

Traces, Cauchy identity, Schur polynomials

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1. Example: $GL_2$
2. $GL_n(\mathbb{C})$ and $U(n)$
3. Decomposing holomorphic polynomials over $GL_n \times GL_n$

The unobvious Cauchy identity[1] is proven by taking a trace of a representation of $GL_n(\mathbb{C})$ in two different ways: one as the sum of traces of irreducible subrepresentations, the other as sum of weight spaces, that is, irreducible subrepresentations for a maximal torus.

The representation whose trace we take is the space of holomorphic polynomials on $n$-by-$n$ complex matrices.

The holomorphic irreducibles occurring are identified by restriction to the realform $U(n)$, invoking the decomposition of the biregular representation of $U(n)$.

Such identities arise in Rankin-Selberg integral representations of $L$-functions. For $GL_2$, a naive, direct computation is sufficient. However, for general $GL_n$ and other higher-rank groups, direct computation is inadequate. Further, connecting local Rankin-Selberg computations to Schur functions usefully connects these computations to the Shintani-Casselman-Shalika formulas for spherical $p$-adic Whittaker functions.

1. Example: $GL_2$

First, we consider the smallest example of the Cauchy identity, namely, for $GL_2(\mathbb{C})$.

The identity[2]

$$\sum_{n=0}^{\infty} z^n \cdot \left( \frac{a^{n+1} - a^{-(n+1)}}{a - a^{-1}} \right) \left( \frac{b^{n+1} - b^{-(n+1)}}{b - b^{-1}} \right) = \frac{1 - z^2}{(1 - z a b)(1 - z a^{-1} b)(1 - z a^{-1} b^{-1})}$$

is easily proven by elementary algebra and summing geometric series. Slightly more generally, absorbing $z$ into the other indeterminates,

$$\sum_{n=0}^{\infty} \left( \frac{a^{n+1} - c^{n+1}}{a - c} \right) \left( \frac{b^{n+1} - d^{n+1}}{b - d} \right) = \frac{1 - abcd}{(1 - ab)(1 - ad)(1 - cb)(1 - cd)}$$

for $a, b, c, d$ small, or viewing everything as taking place in a formal power series ring. The above is the simplest instance of Cauchy’s identity, involving the smallest Schur polynomials

$$\frac{a^{n+1} - c^{n+1}}{a - c} = a^n + a^{n-1}c + a^{n-2}c^2 + \ldots a^2c^{n-2} + ac^{n-1} + c^n$$

In fact, the above identities have meaning: we claim that the rearrangement

$$\sum_{m, n \geq 0} (ac)^m \left( \frac{a^{n+1} - c^{n+1}}{a - c} \right) (bd)^m \left( \frac{b^{n+1} - d^{n+1}}{b - d} \right) = \frac{1}{(1 - ab)(1 - ad)(1 - cb)(1 - cd)}$$

[1] The Cauchy identity is often treated as a combinatorial result, in contexts where Schur functions are defined by formulas, rather than acknowledged to be traces of highest-weight representations.

[2] This identity also arises in the non-archimedean local factors of the $GL_2$ Rankin-Selberg integrals: the two factors in the infinite sum are values of spherical $p$-adic Whittaker functions.
computes the trace of the representation of \( GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \) on the space \( \mathbb{C}[E] \) of holomorphic polynomials on the set \( E \) of two-by-two complex matrices, in two ways. First, let \( g \times h \in GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \) act on a polynomial \( f \) on \( E \) by

\[(g \times h)f(x) = f(g^\top x h)\]

Using transpose rather than inverse in the left action is a convenience: the function \( g \to (g \times h)f(x) \) is thereby a polynomial function in \( g \). On the left-hand side of the identity above,

\[
\begin{pmatrix}
  a & 0 \\
  0 & c
\end{pmatrix}
\to
(ac)^m \left( \frac{a^{n+1} - c^{n+1}}{a - c} \right)
\]

is the character (that is, trace) \( \chi_{m,n} \) of the representation \( \pi_{m,n} = \det^m \otimes \text{Sym}^n(\text{std}) \) of \( GL_2 \), where \( \text{Sym}^n(\text{std}) \) is the \( n^{th} \) symmetric power of the standard representation \( \text{std} \). The representation is polynomial if and only if \( m \geq 0 \). The character of the representation \( \pi_{m,n} \otimes \pi_{m,n} \) of \( GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \) is the product:

\[
\text{trace}(\pi_{m,n}(g) \otimes \pi_{m,n}(h)) = \chi_{m,n}(g) \cdot \chi_{m,n}(h)
\]

