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Fourier expansions of polynomials and values of ζ

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1. Fourier expansions of polynomials
2. Sample computations

This is an example of the impact of harmonic analysis on number theory. We start from the fact that Fourier series expansions of sufficiently good functions represent the functions *pointwise*. More precisely, for present purposes, it suffices to know that for piecewise-C^o periodic functions f, at points x_o where f is C^o and has left and right derivatives,

$$f(x_o) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{2\pi i n x_o} \qquad\qquad (\text{where } \widehat{f}(n) = \textstyle\int_0^1 e^{-2\pi i n x}\, f(x)\, dx)$$

and it is implicit that the two-sided series converges. Applying this to polynomials restricted to $[0,1]$ gives a systematic approach to summing series [1] such as

$$\zeta(2) \;=\; \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots \;=\; \frac{\pi^2}{6}$$

$$\zeta(4) \;=\; \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \ldots \;=\; \frac{\pi^4}{90}$$

Small extensions of the same idea evaluate *certain* more complicated **Dirichlet series** $\sum_{n\geq 1} a_n/n^s$ at positive integers of suitable parity.

More elementary methods *can* produce a few results. For example, from the geometric series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots$$

integrate to obtain

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Letting $x = 1$ gives Leibniz' result

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \;=\; \frac{\pi}{4}$$

However, elementary methods were powerless against $\zeta(2)$.

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1. *Fourier expansions of polynomials*

On the unit interval $[0,1]$, there are two simple types of functions: polynomials, and exponentials $e^{2\pi i n x}$. They are very unlike each other, and expressing one in terms of the other produces interesting information.

The *sawtooth function*

$$f(x) \;= x - [\![x]\!] - \frac{1}{2} \qquad\qquad (\text{where } [\![x]\!] \text{ is the greatest integer} \leq x)$$

[1] Euler summed these series *circa* 1750, although at first he had more a *heuristic* than *proof*. Even a heuristic had eluded the Bernoullis, and Euler's success was one of the things that made a big impression.

is a linear polynomial made periodic. The subtraction of the $1/2$ conveniently makes the 0^{th} Fourier coefficient 0. All the other Fourier coefficients are readily computed by integration by parts:

$$\widehat{f}(n) \;=\; \int_0^1 (x-\frac{1}{2})\, e^{-2\pi i n x}\, dx \;=\; \Big[(x-\frac{1}{2})\,\frac{e^{-2\pi i n x}}{-2\pi i n x}\Big]_0^1 - \int_0^1 \frac{e^{-2\pi i n x}}{-2\pi i n x}\, dx \;=\; \frac{1}{-2\pi i n}$$

This begins an inductive description of a family of polynomials [2] $B_\ell(x)$, arranged to have *easily computed Fourier coefficients*,

[1.0.1] Definition: (Beware: not quite compatible with other normalizations of Bernoulli polynomials)

$$B_1(x) = x - \frac{1}{2} \qquad\qquad \frac{d}{dx}\, B_k(x) = B_{k-1}(x) \qquad\qquad \int_0^1 B_k(x)\, dx = 0$$

Taking $B_k(x)$ so that its derivative is $B_{k-1}(x)$ is entirely motivated by wanting an integration by parts to work nicely:

$$\int_0^1 B_k(x)\, e^{-2\pi i n x}\, dx \;=\; [B_k(x)\,\frac{e^{-2\pi i n x}}{-2\pi i n}]_0^1 \;-\; \int_0^1 B_k'(x)\,\frac{e^{-2\pi i n x}}{-2\pi i n x}\, dx$$

$$=\; [B_k(x)\frac{1}{-2\pi i n}]_0^1 \;+\; \int_0^1 B_k'(x)\,\frac{e^{-2\pi i n x}}{2\pi i n x}\, dx \;=\; \frac{1}{-2\pi i n}\int_0^1 B_k'(x)\, dx \;+\; \int_0^1 B_k'(x)\,\frac{e^{-2\pi i n x}}{2\pi i n x}\, dx$$

$$=\; 0 + \frac{-1}{(2\pi i n)^{k-1}}\cdot\frac{1}{2\pi i n} \;=\; \frac{-1}{(2\pi i n)^k} \qquad\qquad (\text{using } \textstyle\int_0^1 B_{k-1}(x)\, dx = 0)$$

Since $B_k'(x) = B_{k-1}(x)$, the only new thing to be determined in B_k is its constant, which is completely determined by the integral condition. Thus, this description *does* unambiguously specify a sequence of polynomials.

