma factor. Just below, we prove that *every* such global zeta integral has a meromorphic continuation with poles at worst at $s = 1, 0$, with predictable residues, with functional equation

$$Z(s,f) \;=\; Z(1-s, \widehat{f}) \qquad\qquad (\text{for arbitrary } f \in \mathscr{S}(\mathbb{A}))$$

Part of the point is that meromorphic continuation and functional equation of $Z(s,f)$ follow for *all* f, without having determined the best choice of Schwartz function f.

[1.6] Euler products and local zeta integrals

Let $d_v^\times x$ be a multiplication-invariant measure on $\mathbb{Q}_v^\times$ such that the product of all these local measures is the global one: $d^\times x = \prod_v d_v^\times x$. For *monomial* Schwartz functions $f = \prod f_v$, for $\mathrm{Re}\, s > 1$, the zeta integral is

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\, f(x)\, d^\times x \;=\; \prod_v \int_{\mathbb{Q}_v^\times} |x|_v^s\, f_v(x)\, d_v^\times x$$

That is, $Z(s,f)$ is an infinite product of *local* integrals. That is, zeta integrals of *monomial* Schwartz functions have *Euler product* expansions, in the region of convergence. This motivates defining *local zeta integrals* to be those local integrals

$$Z_v(s,f_v) \;=\; \int_{\mathbb{Q}_v^\times} |x|_v^s\, f_v(x)\, d_v^\times x$$

Without clarifying the nature of the local integrals, the Euler product assertion is

$$Z(s,f) \;=\; \prod_v Z_v(s,f_v) \qquad\qquad (\text{for } \mathrm{Re}\, s > 1, \text{ with monomial } f = \textstyle\prod_v f_v)$$

[1.7] The usual Euler factors appear

We see later that a reasonable choice for f, with $\widehat{f} = f$ (and with reasonable measures $d_v^\times x$) produces the standard factors:

$$Z_v(s,f_v) \;=\; \begin{cases} \dfrac{1}{1 - \dfrac{1}{p^s}} & (\text{for finite } v \sim p) \\[2ex] \pi^{-\frac{s}{2}}\Gamma\left(\dfrac{s}{2}\right) & (\text{for } v = \infty) \end{cases}$$

That is, for reasonable choices, in this situation,

$$Z(s,f) \;=\; \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

[1.8] Functional equation of a theta function

The analogue of the *theta function* appearing in Riemann's and Hecke's classical arguments is

$$\theta_f(x) \;=\; \sum_{\alpha\in\mathbb{Q}} f(\alpha x) \qquad\qquad (\text{for } x\in\mathbb{J},\ f\in\mathscr{S}(\mathbb{A}))$$

Adelic Poisson summation will give the functional equation of the theta function, after we compute the adelic Fourier transform of $\alpha\to f(\alpha x)$ for $\alpha\in\mathbb{A}$ and fixed $x\in\mathbb{J}$. First, make the obvious change of variables:

$$\int_{\mathbb{A}} \overline{\psi}(\xi\alpha)\; f(\alpha x)\; d\alpha \;=\; \int_{\mathbb{A}} \overline{\psi}(\xi\alpha/x)\; f(\alpha)\; d(\alpha/x)$$

The change-of-measure is the product of the local changes-of-measure, which are

$$d_v^\times(\alpha/x) \;=\; \frac{1}{|x|_v}\; d_v^\times\alpha \qquad\qquad (\text{for } \alpha, x\in\mathbb{Q}_v^\times)$$

Thus, the adelic change of measure is the idele norm, and

$$\int_{\mathbb{A}} \overline{\psi}(\xi\alpha/x)\; f(\alpha)\; d(\alpha/x) \;=\; \frac{1}{|x|}\int_{\mathbb{A}} \overline{\psi}(\xi\alpha/x)\; f(\alpha)\; d\alpha \;=\; \frac{1}{|x|}\widehat{f}\Big(\frac{\xi}{x}\Big)$$

Then Poisson summation gives the functional equation

$$\theta_f(x) \;=\; \sum_{\alpha\in\mathbb{Q}} f(\alpha x) \;=\; \frac{1}{|x|}\sum_{\alpha\in\mathbb{Q}} \widehat{f}\big(\frac{\alpha}{x}\big) \;=\; \frac{1}{|x|}\;\theta_{\widehat{f}}\big(\frac{1}{x}\big)$$

[1.9] Main argument: analytic continuation and functional equation of global zeta integrals

The analytic continuation and functional equation arise from *winding up*, and breaking the integral into two pieces, and applying the functional equation of θ's, as in the classical scenario: let

$$\mathbb{J}^+ \;=\; \{x\in\mathbb{J}: |x|\ge 1\} \qquad \mathbb{J}^- \;=\; \{x\in\mathbb{J}: |x|\le 1\} \qquad \mathbb{J}^1 \;=\; \{x\in\mathbb{J}: |x|=1\}$$

A notation for θ_f with its constant removed will be convenient: let

$$\theta_f^*(x) \;=\; \theta_f(x) \;-\; f(0) \;=\; \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x) \qquad\qquad (\text{for } x\in\mathbb{J} \text{ and } f\in\mathscr{S}(\mathbb{A}))$$

Wind up the zeta integral, use the product formula, and break the integral into two pieces:

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\, f(x)\, d^\times x \;=\; \int_{\mathbb{J}/\mathbb{Q}^\times} \sum_{\alpha\in\mathbb{Q}^\times} |\alpha x|^s\, f(\alpha x)\, d^\times(\alpha x) \;=\; \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x)\, d^\times x$$

$$=\; \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s\,\theta_f^*(x)\, d^\times x \;=\; \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^s\,\theta_f^*(x)\, d^\times x \;+\; \int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s\,\theta_f^*(x)\, d^\times x$$

The integral over $\mathbb{J}^+/\mathbb{Q}^\times$ is *entire*. The functional equation of θ_f will give a transformation of the integral over $\mathbb{J}^-/\mathbb{Q}^\times$ into an integral over $\mathbb{J}^+/\mathbb{Q}^\times$ plus two elementary terms describing the poles. Replace x by $1/x$, and simplify:

$$\int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s\, \theta^*_f(x)\, d^\times x \;=\; \int_{\mathbb{J}^+/\mathbb{Q}^\times} |1/x|^s\, \theta^*_f(1/x)\, d^\times(1/x)$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{-s} \cdot \Big[|x|\theta_{\widehat{f}}(x) - f(0)\Big]\, d^\times x$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{1-s}\, \theta^*_{\widehat{f}}(x)\, d^\times x \;+\; \widehat{f}(0) \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{1-s}\, d^\times x \;-\; f(0) \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{-s}\, d^\times x$$

The integral of $\theta^*_{\widehat{f}}$ over $\mathbb{J}^+/\mathbb{Q}^\times$ is *entire*. The elementary integrals can be evaluated:

$$\int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{1-s}\, d^\times x \;=\; \text{meas}\,(\mathbb{J}^1/\mathbb{Q}^\times) \cdot \int_1^\infty x^{1-s}\, \frac{dx}{x} \;=\; \text{meas}\,(\mathbb{J}^1/\mathbb{Q}^\times) \cdot \frac{1}{s-1}$$

In this case, the natural measure of $\mathbb{J}^1/\mathbb{Q}^\times$ is 1, so

$$Z(s,f) \;=\; \int_{\mathbb{J}^+/\mathbb{Q}^\times} \Big(|x|^s \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x) + |x|^{1-s} \sum_{\alpha\in\mathbb{Q}^\times} \widehat{f}(\alpha x)\Big)\, d^\times x \;+\; \frac{\widehat{f}(0)}{s-1} \;-\; \frac{f(0)}{s}$$

The integral is entire, so the latter expression gives the *analytic continuation*. Further, there is visible symmetry under

$$s \longleftrightarrow 1-s \qquad \text{and} \qquad f \longleftrightarrow \widehat{f}$$

so we have the *functional equation*

$$Z(s,f) \;=\; Z(1-s, \widehat{f})$$

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2. *Dirichlet L-functions*

We adapt the argument to prove *analytic continuation* and *functional equation* for Dirichlet L-functions from the Iwasawa-Tate viewpoint, with many details still suppressed. One should observe how *few* changes are needed.