Thus, the left-hand side of the identity is

\[
\text{tr} \left( \bigoplus_{m,n \geq 0} \pi_{m,n} \otimes \pi_{m,n} \right) = \sum_{m,n \geq 0} \text{tr}(\pi_{m,n} \otimes \pi_{m,n}) = \sum_{m,n \geq 0} \chi_{m,n} \cdot \chi_{m,n}
\]

Thus, we must verify (below) that

\[
\mathbb{C}[E] \approx \bigoplus_{m,n \geq 0} \pi_{m,n} \otimes \pi_{m,n} \quad \text{(as } GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \text{ representations)}
\]

On the other hand, we can look at the subgroups

\[
M = \{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \} \subset GL_2(\mathbb{C}) \quad \text{and} \quad M \times M \subset GL_2(\mathbb{C}) \times GL_2(\mathbb{C})
\]

and decompose \( \mathbb{C}[E] \) as a representation of \( M \times M \). Monomials

\[
f_{p,q,r,s} \left( \begin{array}{cc}
x & y \\
z & w
\end{array} \right) = x^p y^q z^r w^s
\]

are obvious weight vectors, that is, vector in one-dimensional representations for the action of \( M \times M \):

\[
\left( \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \times \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} \right) f_{p,q,r,s} \left( \begin{array}{cc}
x & y \\
z & w
\end{array} \right) = a^p b^q c^r d^s \cdot f_{p,q,r,s} \left( \begin{array}{cc}
x & y \\
z & w
\end{array} \right)
\]

Thus, it is easy to decompose \( \mathbb{C}[E] \) into one-dimensional irreducibles over \( M \times M \), and the trace is

\[
\sum_{p,q,r,s \geq 0} a^p b^q c^{r+s} d^{q+s} = \sum_p (ab)^p \cdot \sum_q (ad)^q \cdot \sum_r (bc)^r \cdot \sum_s (cd)^s
\]

\[
= \frac{1}{(1-ab)(1-ad)(1-be)(1-cd)} \quad \frac{1}{\det \left( 14 - \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \otimes \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} \right) } \quad \text{(for small } a,b,c,d)\]

Thus, if we verify the claim about decomposition of \( \mathbb{C}[E] \) as an \( GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \) representation, we will have

\[
\sum_{m,n \geq 0} \chi_{m,n}(g) \otimes \chi_{m,n}(h) = \frac{1}{\det \left( 14 - g \otimes h \right) } \quad \text{(for } g,h \text{ diagonal, small)
Both sides are conjugation-invariant, so we have equality for small \textit{diagonalizable} matrices. The density of diagonalizable matrices in $GL_2(\mathbb{C})$ and continuity imply that the identity holds for all small elements of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$.

It is clear that only non-negative weights can appear in a decomposition of the space of polynomials $\mathbb{C}[E]$ as $M \times M$ representation space, but it is less clear that \textit{all} holomorphic irreducibles of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ with exclusively non-negative weights \textit{do} appear. This will be proven after some preparation.

2. $GL_n(\mathbb{C})$ and $U(n)$

The group $K = U(n)$ is a maximal compact subgroup of $G = GL_n(\mathbb{C})$. However, for present purposes, the critical relationship is that any holomorphic function on $G$ vanishing identically when restricted to $K$ is already identically $0$ on $G$. This will imply that the restriction from $G$ to $K$ of an irreducible \textit{holomorphic} representation of $G$ remains irreducible. Further, we will see that \textit{every} irreducible of $K$ occurs as the restriction from $G$ of an irreducible holomorphic representation.

[2.1] \textbf{Realforms} \quad The conclusions about restriction of holomorphic functions will follow from the fact that the Lie algebra $\mathfrak{k} = \mathfrak{u}(n)$ of $K$ is a \textit{realform} \footnote{To assert that a real vector subspace $V$ of a complex vector space $W$ is a \textit{real form} of $W$ is to assert that $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} W$ and that the \textit{complex} span of $V$ is all of $W$. Equivalently, the natural complex-linear map $V \otimes_{\mathbb{R}} \mathbb{C} \to W$ is an isomorphism.} \footnote{Holomorphy of coefficient functions implies holomorphy of the representation for many infinite-dimensional representations, as well, simply because \textit{weak} holomorphy implies (strong) holomorphy for a broad class of topological} of the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. To see that $\mathfrak{k}$ is a realform of $\mathfrak{g}$, observe that $\mathfrak{k}$ consists of skew-hermitian matrices. The $\mathbb{R}$-dimension of $\mathfrak{k}$ is equal to the $\mathbb{C}$-dimension of $\mathfrak{g}$, and any complex $n$-by-$n$ matrix $z$ can be written as a $\mathbb{C}$-linear combination of elements of $\mathfrak{k}$, from the familiar identity

$$z = \frac{z + z^*}{2} + i \cdot \frac{z - z^*}{2i} = i \cdot \frac{z + z^*}{2i} + \frac{z - z^*}{2}$$

where $z \to z^*$ is conjugate-transpose. This proves the realform property.