[1.0.2] Remark: In computing Fourier coefficients, we think either of the *restriction* of $B_k(x)$ to $[0,1]$, or of the *periodicized* version $B_k(x - [\![x]\!])$.

[1.0.3] Claim: The Fourier series of $B_k(x)$ is

$$B_k(x - [\![x]\!]) \;\sim\; \frac{-1}{(2\pi i)^k} \sum_{0\neq n\in\mathbb{Z}} \frac{e^{2\pi i n x}}{n^k}$$

For $k > 1$, $B_k(0) = B_k(1)$.

[1.0.4] Remark: The equality $B_k(0) = B_k(1)$ for $k > 1$ assures that the periodicized version $B_k(x - [\![x]\!])$ of $B_k(x)$ is *continuous*. Piecewise-polynomial functions have left and right derivatives everywhere, so the Fourier series for $B_k(x)$ with $k > 1$ will converge to $B_k(x - [\![x]\!])$ *for all* x. Thus, for *even* k, we evaluate $\zeta(k)$ in terms of $B_k(x)$:

[1.0.5] Corollary:

$$\begin{cases} B_{2k}(0) & = & \dfrac{-2\,\zeta(2k)}{(2\pi i)^{2k}} & (\text{for } 2k \text{ } even) \\ B_{2k+1}(0) & = & 0 & (\text{for } 2k+1 > 1 \text{ } odd) \end{cases}$$

(For k odd, in the previous claim the $\pm n$ terms cancel.) ///

[2] These polynomials are roughly *Bernoulli polynomials*. We are not worrying about conventional normalization or indexing, nor other definitions.

Proof: (of claim) First, the matching at endpoints is easy:

$$B_k(1) - B_k(0) \;=\; \int_0^1 B_{k-1}(t)\,dt = 0 \qquad\qquad (\text{for } k > 1)$$

since, by construction, $B_{k-1}(x)$ has vanishing 0^{th} Fourier coefficient. Integrating by parts, recapitulating earlier observations,

$$\widehat{B}_k(n) \;=\; \int_0^1 B_k(x)\, e^{-2\pi i n x}\, dx \;=\; \left[B_k(x)\, \frac{e^{-2\pi i n x}}{-2\pi i n}\right]_0^1 - \int_0^1 B_{k-1}(x)\, \frac{e^{-2\pi i n x}}{-2\pi i n}\, dx$$

$$= \frac{B_k(1) - B_k(0)}{-2\pi i n} \;-\; \frac{-1}{(2\pi i n)^{k-1}} \cdot \frac{1}{-2\pi i n} \;=\; \frac{-1}{(2\pi i n)^k}$$

by the inductive hypothesis that $\widehat{B}_{k-1}(n) = -1/(2\pi i n)^{k-1}$. ///

[1.0.6] Claim: The polynomials $B_k(x)$ can be characterized by a **generating function**:

$$1 + t\,B_1(x) + t^2\,B_2(x) + t^3\,B_3(x) + \ldots \;=\; \frac{t \cdot e^{tx}}{e^t - 1}$$

[1.0.7] Remark: It is wise to say that we consider this as a *formal power series* in x and t. This is *not* to say that the discussion is a mere heuristic! The formal power series ring $R[[x,t]]$ over a coefficient ring R is a *(projective) limit*[3]

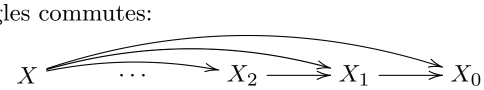
$$\lim_n \mathbb{C}[x,t]/I^n \qquad\qquad (\text{with ideal } I \text{ generated by } x, t))$$

[1.0.8] Remark: The latter relation shows that the polynomials $B_k(x)$ have *rational* coefficients. Rationality also follows from the recursive definition.