[2.1] Dirichlet characters as idele-class characters Fix a Dirichlet character χ_d with conductor N. The main adaptation necessary is rewriting χ_d as a character χ on $\mathbb{J}/k^\times$.

Given an idele α, taking advantage of unique factorization in $\mathbb{Z}$, adjust α by $\mathbb{Q}^\times$ to put its local component inside $\mathbb{Z}_v^\times$ at all finite places. Adjust by ± 1 to make the archimedean component *positive*. Thus, an idele-class character is completely determined by its values on

$$U \;=\; \mathbb{R}^+ \cdot \prod_{v<\infty} \mathbb{Z}_v^\times$$

Further, since the diagonal copy of $\mathbb{Q}^\times$ meets U just at $\{1\}$, there is no risk of ill-definedness. Continuity on U implies continuity on $\mathbb{J}$.

At finite places v corresponding to primes p *not* dividing N, we declare χ to be trivial on the local units:

$$\chi(\mathbb{Z}_v^\times) \;=\; 1 \qquad\qquad (\text{for } v \sim p \text{ not dividing } N)$$

For $v \sim p$ with $N = p^e M$ and $p \not| M$, given $x \in \mathbb{Z}_v^\times$, let $n \in \mathbb{Z}$ such that $n = x \bmod p^e\mathbb{Z}_v$, and $n = 1 \bmod M$, and define $\chi(x) = \chi_d(n)$. Say χ is *unramified* at v when $\chi(\mathbb{Z}_v^\times) = 1$. At finite places v where χ is *non-trivial* on the local units, χ is *ramified*.

[2.2] Global zeta integrals

We consider only idele-class characters χ *trivial* on the copy of positive reals inside $\mathbb{J}$. Define **global zeta integrals**

$$Z(s,\chi,f) \;=\; \int_{\mathbb{J}} |x|^s\,\chi(x)\,f(x)\,d^\times x \qquad\qquad (\text{for } f\in\mathscr{S}(\mathbb{A}),\ s\in\mathbb{C},\ \mathrm{Re}\,s>1)$$

We will see that, for suitable choice of f, the zeta integral is the Dedekind zeta function with its gamma factor, except for technical complications at primes at which χ is *ramified*. Just below, we prove that *every* such global zeta integral has a meromorphic continuation with poles at worst at $s=1,0$, with predictable residues, with functional equation

$$Z(s,\chi,f) \;=\; Z(1-s,\chi^{-1},\widehat{f}) \qquad\qquad (\text{for arbitrary } f\in\mathscr{S}(\mathbb{A}))$$

That is, as expected from the classical argument, in the functional equation

$$\chi \;\leftrightarrow\; \chi^{-1} \qquad\qquad s \;\leftrightarrow\; 1-s \qquad\qquad f \;\leftrightarrow\; \widehat{f}$$

[2.3] Euler products and local zeta integrals

For *monomial* Schwartz functions $f=\prod f_v$, for $\mathrm{Re}\,s>1$, the zeta integral is

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\,\chi(x)\,f(x)\,d^\times x \;=\; \prod_v \int_{k_v^\times} |x|_v^s\,\chi_v(x)\,f_v(x)\,d_v^\times x$$

where χ_v is the restriction of χ to $\mathbb{Q}_v^\times$. That is, $Z(s,f)$ is an infinite product of *local* integrals. That is, zeta integrals of *monomial* Schwartz functions have *Euler product* expansions, in the region of convergence. This motivates defining *local zeta integrals* to be those local integrals

$$Z_v(s,\chi_v,f_v) \;=\; \int_{k_v^\times} |x|_v^s\,\chi_v(x)\;f_v(x)\;d_v^\times x$$

Without clarifying the nature of the local integrals, the Euler product assertion is

$$Z(s,f) \;=\; \prod_v Z_v(s,\chi_v,f_v) \qquad\qquad (\text{for } \mathrm{Re}\,s>1,\text{ with monomial } f=\textstyle\prod_v f_v)$$

[2.4] The usual Euler factors, with a complication

We see later that a reasonable choice of f (and measures $d_v^\times x$) produces the standard factors:

$$Z_v(s,\chi_v,f_v) \;=\; \begin{cases} \dfrac{1}{1-\dfrac{\chi(p)}{p^s}} & (\text{for } v\sim p,\ p\not|N) \\[3ex] \pi^{-\frac{s}{2}}\Gamma\left(\dfrac{s}{2}\right) & (\text{for } v=\mathbb{R} \text{ and } \chi_d(-1)=1) \\[3ex] \pi^{-\frac{s+1}{2}}\Gamma\left(\dfrac{s+1}{2}\right) & (\text{for } v=\mathbb{R} \text{ and } \chi_d(-1)=-1) \end{cases}$$

However, there is a complication due to finite $v\sim$ with $p|N$, namely, that typically there is no choice of Schwartz function f to recover the factor $N^{-s/2}$ appearing in the functional equations

$$N^{s/2}\,\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,L(s,\chi) \;=\; \varepsilon(\chi)\cdot N^{(1-s)/2}\,\pi^{-(1-s)/2}\,\Gamma(\tfrac{1-s}{2})\,L(1-s,\chi^{-1}) \quad (\text{for } \chi \text{ even})$$

$$N^{\frac{s}{2}}\,\pi^{-\frac{(s+1)}{2}}\,\Gamma(\tfrac{s+1}{2})\,L(s,\chi) \;=\; \varepsilon(\chi)\cdot N^{\frac{(1-s)}{2}}\,\pi^{-\frac{(2-s)}{2}}\,\Gamma(\tfrac{2-s}{2})\,L(1-s,\chi^{-1}) \qquad (\text{for } \chi \text{ odd})$$

$$\text{with} \qquad |\varepsilon(\chi)| \;=\; 1, \qquad \varepsilon(\chi)\cdot\varepsilon(\chi^{-1}) \;=\; 1$$

Nevertheless, a reasonable choice will produce $Z(s,\chi,f)$ and $Z(s,\chi^{-1},\widehat{f})$ such that, letting $\Lambda(s,\chi)$ be the L-function with its gamma factor and *with* factor of $N^{s/2}$,

$$Z(s,\chi,f) \;=\; N^{-s/2}\cdot\Lambda(s,\chi) \qquad\text{and}\qquad Z(1-s,\chi^{-1},\widehat{f}) \;=\; \varepsilon\cdot N^{-s/2}\cdot\Lambda(1-s,\chi^{-1})$$

with $|\varepsilon|=1$. Thus, from $Z(s,\chi,f)=Z(1-s,\chi^{-1},f)$ the symmetrical functional equation can be obtained.