[2.2] \textbf{Restrictions of holomorphic functions} \quad We claim that a holomorphic function on $G$ vanishes identically on $K$ if and only if it is identically zero on $G$. Subordinate to this, we claim that a holomorphic function on $\mathfrak{g}$ vanishes identically on $\mathfrak{k}$ if and only if it is identically zero on $\mathfrak{g}$.

Since $G$ is \textit{connected}, a holomorphic function on it is completely determined by its power series expansion at the identity $e \in G$. The exponential map $\mathfrak{g} \to G$ is holomorphic, and is a local isomorphism near $0 \in \mathfrak{g}$ and $e \in G$. Thus, a holomorphic function $f$ near $e \in G$ gives a holomorphic function $F(x) = f(e^x)$ for $x$ near $0$ in $\mathfrak{g}$, and vice-versa. That is, $f$ is identically $0$ if and only if $F$ is identically $0$. This reduces the first claim about restriction from $G$ to $K$ to the second claim about restriction from $\mathfrak{g}$ to $\mathfrak{k}$.

The coefficients of the complex power series expansion at $0$ of a holomorphic function on a complex vector space $V$ are essentially complex derivatives of the function. All these complex derivatives can be computed along a realform $V_0$ of $V$, exactly because the realform spans the complex vectorspace. Thus, if the restriction of a holomorphic function to $V_0$ is identically $0$, all its real derivatives along $V_0$ are $0$, so its complex derivatives at $0$ are $0$. Thus, its complex power series expansion at $0$ is identically $0$. Since $V$ is connected, the function is identically $0$.

[2.3] \textbf{Irreducibility criterion via coefficient functions} \quad We say that a representation $\pi$ of $G = GL_n(\mathbb{C})$ is \textit{holomorphic} when $g \to g \cdot v$ is a holomorphic $\pi$-valued function on $G$, for every $v \in \pi$. For finite-dimensional $\pi$ this property is immediately equivalent \footnote{Holomorphy of coefficient functions implies holomorphy of the representation for many infinite-dimensional representations, as well, simply because \textit{weak} holomorphy implies (strong) holomorphy for a broad class of topological} to holomorphy of the $\mathbb{C}$-valued coefficient functions

$$e_{v,\mu}^\pi(g) = \langle g \cdot v, \mu \rangle = \mu(gv) \quad \text{(for } v \in \pi \text{ and } \mu \in \pi^*)$$

Thus, a holomorphic function on $G$ vanishes identically on $K$ if and only if it vanishes identically on $\mathfrak{g}$. This proves the realform property.
By the above discussion, for \( v \neq 0 \) and \( \mu \neq 0 \), the non-zero coefficient function \( c_{v,\mu}^\pi \) on \( G \) restricts to a non-zero function on \( K \). This restriction is obviously the \( v,\mu \) coefficient function of the restriction \( \text{Res}_K^G \pi \) of \( \pi \) to \( K \).

We claim that a representation \( \tau \) is irreducible if and only if no one of its coefficients \( c_{v,\mu}^\tau \) vanishes identically, of course for \( v \neq 0 \) and \( \mu \neq 0 \). Indeed, for such \( v,\mu \), if \( c_{v,\mu}^\tau \) were identically 0, then the span of images \( gv \) would be inside the kernel of \( \mu \), a proper closed subspace of \( \tau \), contradicting the irreducibility of \( \tau \). Conversely, if \( \tau \) had a proper closed subspace, there would be a continuous linear functional \( \mu \) whose kernel contains that subspace, so \( c_{v,\mu}^\tau \) would be identically 0 for \( v \) in that subspace. This proves the irreducibility criterion.

[2.4] Restriction of holomorphic representations from \( G \) to \( K \) Thus, since non-zero coefficient functions of irreducible holomorphic representations \( \pi \) of \( G \) restrict to non-zero functions on \( K \), the restriction \( \text{Res}_K^G \pi \) of \( \pi \) to \( K \) remains irreducible.