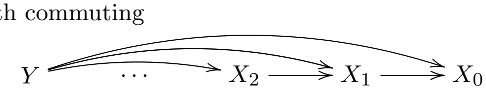
Proof: Let

$$f(t,x) \;=\; 1 + t\,B_1(x) + t^2\,B_2(x) + t^3\,B_3(x) + \ldots$$

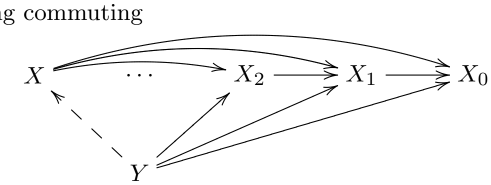
[3] Recall (!) that a projective limit X of a family $\ldots \to X_2 \to X_1 \to X_0$ is an object with maps $X \to X_i$ such that the natural diagram of curvy trangles commutes:



and, for all collections $Y \to X_i$ with commuting



there exists a *unique* $Y \to X$ giving commuting



The most basic naive-category-theory ideas show that the object X (and maps $X \to X_i$) is unique up to unique isomorphism. This should be very reassuring. Unfortunately, this way of thinking is not as common as it should be, so one may be put off by this description of formal power series rings. If so, one should correct this weakness, since such descriptions/explanations are essential in higher mathematics.

Applying a standard idea, differentiate [4] in x, use $B'_\ell(x) = B_{\ell-1}(x)$, and see what happens:

$$\frac{\partial}{\partial x} f(t,x) \;=\; t\cdot 1 + t^2\, B_1(x) + t^3\, B_2(x) + \ldots \;=\; t\cdot f(t,x)$$

Therefore, [5]

$$f(t,x) \;=\; C(t)\cdot e^{tx} \qquad\qquad (\text{for some } C(t)\in\mathbb{C}[[t]], \text{ independent of } x)$$

The conditions [6] $\int_0^1 B_k(x)\,dx = 0$ give, on one hand,

$$\int_0^1 C(t)\cdot e^{tx}\,dx = \int_0^1 (1 + t\,B_1(x)+\ldots)\,dx = \int_0^1 1\,dt + \sum_{\ell\ge 1} t^\ell \int_0^1 B_\ell(x)\,dx = 1$$

On the other hand,

$$\int_0^1 C(t)\cdot e^{tx}\,dx = C(t)\int_0^1 e^{tx}\,dx = C(t)\cdot\frac{e^t-1}{t}$$

Thus,

$$C(t) = \frac{t}{e^t-1}$$

Therefore,

$$1 + t\,B_1(x) + t^2\,B_2(x) + \ldots = \frac{t\cdot e^{tx}}{e^t-1}$$

as claimed. ///

[1.0.9] Corollary:

$$t^2\cdot\frac{2\,\zeta(2)}{(2\pi i)^2} \;+\; t^4\cdot\frac{2\,\zeta(4)}{(2\pi i)^4} \;+\; t^6\cdot\frac{2\,\zeta(6)}{(2\pi i)^6} \;+\; t^8\cdot\frac{2\,\zeta(8)}{(2\pi i)^8} \;+\;\ldots \;=\; 1 - \frac{t}{2} - \frac{t}{e^t-1}$$

Proof: Subtract $1 + tB_1(x) = 1 + t(x-\frac{1}{2})$ from both sides of the identity

$$1 + t\,B_1(x) + t^2\,B_2(x) + t^3\,B_3(x) + \ldots \;=\; \frac{t\cdot e^{tx}}{e^t-1}$$

evaluate at $x=0$, and multiply through by -1, noting that, from above, $B_k(0)=0$ for odd $k>1$, while $B_{2k}(0) = -2\zeta(2k)/(2\pi i)^{2k}$. ///

[4] When we've dodged convergence issues by saying *formal power series ring*, we can't talk about differentiation as the usual sort of limit. Instead, define a *derivation* D on $\mathbb{C}[x,t]$ or $\mathbb{C}[[x,t]]$ by requiring that D be a linear map such that $Dx = 1$ and $Dt = 0$, and satisfying *Leibniz' rule*

$$D(fg) \;=\; Df\cdot g \;+\; f\cdot Dg$$

An induction proves $Dx^n = nx^{n-1}$. A bit of thought shows that the limit-taking version of *derivative* was not essential. Some of the *properties* are essential: linearity, Leibniz' rule, annihilation of constants.

[5] The differential equation $\partial f(t,x)/\partial x = t\cdot f(t,x)$ has the *convergent* power series solution e^{tx}. Further, the obvious induction on coefficients proves that, given $C(t)\in\mathbb{C}[[t]]$, there is a unique solution $f(t,x)$ with $f(t,0)=C(t)$.