[2.5] Functional equation of a theta function

The *theta function* attached to a Schwartz function is

$$\theta_f(x) \;=\; \sum_{\alpha\in k} f(\alpha x) \qquad\qquad (\text{for } x\in\mathbb{J},\ f\in\mathscr{S}(\mathbb{A}))$$

and Poisson summation gives the functional equation

$$\theta_f(x) \;=\; \sum_{\alpha\in k} f(\alpha x) \;=\; \frac{1}{|x|}\sum_{\alpha\in k}\widehat{f}\Big(\frac{\alpha}{x}\Big) \;=\; \frac{1}{|x|}\,\theta_{\widehat{f}}\Big(\frac{1}{x}\Big)$$

[2.6] Main argument: analytic continuation and functional equation of global zeta integrals

The analytic continuation and functional equation arise from *winding up*, and breaking the integral into two pieces, and applying the functional equation of θ's, as in the classical scenario. For non-trivial χ, the Schwartz function f can be taken so that

$$f(0) \;=\; 0 \qquad\quad (\text{and})\widehat{f}(0) \;=\; 0$$

relieving us of tracking those values, and giving the simpler presentation

$$\theta_f(x) \;=\; \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x) \qquad\qquad (\text{for } x\in\mathbb{J} \text{ and } f\in\mathscr{S}(\mathbb{A}))$$

Wind up the zeta integral, use the product formula and $\mathbb{Q}^\times$-invariance of χ, and break the integral into two pieces:

$$Z(s,\chi,f) \;=\; \int_{\mathbb{J}} |x|^s\,\chi(x)\,f(x)\,d^\times x \;=\; \int_{\mathbb{J}/\mathbb{Q}^\times}\sum_{\alpha\in k^\times} |\alpha x|^s\,\chi(\alpha x)\,f(\alpha x)\,d^\times(\alpha x)$$

$$= \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s\,\chi(x)\sum_{\alpha\in k^\times} f(\alpha x)\,d^\times x \;=\; \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s\,\chi(x)\,\theta_f(x)\,d^\times x$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^s\,\chi(x)\,\theta_f(x)\,d^\times x \;+\; \int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s\,\chi(x)\,\theta_f(x)\,d^\times x$$

The integral over $\mathbb{J}^+/\mathbb{Q}^\times$ is *entire*. The functional equation of θ_f will give a transformation of the integral over $\mathbb{J}^-/\mathbb{Q}^\times$ into an integral over $\mathbb{J}^+/\mathbb{Q}^\times$. Replace x by $1/x$, and simplify:

$$\int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s\,\chi(x)\,\theta_f(x)\,d^\times x \;=\; \int_{\mathbb{J}^+/\mathbb{Q}^\times} |1/x|^s\,\chi(1/x)\,\theta_f(1/x)\,d^\times(1/x)$$

$$= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{-s}\,\chi^{-1}(x)\cdot|x|\theta_{\widehat{f}}(x)\,d^\times x \;=\; \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{1-s}\,\chi^{-1}(x)\,\theta_{\widehat{f}}(x)\,d^\times x$$

The integral of $\theta_{\widehat{f}}$ over $\mathbb{J}^+/\mathbb{Q}^\times$ is *entire*. Thus,

$$Z(s,\chi,f) \;=\; \int_{\mathbb{J}^+/\mathbb{Q}^\times}\Big(|x|^s\,\chi(x)\sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x) + |x|^{1-s}\,\chi^{-1}(x)\sum_{\alpha\in\mathbb{Q}^\times}\widehat{f}(\alpha x)\Big)\,d^\times x$$

The integral is entire, and gives the *analytic continuation.* Further, there is visible symmetry under

$$\chi \leftrightarrow \chi^{-1} \qquad\qquad s \leftrightarrow 1-s \qquad\qquad f \leftrightarrow \widehat{f}$$

Thus, we have the *functional equation*

$$Z(s,\chi,f) \;=\; Z(1-s,\chi^{-1},\widehat{f})$$

[2.6.1] Remark: There was no compulsion to track of $|x|^s$ and $\chi(x)$ separately in the above argument. Thus, we could rewrite the above to treat an *arbitrary* χ on $\mathbb{J}/\mathbb{Q}^\times$, define

$$Z(\chi,f) \;=\; \int_{\mathbb{J}} \chi(x)\, f(x)\, d^\times x$$

and obtain the slightly cleaner functional equation

$$Z(\chi,f) \;=\; Z(|.|\,\chi^{-1},\widehat{f})$$

That is, rather than $s \to 1-s$ and $\chi \to \chi^{-1}$, simply replace χ by $x \to |x|\cdot\chi^{-1}(x)$.

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3. *Dedekind zetas of number fields*

We run through the argument again, proving *analytic continuation* and *functional equation* for Dedekind zeta functions of number fields from the Iwasawa-Tate viewpoint, noting changes from the rational integers, but with many details still suppressed. At the same time, one should observe how *few* changes are needed.

[3.1] Quick version of v-adics, adeles, ideles

Let k be an algebraic number field with algebraic integers $\mathfrak{o}$. Let v index the completions k_v of k, and for non-archimedean v the local ring of integers

$$\mathfrak{o}_v \;=\; \{x \in k_v \;:\; |x|_v \le 1\}$$

The *adeles* $\mathbb{A} = \mathbb{A}_k$ of k consist of elements $\alpha = \{\alpha_v\} \in \prod_v k_v$ such that $\alpha \in \mathfrak{o}_v$ for all but finitely-many places v. The *ideles* $\mathbb{J} = \mathbb{J}_k$ of k consist of elements $\alpha = \{\alpha_v\} \in \prod_v k_v^\times$ such that $\alpha \in \mathfrak{o}_v^\times$ for all but finitely-many places v.

[3.2] Schwartz functions

The spaces $\mathscr{S}(\mathbb{R})$ and $\mathscr{S}(\mathbb{C})$ of Schwartz functions on $\mathbb{R}$ and $\mathbb{C}$ consists of the infinitely differentiable functions of rapid decay, with all derivatives also of rapid decay.

For v non-archimedean, the space of Schwartz functions $\mathscr{S}(k_v)$ consists of compactly supported, locally constant functions on k_v. A *monomial* Schwartz function f on $\mathbb{A}$ is a function of the form $f(\alpha) = \prod_v f_v(\alpha_v)$ for adeles α, where f_v is the characteristic function of the v-adic integers $\mathfrak{o}_v$ for all but finitely-many places v. The space $\mathscr{S}(\mathbb{A})$ of complex-valued Schwartz functions on $\mathbb{A}$ is the vector space of *finite* linear combinations of monomial Schwartz functions.

[3.3] Global Fourier transform and Poisson summation

Let ψ be a non-trivial continuous *global* character $\psi : \mathbb{A} \to \mathbb{C}^\times$ trivial on the diagonal copy of k in $\mathbb{A}$. Thus, ψ gives a non-trivial character on $\mathbb{A}/k$. For an addition-invariant measure dx on $\mathbb{A}$, the global Fourier transform is

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{A}} \overline{\psi}(\xi x)\, f(x)\, dx \qquad\qquad (\text{for } f \in \mathscr{S}(\mathbb{A}))$$

Normalize the measure so that the total measure of the compact quotient $\mathbb{A}/k$ is 1. With such ψ and dx, *Fourier inversion*

$$f(x) \;=\; \int_{\mathbb{A}} \psi(\xi x)\,\widehat{f}(\xi)\,d\xi$$

holds. Adelic *Poisson summation* is

$$\sum_{x\in k} f(x) \;=\; \sum_{\xi\in k} \widehat{f}(\xi) \qquad\qquad (\text{for } f\in\mathscr{S}(\mathbb{A}))$$

[3.4] Idele norm, product formula

Each completion k_v is a *finite* extension of a completion $\mathbb{Q}_w$ of $\mathbb{Q}$. Normalize the norm $|\cdot|_v$ on k_v by

$$|\alpha|_v \;=\; |\mathrm{Norm}^{k_v}_{\mathbb{Q}_w}\,\alpha|_w$$

with $|\cdot|_w$ the standard norm on $\mathbb{Q}_w$. The *idele norm* $|\cdot|:\mathbb{J}\to\mathbb{C}^\times$ is

$$|\alpha| \;=\; \prod_v |\alpha_v|_v \qquad\qquad (\text{for an idele } \alpha=\{\alpha_v\})$$

The *product formula* asserts that the idele norm is trivial on the diagonal copy of $k^\times$ inside $\mathbb{J}$, that is,

$$1 \;=\; |a| \;=\; \prod_v |a|_v \qquad\qquad (\text{for } a\in k^\times)$$

Thus, the idele norm descends to a character $\mathbb{J}/k^\times\to\mathbb{C}^\times$.