We will show just below that every irreducible of \( K \) occurs as the restriction from \( G \) of a holomorphic representation.\[6\]

[2.5] Highest weights The irreducibles of \( K = U(n) \) are parametrized by highest weights \( \lambda \), in the following sense. Specifically, let \( a \) be the standard Cartan subalgebra of diagonal elements of the complexification \( g = gl_n(\mathbb{C}) \) of \( \mathfrak{k} \), and \( n \) the strictly upper-triangular elements of \( g \). The irreducible complex representations of \( K \) are finite-dimensional, so \( \mathfrak{k} \) and \( \mathfrak{g} \) act on them\[7\] by taking derivatives: for \( \theta \in \mathfrak{k}, v \in \tau \),

\[
\theta \cdot v = \frac{\partial}{\partial t} \bigg|_{t=0} e^{t\theta} \cdot v
\]

The action of \( \mathfrak{g} \) is the \( \mathbb{C} \)-linear extension.

In an irreducible \( \tau \) of \( K \), the space \( \tau^n \) of vectors annihilated by \( n \) is one-dimensional. Necessarily \( a \) acts on \( \tau^n \) by a character \( \lambda = \lambda_{\tau} : a \to \mathbb{C} \). The subspace \( \tau^n \) is the space of highest weight vectors in \( \tau \), and \( \lambda \) is the highest weight.

The map \( \tau \to \lambda \) is injective: the highest weight of an irreducible determines the isomorphism class of the irreducible.

Not every \( \lambda \in a^* \) appears as a highest weight of an irreducible of \( K \). There are two conditions necessary (and sufficient) for \( \lambda \) to appear as a highest weight. Any \( \lambda \in a^* \) is of the form

\[
\lambda = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \lambda_1 \alpha_1 + \ldots + \lambda_n \alpha_n \quad \text{(with } \lambda_i \in \mathbb{C} \text{)}
\]
Then \( \lambda \) appears as a highest weight of an irreducible of \( K \) if and only if

\[
\begin{cases}
\text{all } \lambda_j \in \mathbb{Z} & \text{(Integrality)} \\
\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n & \text{(Dominance)}
\end{cases}
\]

[2.6] Surjectivity of restriction from \( G \) to \( K \)

Now we can show that every irreducible \( \tau \) of \( K \) is the restriction of a holomorphic representation \( \pi \) of \( G \), and uniquely so. Specifically, given a dominant, integral highest weight \( \lambda \), we construct a finite-dimensional holomorphic representation \( \pi_\lambda \) of \( G \) which restricts to the irreducible \( \tau_\lambda \) of \( K \) having highest weight \( \lambda \).

Since \( \lambda \) is integral, we can tensor \( \tau_\lambda \) with an integer power of \( \det \) to assume that \( \lambda_n \geq 0 \).

Let \( G \) act on holomorphic polynomials \( f \) on \( n \)-by-\( n \) complex matrices \( x \) by right translation:

\[
(g \cdot f)(x) = f(xg)
\]

Letting \( x^{(j)} \) be the upper left \( j \)-by-\( j \) minor of \( x \), let

\[
f_\lambda(x) = (\det x^{(1)})^{\lambda_1} \cdot (\det x^{(2)})^{\lambda_2} \cdot \ldots \cdot (\det x^{(n)})^{\lambda_n}
\]

Note that the integrality of \( \lambda \), and \( \lambda_n \geq 0 \), makes \( f_\lambda \) a genuine polynomial. By design, for \( p \) in the standard minimal parabolic \( P \) of upper-triangular matrices,

\[
(p \cdot f_\lambda)(x) = f_\lambda(xp) = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdot \ldots \cdot p_n^{\lambda_n} \cdot f_\lambda(x)
\]

where \( p_j \) is the \( j \)-th diagonal entry of \( p \). Thus, \( \pi \) annihilates \( f_\lambda \), and \( \pi \) acts on \( f_\lambda \) by the weight \( \lambda = (\lambda_1, \ldots, \lambda_n) \).

Take \( \pi_\lambda \) to be the \( G \)-subrepresentation of the space of holomorphic polynomials generated by \( f_\lambda \).

Thus, by construction, \( \text{Res}^G_K \pi_\lambda \) contains a copy of the irreducible \( \tau_\lambda \) of \( K \) with highest weight \( \lambda \), with highest weight vector \( f_\lambda \). We claim that the restriction is exactly \( \tau_\lambda \).

Since the whole space of holomorphic polynomials in \( x \in E \) of a given total degree is finite-dimensional, certainly \( \pi_\lambda \) is finite-dimensional. Use the Iwasawa decomposition

\[
G = K \cdot A^+ \cdot N
\]

where \( A^+ \) is positive real diagonal elements of \( G \) and \( N \) is upper-triangular unipotent elements. The exponential map from the Lie algebra of \( A^+ \) to \( A^+ \) is obviously surjective. It is less obvious, but an easy induction, that the exponential map from the Lie algebra \( \mathfrak{n} \) to \( N \) surjects. Thus, \( \pi_\lambda \) is finite-dimensional. Use the Iwasawa decomposition

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constructed restricts to $\tau_\lambda$ on $K$.