[6] These definite integrals can be rewritten as linear functionals on vector spaces of polynomials, again avoiding any genuine limit-taking.

[1.0.10] Remark: Note that the above gives no information about the $\zeta(2k+1)$ for positive odd integers $2k+1$.

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2. *Sample computations*

Reasonable efficiency allows numerical computation of some small-index polynomials $B_k(x)$.

[2.1] Symmetry/skew-symmetry We discover that it is convenient to express the polynomials $B_\ell(x)$ as a sum of monomials $(x-\frac{1}{2})^m$ rather than x^m. The recursive description, starting with $B_1(x) = x - \frac{1}{2}$, can be written in two steps: use an auxiliary polynomial $C_\ell(x)$ which needs its constant corrected to be $B_\ell(x)$:

$$C_\ell(x) \;=\; \int_{\frac{1}{2}}^x B_{\ell-1}(t)\,dt \qquad\qquad B_\ell(x) \;=\; C_\ell(x) - \int_0^1 C_\ell(t)\,dt$$

Thus, with $B_{\ell-1}(x) = \sum_n c_n (x-\frac{1}{2})^n$,

$$C_\ell(x) \;=\; \sum_n \frac{c_n}{n+1}(x-\tfrac{1}{2})^{n+1} \qquad B_\ell(x) \;=\; \sum_n \frac{c_n}{n+1}(x-\tfrac{1}{2})^{n+1} \;-\; \sum_n \frac{c_n}{(n+1)(n+2)}\frac{1-(-1)^{n+2}}{2^{n+2}}$$

Looking at the beginning of the numerical computation below, we discover that when expressed in monomials in $x-\frac{1}{2}$, polynomials $B_\ell(x)$ *apparently* have only odd-degree monomials for ℓ odd, and only even-degree monomials for ℓ even. That is, $B_\ell(x)$ *apparently* has the symmetry/skew-symmetry

$$B_\ell(1-x) \;=\; (-1)^\ell \cdot B_\ell(x) \qquad\qquad \text{(skew/symmetry about the point } x = \tfrac{1}{2})$$

Indeed, the constant-term correction *vanishes* when $B_{\ell-1}(x)$ consists entirely of even-degree monomials in $x-\frac{1}{2}$. This gives an induction in ℓ proving the skew/symmetric property.

[2.2] Rewritten recursive description Expressing $B_{\ell-1}(x)$ as a sum of monomials in $x-\frac{1}{2}$, say $B_{\ell-1}(x) = \sum c_n\,(x-\frac{1}{2})^n$, the odd-even property reduces the complexity:

$$B_\ell(x) \;=\; \begin{cases} \sum_n \dfrac{c_n}{n+1}(x-\tfrac{1}{2})^{n+1} \;-\sum_n \dfrac{c_n}{(n+1)(n+2)}\cdot\dfrac{1}{2^{n+1}} & (\ell \text{ even}) \\ \sum_n \dfrac{c_n}{n+1}(x-\tfrac{1}{2})^{n+1} & (\ell \text{ odd}) \end{cases} \qquad (\text{for } B_{\ell-1}(x) = \textstyle\sum c_n\,(x-\frac{1}{2})^n)$$

[2.3] Values $B_{2\ell}(\frac{1}{2})$ We saw above that

$$B_{2\ell}(0) \;=\; \frac{(-1)^{\ell+1}\zeta(2\ell)}{2^{2\ell-1}\,\pi^{2\ell}}$$

Expressing $B_{2\ell}(x)$ in powers of $x-\frac{1}{2}$ makes evaluation at $\frac{1}{2}$ simpler. Thus, rearrange

$$B_{2\ell}(\tfrac{1}{2}) \;=\; \sum_{n\neq 0} \frac{-e^{\pi i n}}{(2\pi i n)^{2\ell}} \;=\; \frac{(-1)^\ell}{2^{2\ell-1}\,\pi^{2\ell}} \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^{2\ell}}$$

In the region of convergence

$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^s} \;=\; \zeta(s) - 2\cdot\frac{1}{2^s}\cdot\zeta(s) \;=\; \big(1 - \frac{1}{2^{s-1}}\big)\cdot\zeta(s)$$