[3.5] Global zeta integrals

Let $d^\times x$ be a multiplication-invariant measure on $\mathbb{J}$. Define **global zeta integrals**

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\,f(x)\,d^\times x \qquad\qquad (\text{for } f\in\mathscr{S}(\mathbb{A}),\ s\in\mathbb{C},\ \mathrm{Re}\,s>1)$$

We will see that, for suitable choice of f, the zeta integral is the Dedekind zeta function with its gamma factor. Just below, we prove that *every* such global zeta integral has a meromorphic continuation with poles at worst at $s=1,0$, with predictable residues, with functional equation

$$Z(s,f) \;=\; Z(1-s,\widehat{f}) \qquad\qquad (\text{for arbitrary } f\in\mathscr{S}(\mathbb{A}))$$

[3.6] Euler products and local zeta integrals

Let $d_v^\times x$ be a multiplication-invariant measure on $k_v^\times$ such that the product of all these local measures is the global one: $d^\times x=\prod_v d_v^\times x$. For *monomial* Schwartz functions $f=\prod f_v$, for $\mathrm{Re}\,s>1$, the zeta integral is

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\,f(x)\,d^\times x \;=\; \prod_v \int_{k_v^\times} |x|_v^s\,f_v(x)\,d_v^\times x$$

That is, $Z(s,f)$ is an infinite product of *local* integrals. That is, zeta integrals of *monomial* Schwartz functions have *Euler product* expansions, in the region of convergence. This motivates defining *local zeta integrals* to be those local integrals

$$Z_v(s,f_v) \;=\; \int_{k_v^\times} |x|_v^s\,f_v(x)\,d_v^\times x$$

Without clarifying the nature of the local integrals, the Euler product assertion is

$$Z(s,f) \;=\; \prod_v Z_v(s,f_v) \qquad\qquad \text{(for Re } s>1\text{, with monomial } f=\textstyle\prod_v f_v)$$

[3.7] The usual Euler factors, with a complication

We see later that a reasonable choice of f (and measures $d_v^\times x$) produces the standard factors: with q_v the cardinality of the residue field for non-archimedean v,

$$Z_v(s,f_v) \;=\; \begin{cases} \dfrac{1}{1-\dfrac{1}{q_v^s}} & \text{(for } k_v \text{ unramified over } \mathbb{Q}_w) \\[2ex] \pi^{-\frac{s}{2}}\Gamma\left(\dfrac{s}{2}\right) & \text{(for } v\approx\mathbb{R}) \\[2ex] (2\pi)^{-s}\Gamma(s) & \text{(for } v\approx\mathbb{C}) \end{cases}$$

However, there is a complication due to finite v with $k_v/\mathbb{Q}_w$ *ramified.* The Dedekind zeta function of k is

$$\zeta_k(s) \;=\; \prod_{v<\infty} \frac{1}{1-\dfrac{1}{q_v^s}}$$

Let

$$\Gamma_{\mathbb{R}}(s) \;=\; \pi^{-\frac{s}{2}}\,\Gamma\big(\frac{s}{2}\big) \qquad\qquad \Gamma_{\mathbb{C}}(s) \;=\; (2\pi)^{-s}\,\Gamma(s)$$

and let r_1,r_2 be the number or real and complex places, respectively. Hecke found that the functional equation of the Dedekind zeta function $\zeta_k(s)$ involves the *discriminant* D_k of $\mathfrak{o}_k$ over $\mathbb{Z}$: it has a symmetrical form

$$\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{R}}(s)^{r_2}\cdot|D_k|^{-\frac{s}{2}}\cdot\zeta_k(s) \;=\; \Gamma_{\mathbb{R}}(1-s)^{r_1}\Gamma_{\mathbb{R}}(1-s)^{r_2}\cdot|D_k|^{-\frac{1-s}{2}}\cdot\zeta_k(1-s)$$

The factor $|D_k|^{-\frac{s}{2}}$ *can* be obtained as a product of local contributions, as follows. The absolute value of the discriminant is the ideal-norm of the *different*, which is essentially the product of *local* differents. The local different at v is the *inverse* of the fractional ideal

$$\mathfrak{o}_v^* \;=\; \{x\in k_v \;:\; \text{Trace}_{\mathbb{Q}_w}^{k_v}\, xy\in\mathbb{Z}_w,\ \text{for all } y\in\mathfrak{o}_v\}$$

Thus, the local factor at ramified v to have the functional equation should be

$$\frac{[\mathfrak{o}_v^*:\mathfrak{o}_v]^{-\frac{s}{2}}}{1-\dfrac{1}{q_v^s}}$$

However, typically, there is no choice of f or local component f_v to produce this Euler factor as a local zeta integral! In fact, typically, there is no choice of f such that $\widehat{f}=f$, because, typically, at ramified v there is no $f_v\in\mathscr{S}(k_v)$ with $\widehat{f}_v=f_v$. That is, there is no choice of Schwartz function to make the local zeta functions $Z_v(s,f_v)$ and $Z_v(s,\widehat{f}_v)$ the same. That is, while the expected functional equation

$$Z(s,f) \;=\; Z(1-s,\widehat{f})$$

will still hold, there is simply no choice of f to make the functional equation obviously relate a zeta integral to itself.

However, there are other options. A reasonable choice of $f = \prod_v f_v$ will produce the expected factors at archimedean and unramified finite places, and at ramified finite v will produce

$$Z_v(s, f_v) \;=\; \frac{[\mathfrak{o}_v^* : \mathfrak{o}_v]^{-\frac{1}{2}}}{1 - \dfrac{1}{q^s}} \qquad\qquad Z_v(s, \widehat{f}_v) \;=\; \frac{[\mathfrak{o}_v^* : \mathfrak{o}_v]^{s-\frac{1}{2}}}{1 - \dfrac{1}{q^s}}$$

Thus,

$$Z(s, f) \;=\; |D_k|^{\frac{1}{2}} \cdot \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{R}}(s)^{r_2} \cdot \zeta_k(s) \qquad\qquad Z(s, \widehat{f}) \;=\; |D_k|^{s-\frac{1}{2}} \cdot \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{R}}(s)^{r_2} \cdot \zeta_k(s)$$

From $Z(s, f) = Z(1-s, \widehat{f})$,

$$|D_k|^{\frac{1}{2}} \cdot \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{R}}(s)^{r_2}\, \zeta_k(s) \;=\; |D_k|^{(1-s)-\frac{1}{2}} \cdot \Gamma_{\mathbb{R}}(1-s)^{r_1} \Gamma_{\mathbb{R}}(1-s)^{r_2} \cdot \zeta_k(1-s)$$

and divide through by $|D_k|^{s/2}$ to obtain the symmetrical form of the functional equation for $\zeta_k(s)$.

[3.7.1] Remark: The asymmetry in zeta integrals cannot be avoided, in general. Thus, zeta *functions*, including optimized gamma factors and powers of discriminants, are *not exactly* given by zeta *integrals*. Nevertheless, the zeta integrals *are* inevitably correct at all but finitely-many places.

[3.8] Functional equation of a theta function

The analogue of the *theta function* appearing in Riemann's and Hecke's classical arguments is

$$\theta_f(x) \;=\; \sum_{\alpha \in k} f(\alpha x) \qquad\qquad (\text{for } x \in \mathbb{J},\ f \in \mathscr{S}(\mathbb{A}))$$

Adelic Poisson summation will give the functional equation of the theta function, after we compute the adelic Fourier transform of $\alpha \to f(\alpha x)$ for $\alpha \in \mathbb{A}$ and fixed $x \in \mathbb{J}$. First, make the obvious change of variables:

$$\int_{\mathbb{A}} \overline{\psi}(\xi\alpha)\; f(\alpha x)\; d\alpha \;=\; \int_{\mathbb{A}} \overline{\psi}(\xi\alpha/x)\; f(\alpha)\; d(\alpha/x)$$

The change-of-measure is the product of the local changes-of-measure, which are

$$d_v^{\times}(\alpha/x) \;=\; \frac{1}{|x|_v}\; d_v^{\times}\alpha \qquad\qquad (\text{for } \alpha, x \in k_v^{\times})$$

Thus, the adelic change of measure is the idele norm, and

$$\int_{\mathbb{A}} \overline{\psi}(\xi\alpha/x)\; f(\alpha)\; d(\alpha/x) \;=\; \frac{1}{|x|} \int_{\mathbb{A}} \overline{\psi}(\xi\alpha/x)\; f(\alpha)\; d\alpha \;=\; \frac{1}{|x|} \widehat{f}\Big(\frac{\xi}{x}\Big)$$

Then Poisson summation gives the functional equation

$$\theta_f(x) \;=\; \sum_{\alpha \in k} f(\alpha x) \;=\; \frac{1}{|x|} \sum_{\alpha \in k} \widehat{f}\big(\frac{\alpha}{x}\big) \;=\; \frac{1}{|x|}\; \theta_{\widehat{f}}\big(\frac{1}{x}\big)$$

[3.9] Main argument: analytic continuation and functional equation of global zeta integrals

The analytic continuation and functional equation arise from *winding up*, and breaking the integral into two pieces, and applying the functional equation of θ's, as in the classical scenario: let

$$\mathbb{J}^+ \;=\; \{x \in \mathbb{J} : |x| \geq 1\} \qquad\quad \mathbb{J}^- \;=\; \{x \in \mathbb{J} : |x| \leq 1\} \qquad\quad \mathbb{J}^1 \;=\; \{x \in \mathbb{J} : |x| = 1\}$$

A notation for θ_f with its constant removed will be convenient: let

$$\theta^*_f(x) \;=\; \theta_f(x) \;-\; f(0) \;=\; \sum_{\alpha\in k^\times} f(\alpha x) \qquad\qquad \text{(for } x\in\mathbb{J} \text{ and } f\in\mathscr{S}(\mathbb{A}))$$

Wind up the zeta integral, use the product formula, and break the integral into two pieces:

$$Z(s,f) \;=\; \int_{\mathbb{J}} |x|^s\, f(x)\, d^\times x \;=\; \int_{\mathbb{J}/k^\times} \sum_{\alpha\in k^\times} |\alpha x|^s\, f(\alpha x)\, d^\times(\alpha x) \;=\; \int_{\mathbb{J}/k^\times} |x|^s \sum_{\alpha\in k^\times} f(\alpha x)\, d^\times x$$

$$= \int_{\mathbb{J}/k^\times} |x|^s\, \theta^*_f(x)\, d^\times x \;=\; \int_{\mathbb{J}^+/k^\times} |x|^s\, \theta^*_f(x)\, d^\times x \;+\; \int_{\mathbb{J}^-/k^\times} |x|^s\, \theta^*_f(x)\, d^\times x$$

The integral over $\mathbb{J}^+/k^\times$ is *entire*. The functional equation of θ_f will give a transformation of the integral over $\mathbb{J}^-/k^\times$ into an integral over $\mathbb{J}^+/k^\times$ plus two elementary terms describing the poles. Replace x by $1/x$, and simplify:

$$\int_{\mathbb{J}^-/k^\times} |x|^s\; \theta^*_f(x)\, d^\times x \;=\; \int_{\mathbb{J}^+/k^\times} |1/x|^s\; \theta^*_f(1/x)\, d^\times(1/x)$$

$$= \int_{\mathbb{J}^+/k^\times} |x|^{-s}\cdot \Big[|x|\theta_{\widehat{f}}(x) - f(0)\Big]\, d^\times x$$

$$= \int_{\mathbb{J}^+/k^\times} |x|^{1-s}\; \theta^*_{\widehat{f}}(x)\, d^\times x \;+\; \widehat{f}(0) \int_{\mathbb{J}^+/k^\times} |x|^{1-s}\, d^\times x \;-\; f(0) \int_{\mathbb{J}^+/k^\times} |x|^{-s}\, d^\times x$$

The integral of $\theta^*_{\widehat{f}}$ over $\mathbb{J}^+/k^\times$ is *entire*. The elementary integrals can be evaluated:

$$\int_{\mathbb{J}^+/k^\times} |x|^{1-s}\, d^\times x \;=\; \text{meas}\,(\mathbb{J}^1/k^\times)\cdot \int_1^\infty x^{1-s}\, \frac{dx}{x} \;=\; \text{meas}\,(\mathbb{J}^1/k^\times)\cdot \frac{1}{s-1}$$

In this case, the natural measure of $\mathbb{J}^1/k^\times$ is

$$\text{meas}\,(\mathbb{J}^1/k^\times) \;=\; \frac{2^{r_1}\,(2\pi)^{r_2}\,h\,R}{|D_k|^{\frac{1}{2}}\,w}$$

where r_1, r_2 are the numbers of real and complex places, respectively, h is the class number of $\mathfrak{o}$, R is the *regulator*, D_k is the *discriminant*, and w is the number of roots of unity in k. Thus,

$$Z(s,f) \;=\; \int_{\mathbb{J}^+/k^\times} \Big(|x|^s \sum_{\alpha\in k^\times} f(\alpha x) + |x|^{1-s} \sum_{\alpha\in k^\times} \widehat{f}(\alpha x)\Big)\, d^\times x \;+\; \text{meas}\,(J^1/k^\times)\cdot \Big(\frac{\widehat{f}(0)}{s-1} \;-\; \frac{f(0)}{s}\Big)$$

The integral is entire, so the latter expression gives the *analytic continuation*. Further, there is visible symmetry under

$$s \longleftrightarrow 1-s \qquad\quad \text{and} \qquad\quad f \longleftrightarrow \widehat{f}$$

so we have the *functional equation*

$$Z(s,f) \;=\; Z(1-s,\widehat{f})$$

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4. *General case: Hecke L-functions*

Let χ be a character on the idele class group $\mathbb{J}/k^\times$ of k, trivial on the diagonal copy of $\mathbb{R}^+ = (0,+\infty)$ in archimedean factors inside $\mathbb{J}$. In particular,

$$\mathbb{J}/k^\times \;\approx\; \mathbb{J}^1/k^\times \;\times\; \mathbb{R}^+$$

and $|x|^s$ is trivial on $\mathbb{J}^1/k^\times$. Let f be a Schwartz function on the adeles $\mathbb{A}$ of a number field k. The Iwasawa-Tate *global zeta integral* is

$$Z(s,\chi,f) \;=\; \int_{\mathbb{J}} |x|^s\,\chi(x)\,f(x)\,d^\times x$$

for Haar measure $d^\times x$ on $\mathbb{J}$. Let

$$\kappa \;=\; \text{meas}\,(\mathbb{J}^1/k^\times) \;=\; \frac{2^{r_1}\,(2\pi)^{r_2}\,h\,R}{|D_k|^{\frac{1}{2}}\,w}$$

[4.0.1] Theorem: The zeta integral has a *meromorphic continuation* in s to a meromorphic function on $\mathbb{C}$, with poles at most at $s=0$ and $s=1$, with respective residues

$$\text{Res}_{s=1} Z(s,\chi,f) \;=\; \begin{cases} \kappa\cdot\widehat{f}(0) & \text{(for } \chi \text{ trivial)} \\ 0 & \text{(for } \chi \text{ non-trivial)} \end{cases}$$

$$\text{Res}_{s=0} Z(s,\chi,f) \;=\; \begin{cases} \kappa\cdot f(0) & \text{(for } \chi \text{ trivial)} \\ 0 & \text{(for } \chi \text{ non-trivial)} \end{cases}$$

There is the *functional equation*

$$Z(s,\chi,f) \;=\; Z(1-s,\chi^{-1},\widehat{f})$$

For *monomial* Schwartz functions $f=\prod f_v$, for Re $s>1$, the zeta integral has an Euler product

$$Z(s,\chi,f) \;=\; \int_{\mathbb{J}} |x|^s\,\chi(x)\,f(x)\,d^\times x \;=\; \prod_v \int_{k_v^\times} |x|_v^s\,\chi_v(x)\,f_v(x)\,d_v^\times x$$

where χ_v is the restriction of χ to $k_v^\times$.