Thus,

$$B_{2\ell}(\tfrac{1}{2}) \;=\; \frac{(-1)^\ell}{2^{2\ell-1}\,\pi^{2\ell}}\big(1-\frac{1}{2^{2\ell-1}}\big)\cdot\zeta(2\ell)$$

[2.4] Numerical determination of B_ℓ, $\ell \le 6$ Numerical computation of the first few $B_\ell(x)$ is easy: listing coefficients of monomials of in $x-\frac{1}{2}$ in ascending order, starting with $B_1(x)$,

0	1					
$-\frac{1}{24}$	0	$\frac{1}{2}$				
0	$-\frac{1}{24}$	0	$\frac{1}{6}$			
$\frac{7}{6!\cdot 8}$	0	$-\frac{1}{48}$	0	$\frac{1}{24}$		
0	$\frac{7}{6!\cdot 8}$	0	$-\frac{1}{144}$	0	$\frac{1}{120}$	
$\frac{-31}{8!\cdot 12}$	0	$\frac{7}{6!\cdot 16}$	0	$-\frac{1}{576}$	0	$\frac{1}{720}$

where, as above, the constant terms for even-degree cases are determined by

$$\text{constant coefficient for } B_{2k}(x) \;=\; -\sum_n \frac{c_n}{(n+1)(n+2)}\cdot\frac{1}{2^{n+1}} \qquad (\text{for } B_{2k-1}(x) = \sum c_n\,(x-\tfrac{1}{2})^n)$$

$$\begin{cases} -\dfrac{1}{(1+1)(1+2)}\cdot\dfrac{1}{2^{1+1}} & = \; -\dfrac{1}{24} \quad (\text{going from } \ell=1 \text{ to } \ell=2) \\[2ex] \dfrac{1}{24\cdot(1+1)(1+2)}\cdot\dfrac{1}{2^{1+1}} - \dfrac{1}{6\cdot(3+1)(3+2)}\cdot\dfrac{1}{2^{3+1}} & = \; \dfrac{7}{6!\cdot 8} \quad (\text{going from } \ell=3 \text{ to } \ell=4) \end{cases}$$

and

$$-\frac{7}{6!\cdot 8\cdot(1+1)(1+2)}\cdot\frac{1}{2^{1+1}} + \frac{1}{144\cdot(3+1)(3+2)}\cdot\frac{1}{2^{3+1}} - \frac{1}{120\cdot(5+1)(5+2)}\cdot\frac{1}{2^{5+1}}$$

$$= -\frac{7}{6!\cdot 3\cdot 64} + \frac{1}{6!\cdot 64} - \frac{1}{7!\cdot 64} \;=\; \frac{-49+21-3}{7!\cdot 3\cdot 64}$$

$$= -\frac{31}{8!\cdot 24} \qquad (\text{going from } \ell=5 \text{ to } \ell=6)$$

[2.5] Values of ζ Putting this together,

$$-\frac{1}{24} \;=\; B_2(\tfrac{1}{2}) \;=\; -\frac{1}{2\,\pi^2}\big(1-\frac{1}{2}\big)\cdot\zeta(2) \;=\; -\frac{\zeta(2)}{4\cdot\pi^2}$$

which gives

$$\zeta(2) \;=\; \frac{\pi^2}{6}$$

Next,

$$\frac{7}{6!\cdot 8} \;=\; B_4(\tfrac{1}{2}) \;=\; \frac{(-1)^2}{2^{2\cdot 2-1}\,\pi^{2\cdot 2}}\big(1-\frac{1}{2^{2\cdot 2-1}}\big)\cdot\zeta(4) \;=\; \frac{7\cdot\zeta(4)}{64\cdot\pi^4}$$

which gives

$$\zeta(4) \;=\; \frac{\pi^4}{90}$$

Finally,

$$-\frac{31}{8!\cdot 24} \;=\; B_6(\tfrac{1}{2}) \;=\; \frac{(-1)^3}{2^{2\cdot 3-1}\,\pi^{2\cdot 3}}\big(1-\frac{1}{2^{2\cdot 3-1}}\big)\cdot\zeta(6)$$

which gives

$$\zeta(6) \;=\; \frac{\pi^6}{3^3\cdot 5\cdot 7} \;=\; \frac{\pi^6}{945}$$

All this was known long ago.