Proof: The *theta function* attached to a Schwartz function is

$$\theta_f(x) \;=\; \sum_{\alpha\in k} f(\alpha x) \qquad\qquad \text{(for } x\in\mathbb{J},\ f\in\mathscr{S}(\mathbb{A})\text{)}$$

and Poisson summation gives the functional equation

$$\theta_f(x) \;=\; \sum_{\alpha\in k} f(\alpha x) \;=\; \frac{1}{|x|}\sum_{\alpha\in k}\widehat{f}\Big(\frac{\alpha}{x}\Big) \;=\; \frac{1}{|x|}\,\theta_{\widehat{f}}\Big(\frac{1}{x}\Big)$$

The analytic continuation and functional equation arise from *winding up*, breaking the integral into two pieces, and applying the functional equation of θs. Let

$$\theta_f^*(x) \;=\; \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x) \;=\; \theta_f(x)-f(0) \qquad\qquad \text{(for } x\in\mathbb{J} \text{ and } f\in\mathscr{S}(\mathbb{A})\text{)}$$

Computing directly,

$$Z(s,\chi,f) \;=\; \int_{\mathbb{J}} |x|^s\,\chi(x)\,f(x)\,d^\times x \;=\; \int_{\mathbb{J}/k^\times}\sum_{\alpha\in k^\times} |\alpha x|^s\,\chi(\alpha x)\,f(\alpha x)\,d^\times(\alpha x)$$

$$=\; \int_{\mathbb{J}/k^\times} |x|^s\,\chi(x)\sum_{\alpha\in k^\times} f(\alpha x)\,d^\times x \;=\; \int_{\mathbb{J}/k^\times} |x|^s\,\chi(x)\,\theta_f^*(x)\,d^\times x$$

$$=\; \int_{\mathbb{J}^+/k^\times} |x|^s\,\chi(x)\,\theta_f^*(x)\,d^\times x \;+\; \int_{\mathbb{J}^-/k^\times} |x|^s\,\chi(x)\,\theta_f^*(x)\,d^\times x$$

The integral over $\mathbb{J}^+/k^\times$ is *entire*. The functional equation of θ_f will give a transformation of the integral over $\mathbb{J}^-/k^\times$ into an integral over $\mathbb{J}^+/k^\times$ plus two elementary terms describing the poles. Replace x by $1/x$, and simplify:

$$\int_{\mathbb{J}^-/k^\times} |x|^s\,\chi(x)\,\theta_f^*(x)\,d^\times x \;=\; \int_{\mathbb{J}^+/k^\times} |1/x|^s\,\chi(1/x)\,\theta_f^*(1/x)\,d^\times(1/x)$$

$$= \int_{\mathbb{J}^+/k^\times} |x|^{-s}\,\chi^{-1}(x)\cdot\Big[|x|\theta_{\widehat{f}}(x) - f(0)\Big]\,d^\times x$$

$$= \int_{\mathbb{J}^+/k^\times} |x|^{1-s}\,\chi^{-1}(x)\,\theta_{\widehat{f}}^*(x)\,d^\times x \;+\; \widehat{f}(0)\int_{\mathbb{J}^+/k^\times} |x|^s\,\chi(x)\,d^\times x \;-\; f(0)\int_{\mathbb{J}^+/k^\times} |x|^{1-s}\,\chi^{-1}(x)\,d^\times x$$

The last two integrals are elementary:

$$\widehat{f}(0)\int_{\mathbb{J}^+/k^\times} |x|^s\,\chi(x)\,d^\times x \;-\; f(0)\int_{\mathbb{J}^+/k^\times} |x|^{1-s}\,\chi^{-1}(x)\,d^\times x \;=\; \frac{\kappa\,\widehat{f}(0)}{s-1} \;-\; \frac{\kappa\,f(0)}{s}$$

The integral of $\theta_{\widehat{f}}$ over $\mathbb{J}^+/k^\times$ is *entire*. Thus,

$$Z(s,\chi,f) \;=\; \int_{\mathbb{J}^+/\mathbb{Q}^\times} \Big(|x|^s\,\chi(x)\,\theta_f^*(x) + |x|^{1-s}\,\chi^{-1}(x)\theta_{\widehat{f}}^*(x)\Big)\,d^\times x \;+\; \frac{\kappa\,\widehat{f}(0)}{s-1} \;-\; \frac{\kappa\,f(0)}{s}$$

The integral is entire, and gives the *analytic continuation*. Further, there is visible symmetry under

$$\chi \;\leftrightarrow\; \chi^{-1} \qquad\qquad s \;\leftrightarrow\; 1-s \qquad\qquad f \;\leftrightarrow\; \widehat{f}$$

Thus, we have the *functional equation*

$$Z(s,\chi,f) \;=\; Z(1-s,\chi^{-1},\widehat{f})$$

This proves the analytic continuation and functional equation. ///

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5. *A few details acknowledged*

We retrace the argument, noting what needs eventual justification, and elaborating.

Dirichlet characters, ideal-class characters, and, generally, Hecke's Grössencharakters all need to be redescribed as idele-class characters.

One should not worry too much about the explicit form of invariant measures on $\mathbb{J}$ and $k_v^\times$.

Once a single non-trivial additive character ψ_v is chosen, local fields k_v are isomorphic to their own unitary duals. The same is true of $\mathbb{A}$.

Notions of Schwartz functions on $\mathbb{R}$ and $\mathbb{C}$ are well-documented and reasonably well-known, but the p-adic versions deserve amplification.

After a discussion of characters ψ on $\mathbb{A}/k$ as well as ψ_v on k_v, Fourier transforms should be shown to stabilize the *local* Schwartz spaces. Local and global Fourier inversion can be proved without direct mention of the self-duality of local fields k_v and adeles $\mathbb{A}$, and makes a different point. Fourier inversion for non-archimedean local fields is much easier than for $\mathbb{R}$.

One should certify the absolute convergence in Re $s > 1$ of the global zeta integrals

$$Z(s,\chi,f) \;=\; \int_{\mathbb{J}} |x|^s\,\chi(x)\,f(x)\,d^\times x \qquad\qquad (\text{for } f\in\mathscr{S}(\mathbb{A}),\ s\in\mathbb{C},\ \text{Re } s>1)$$

The good-prime finite-prime part of any of Hecke's L-functions looks like

$$L^S(s,\chi) \;=\; \prod_{v\not\in S} \frac{1}{1-q_v^{-(s+it_v)}}$$

where S is a finite set of places, including archimedean ones, to be excluded, and the real number t_v is determined by χ_v. It is relatively straightforward to produce these local factors as local zeta integrals. However, the global zeta integrals inevitably contain bad-prime factors, and, even when f is chosen so that for simple reasons $Z(s,\chi,f)$ is well behaved at bad primes, the adelic Fourier transform $\widehat{f}$ usually frustrates a similar trivial analysis. Thus, local zeta integrals

$$Z_v(s,\chi_v,f_v) \;=\; \int_{k_v^\times} |x|_v^s\,\chi_v(x)\;f_v(x)\;d_v^\times x$$

with *arbitrary* data f_v must be proven meromorphic, or else meromorphic continuation of $L^S(s,\chi)$ does not follow! We do such a computation in the next section.

Poisson summation is not trivial. Indeed, the *classical* form of Poisson summation depends on pointwise convergence of (classical) Fourier series, which is less trivial than one might imagine. Nevertheless, the adelic form, its consequence for functional equations of theta functions, and consequences for functional equations of zeta integrals, provides a viewpoint that completely avoids concerns over congruence conditions and ideal classes.

Fujisaki's compactness lemma is non-trivial, and all the more mysterious since its proof is unexpected. We give this below.

In the proof of analytic continuation (and functional equation), one really should verify that the half-zeta integrals, that is, over ideles of norm ≥ 1, really *are* nicely convergent for all $s\in\mathbb{C}$, *and* that this entails entire-ness. Factoring the elementary integrals (for trivial χ) over the product of $\mathbb{J}^1/k^\times$ and the ray $\mathbb{R}^+$ is less serious. We give a simple argument below.

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6. *Local functional equations*

The first point of *local functional equations* is to prove that local zeta integrals with any Schwartz data whatsoever are meromorphic functions, granting that any given (non-zero) local zeta integral at v is meromorphic.

Let f be a Schwartz function on a *local* field k, ψ a non-trivial additive character on k. Fix an additive Haar measure dx on k, and define the local Fourier transform

$$\widehat{f}(\xi) \;=\; \int_k f(x)\,\psi(x\xi)\,dx$$

Suppose that his additive Haar measure is normalized so that Fourier inversion holds:

$$\widehat{\widehat{f}}\,(x) \;=\; f(-x)$$

Let $d^\times x$ be multiplicative Haar measure on $k^\times$. For a complex number s, define the simplest class of **local zeta integral** by

$$Z(s,f) \;=\; \int_{k^\times} |x|^s\,f(x)\;d^\times x$$

[6.0.1] Theorem: *(Local functional equation)* For Schwartz functions f,g on the local field k, the local zeta integral $Z(s,f)$ is absolutely convergent for $\mathrm{Re}(s)>0$, has a meromorphic continuation to $s\in\mathbb{C}$, and

$$Z(s,f)\,Z(1-s,\widehat{g}) \;=\; Z(1-s,\widehat{f})\,Z(s,g)$$

Equivalently,

$$\frac{Z(s,f)}{Z(1-s,\widehat{f})} = \frac{Z(s,g)}{Z(1-s,\widehat{g})}$$

Proof: Since Schwartz functions are integrable, so the only issue in convergence of local zeta integrals occurs at 0, due to the potential blow-up of $|x|^s \cdot d^\times x$ there. Since Schwartz functions are locally constant near 0, it suffices to observe the finiteness of

$$\int_{\mathrm{ord}x\geq 0} |x|^{\mathrm{Re}\,(s)}\, d^\times x \;=\; \mathrm{const}\cdot \sum_{\ell\geq 0} q^{-\ell\cdot \mathrm{Re}\,(s)} \;<\; \infty \qquad\qquad (\text{for } \mathrm{Re}\,(s) > 0)$$

where q is the residue class field cardinality.

For the functional equation, take $0 < \mathrm{Re}\,(s) < 1$, so that the integrals for both $Z(s,f)$ and $Z(1-s,\widehat{g})$ are absolutely convergent. Then the local functional equation is a direct computation, as follows. Fubini is invoked throughout to change orders of integration. Some reflection may indicate that the crucial step is to expand the definition and replace y by yx/η:

$$\begin{aligned} Z(s,f)\, Z(1-s,\widehat{g}) &= \int_{k^\times}\int_{k^\times} |x|^s\, f(x)\, |y|^{1-s}\, \widehat{g}(y)\; d^\times x\; d^\times y \\ &= \int_k \int_{k^\times}\int_{k^\times} \overline{\psi}(y\eta)\, |x|^s\, f(x)\, |y|^{1-s}\, g(\eta)\; d^\times y d^\times x\;\; d\eta \\ &= \int_k \int_{k^\times}\int_{k^\times} \overline{\psi}(yx)\, |x|^s\, f(x)\, |yx/\eta|^{1-s}\, g(\eta)\; d^\times y\; d^\times x\; d\eta \\ &= \int_k \int_{k^\times}\int_{k^\times} \overline{\psi}(yx)\, |x|^1\, f(x)\, |y|^{1-s}\, |\eta|^{s-1}\, g(\eta)\; d^\times y\; d^\times x\; d\eta \end{aligned}$$

All that's left is to simplify and re-pack the integrals: the measure $|x|\cdot d^\times x$ is a constant multiple of the additive Haar measure dx. The precise constant is irrelevant, since it cancels itself in the necessary rearrangement:

$$|x|\, |\eta|^{-1}\; d^\times x\; d\eta \;=\; dx\; d^\times \eta$$

Also, it is convenient that for local fields k and $k^\times$ differ by a single point, of additive measure 0. Thus, continuing the previous computation,

$$\begin{aligned} Z(s,f)\, Z(1-s,\widehat{g}) &= \int_{k^\times}\int_k \int_{k^\times} \overline{\psi}(yx)\, f(x)\, |y|^{1-s}\, |\eta|^s\, g(\eta)\; d^\times y\; dx\; d^\times \eta \\ &= \int_{k^\times}\int_{k^\times} \widehat{f}(y)\, |y|^{1-s}\, |\eta|^s\, g(\eta)\; d^\times y\; d^\times \eta \\ &= Z(1-s,\widehat{f})\, Z(s,g) \end{aligned}$$

This proves the local functional equation in the region $0 < \mathrm{Re}\,(s) < 1$, so the general assertion follows from the Identity Principle from complex analysis. ///

[6.0.2] Remark: The local functional equations *locally everywhere* do not yield the *global* functional equation. Rather, the point of the local functional equations is to prove the *irrelevance* of the choice of local data f, g, as well as the *meromorphic continuation* of local zeta integrals for arbitrary data.

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7. *Fujisaki's lemma, units, finiteness of class number*

Fujisaki's lemma asserts the *compactness* of $\mathbb{J}^1/k^\times$, where $\mathbb{J}^1$ is the ideles of idele-norm 1 of a number field k. This compactness is a basic necessity. and is a corollary of existence and uniqueness of Haar measure. Surprisingly, as below, this compactness implies both *finiteness of class number* and the *units theorem*.

[7.1] Fujisaki's compactness lemma

[7.1.1] Theorem: For a number field k let

$$\mathbb{J}^1 \;=\; \{\alpha \in \mathbb{J}_k \;:\; |\alpha| = 1\}$$

Then the quotient $k^\times\backslash\mathbb{J}^1$ is *compact*.

Proof: Give $\mathbb{A} = \mathbb{A}_k$ a Haar measure so that $k\backslash\mathbb{A}$ has measure 1. First, we have the Minkowski-like claim that a compact subset C of $\mathbb{A}$ with measure greater than 1 cannot *inject* to the quotient $k\backslash\mathbb{A}$. Indeed, suppose, to the contrary, that C injects to the quotient. Letting f be the characteristic function of C,

$$1 < \int_{\mathbb{A}} f(x)\,dx = \int_{k\backslash\mathbb{A}} \sum_{\gamma\in k} f(\gamma+x)\,dx \le \int_{k\backslash\mathbb{A}} 1\,dx = 1 \qquad \text{(last inequality by injectivity)}$$

contradiction, proving the claim. Fix compact $C \subset \mathbb{A}$ with measure greater than 1. For idele α, the change-of-measure on $\mathbb{A}$ is

$$\frac{d(\alpha x)}{dx} = |\alpha|$$

Thus, neither αC nor $\alpha^{-1}C$ inject to the quotient $k\backslash\mathbb{A}$.

So there are $x \neq y$ in k so that $x + \alpha C = y + \alpha C$. Subtracting, $x - y \in \alpha(C - C) \cap k$. Since $x - y \neq 0$ and k is a field, $k^\times \cap \alpha(C - C) \neq \phi$. Likewise, $k^\times \cap \alpha^{-1}(C - C) \neq \phi$.

Thus, there are a, b in $k^\times$ such that

$$a\cdot\alpha \in C - C \qquad\qquad b\cdot\alpha^{-1} \in C - C$$

There is an obvious constraint

$$ab \;=\; (a\cdot\alpha)(b\cdot\alpha^{-1}) \;\in\; (C-C)^2 \cap k^\times \;=\; \text{compact} \cap \text{discrete} \;=\; \text{finite}$$

Let Ξ be the latter finite set. That is, given $|\alpha| = 1$, there is $a \in k^\times$ such that $a\cdot\alpha \in C - C$, and $\xi \in \Xi$ (ξ is ab just above) such that $(\xi a^{-1})\cdot\alpha^{-1} \in C - C$. That is,

$$(a\cdot\alpha,\; (a\cdot\alpha)^{-1}) \;\in\; (C-C)\times\xi^{-1}(C-C)$$

The topology on $\mathbb{J}$ is obtained by imbedding $\mathbb{J} \to \mathbb{A}\times\mathbb{A}$ by $\alpha \to (\alpha, \alpha^{-1})$ and taking the subset topology. Thus, for each $\xi \in \Xi$,

$$\Big((C-C)\times\xi^{-1}(C-C)\Big) \;\cap\; \mathbb{J} \;=\; \text{compact in } \mathbb{J}$$

The continuous image in $k^\times\backslash\mathbb{J}$ of each of these finitely-many compacts is compact, and their union covers the closed subset $k^\times\backslash\mathbb{J}^1$, so the latter is compact. ///

[7.2] Units theorem, finiteness of class number

The compactness of $\mathbb{J}^1/k^\times$ can be proven as a corollary of finiteness of class numbers and Dirichlet's units theorem. However, the compactness *implies* both the finiteness of class numbers and the units theorem, in a straightforward manner.

Let i be the *ideal map* from ideles to non-zero fractional ideals of the integers $\mathfrak{o}$ of k. That is,

$$i(\alpha) \;=\; \prod_{v<\infty} \mathfrak{p}_v^{\mathrm{ord}_v\alpha} \qquad\qquad (\text{for } \alpha \in \mathbb{J})$$

where $\mathfrak{p}_v$ is the prime ideal in $\mathfrak{o}$ attached to the place v. Certainly the subgroup $\mathbb{J}^1$ of $\mathbb{J}$ still surjects to the group of non-zero fractional ideals. The kernel in $\mathbb{J}$ of the ideal map is

$$H \;=\; \prod_{v|\infty} k_v^\times \;\times\; \prod_{v<\infty} \mathfrak{o}_v^\times$$

and the kernel on $\mathbb{J}^1$ is $H^1 = H \cap J^1$. The principal ideals are the image $i(k^\times)$. The map of $\mathbb{J}^1$ to the ideal class group factors through the idele class group $\mathbb{J}^1/k^\times$, noting as usual that the product formula implies that $k^\times \subset \mathbb{J}^1$.

The intersection $H^1 = H \cap \mathbb{J}^1$ is open in $\mathbb{J}^1$, so its image K in the quotient $\mathbb{J}^1/k^\times$ is open. The cosets of K cover $\mathbb{J}^1/k^\times$, and by compactness there is a finite subcover. Thus, the quotient $\mathbb{J}^1/k^\times K$ is finite, and this finite group is the absolute ideal class group.

Since K is open, its cosets are open. Thus, K is closed. Since $\mathbb{J}^1/k^\times$ is Hausdorff and compact, K is compact. That is, we have compactness of

$$K \;=\; (H^1 \cdot k^\times)/k^\times \;\approx\; H^1/(k^\times \cap H^1) \;=\; H^1/\mathfrak{o}^\times$$

with the global units $\mathfrak{o}^\times$ imbedded on the diagonal. Since

$$\prod_{v<\infty} \mathfrak{o}_v^\times$$

is compact, its image U under the continuous map to $H^1/\mathfrak{o}^\times$ is compact. By Haudorff-ness, the image U is closed. Thus, we can take a further (Hausdorff) quotient by U,

$$H^1/(U \cdot \mathfrak{o}^\times) \;=\; \text{(compact)}$$

Let

$$k_\infty^1 \;=\; \{\alpha \in \prod_{v|\infty} k_v^\times \;:\; \prod_v |\alpha_v|_v = 1\}$$

Then

$$k_\infty^1/\mathfrak{o}^\times \;\approx\; H^1/(U \cdot \mathfrak{o}^\times) \;=\; \text{(compact)}$$

This compactness is the units theorem. ///

[7.2.1] Remark: To compare with classical presentations, one might also want to prove the accompanying result that a discrete (closed) additive subgroup L of $\mathbb{R}^n$ such that $\mathbb{R}^n/L$ is *compact* is a free $\mathbb{Z}$-module on n generators.

The same arguments prove finiteness of generalized ideal class groups and prove generalized units theorems.

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8. *Convergence of half-zeta integrals*

The point is to genuinely prove convergence of the half-zeta integrals

$$\int_{|y|\geq 1} |y|^s \, f(y)\, dy$$

with f a Schwartz function on the adeles, for *all* $s \in \mathbb{C}$.

Since f is at worst a *finite* sum of monomials $\otimes_v f_v$, without loss of generality we take it to be such a monomial, with f_v Schwartz on k_v. Since f is Schwartz, for all N there is a constant C_N (depending on f) such that

$$|f(x)| \;\leq\; C_N \cdot \prod_v \sup(|x_v|_v, 1)^{-2N} \qquad \text{(for adele } x = \{x_v\})$$

For an *idele* y define the **gauge** [2]

$$\nu(y) \;=\; \prod_v \sup\{|y_v|_v,\, |\frac{1}{y_v}|_v\}$$

Almost all factors on the right-hand side are 1, so there is no issue of convergence. Further, note that

$$\big(\sup\{a,\, 1\}\big)^2 \;=\; \sup\{a^2,\, 1\} \;=\; a\cdot\sup\{a,\, \frac{1}{a}\} \qquad\qquad (\text{for } a>0)$$

Applying the latter equality to every factor,

$$\prod_v \sup(|y_v|_v, 1)^{-2N} \;=\; |y|^{-N}\prod_v \sup(|y_v|_v,\, \frac{1}{|y_v|_v})^{-N} \;=\; |y|^{-N}\nu(y)^{-N}$$

Thus, on the set of ideles $\{|y|\ge 1\}$,

$$\prod_v \sup(|y_v|_v, 1)^{-2N} \;=\; |y|^{-N}\nu(y)^{-N} \;\le\; \nu(y)^{-N} \qquad\qquad (\text{when } |y|\ge 1,\, N\ge 0)$$

Thus, with $\sigma = \mathrm{Re}\, s$, for every $N\ge 0$

$$\Big|\int_{|y|\ge 1} |y|^s\, f(y)\, dy\Big| \;\ll\; \int_{|y|\ge 1} |y|^\sigma\, \nu(y)^{-N}\, dy \;\ll\; \int_{\mathbb{J}} |y|^\sigma\, \nu(y)^{-N}\, dy \;=\; \prod_v \Big(\int_{k_v^\times} |y|^\sigma\, \sup(|y|, \frac{1}{|y|})^{-N}\, dy\Big)$$

For $N>|\sigma|$, the non-archimedean local integrals are absolutely convergent:

$$\int_{k_v^\times} |y|^\sigma\, \sup(|y|, \frac{1}{|y|})^{-N}\, dy \;=\; \sum_{\ell=0}^\infty q_v^{-\sigma-N} \;+\; \sum_{\ell=1}^\infty q_v^{\sigma-N}$$

$$= \; \frac{1}{1-q^{-\sigma-N}} + \frac{q^{\sigma-N}}{1-q^{\sigma-N}} \;=\; \frac{1-q^{-2N}}{(1-q^{-\sigma-N})\,(1-q^{\sigma-N})}$$

The archimedean integrals are convergent for similarly over-whelming reasons. For $2N>1$ and $N>|\sigma|+1$, the product over places is dominated by the Euler product for the completed zeta functions $\xi_k(N+\sigma)\xi_k(N-\sigma)/\xi_k(2N)$, which converges absolutely.

Bibliographical comments

The idea to recast Hecke's discussion of zeta and L-functions of number fields using Chevalley's adeles and ideles was evidently in circulation by the mid 1940s. E. Artin's student Margaret Matchett's 1946 Ph.D. thesis [Matchett 1946] predated [Iwasawa 1950/1952], [Iwasawa 1952/1992], and [Tate 1950/1967]. Both Iwasawa and Tate gave more robust treatments circa 1950, but neither appeared in print throughout the 1950s. Iwasawa's contributions on this subject are less well known than Tate's, as is visible in the common reference to *Tate's thesis* for what should arguably be *Iwasawa-Tate* theory.

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[2] Such a gauge is often called a *group norm*. The latter terminology is mildly unfortunate, since it is not a norm in the vector space sense. Nevertheless, the terminology is standard.

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