That is, given the irreducible $\tau_\lambda$ of $K$ with (dominant integral) highest weight $\lambda$, with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, the holomorphic polynomial representation $\pi_\lambda$ constructed above restricts to $\tau_\lambda$.

Dropping the assumption $\lambda_n \geq 0$, a similar conclusion follows by tensoring with integer powers of determinant, but the polynomial character of $\pi_\lambda$ seems to be lost when $\lambda_n < 0$, whether or not $\lambda$ is dominant integral.

That is, whether or not $\lambda_n \geq 0$, every irreducible $\tau_\lambda$ of $K$ is the restriction of a holomorphic irreducible $\pi_\lambda$ of $G$. If $\lambda_n \geq 0$, then $\pi_\lambda$ occurs as a subrepresentation of holomorphic polynomials $\mathbb{C}[E]$ on $n$-by-$n$ complex matrices $E$.

[2.7] Extendability of representations of $G$  Since $G = GL_n(\mathbb{C})$ is a dense open subset of the complex vector space $E$ of $n$-by-$n$ complex matrices, for holomorphic (or other) representations $\pi$ of $G$ it is reasonable to ask whether $\pi$ extends continuously to a holomorphic $\text{End}_{\mathbb{C}}(\pi)$-valued function on $E$.

For finite-dimensional holomorphic representations, extendability of $\pi$ is equivalent to extendability of all coefficient functions of $\pi$.

In fact, for finite-dimensional holomorphic representations, the highest-weight classification above shows that for any finite-dimensional holomorphic $\pi$, there is an integer $\ell$ such that $\det^\ell \otimes \pi$ is polynomial, so certainly extends to a holomorphic function on $E$.

Specifically, the finite-dimensional holomorphic representations $\pi_\lambda$ with dominant integral highest weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_\ell \geq 0$ are polynomial, and extend to $E$.

We claim that $\pi_\lambda$ does not extend when $\lambda_n < 0$. Recall that the rational function

$$f_\lambda(x) = (\det x^{(1)})^{\lambda_1 - \lambda_2} (\det x^{(2)})^{\lambda_2 - \lambda_3} (\det x^{(3)})^{\lambda_3 - \lambda_4} \cdots (\det x^{(n-1)})^{\lambda_{n-1} - \lambda_n} (\det x^{(n)})^{\lambda_n}$$

generates $\pi_\lambda$ under the right action of $G$, regardless of the sign of $\lambda_n$. Specifically, for upper-triangular $p$,

$$f_\lambda(xp) = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_n^{\lambda_n} \cdot f_\lambda(x)$$

where $p_i$ is the $i$th diagonal entry of $p$. For $\lambda_n < 0$, this function cannot extend to $E$, because for fixed non-zero $p_1, \ldots, p_{n-1}$ it blows up as $p_n$ goes to 0. This proves the claim.

Thus, further, if a single (non-zero) coefficient function of $\pi_\lambda$ extends, then all coefficient functions extend. Indeed, a coefficient function $c_{\pi_\mu}(g)$ that does not extend is not polynomial, so $\lambda_n < 0$, and $\pi_\lambda$ does not extend.

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3. Decomposing holomorphic polynomials over $GL_n \times GL_n$

The Cauchy identity is obtained by taking the trace of a representation in two ways. The analogue of the Cauchy identity for bigger groups $GL_n(\mathbb{C})$ cannot use explicit elementary expressions for characters of finite-dimensional irreducibles, as did the $GL_2(\mathbb{C})$ example.

[3.1] Schur polynomials  Without trying to write an explicit expression\[10\] for it, take the $\lambda$th Schur polynomial $s_\lambda(t_1, \ldots, t_n)$ to be the character

$$s_\lambda(t_1, \ldots, t_n) = \text{trace} \left( \pi_\lambda \begin{pmatrix} t_1 & \cdots & \cdots & t_n \\ & & & \\ & & & \\ \end{pmatrix} \right)$$

\[10\] The Weyl character formula gives a sort of explicit expression for the Schur polynomials, but that expression is not useful in proving the Cauchy identity here.
As usual, a vector \( \pi_\lambda \) of \( GL_n(\mathbb{C}) \) with highest weight\(^{[11]} \) \( \lambda \), evaluated on diagonal matrices with entries \( t_1, \ldots, t_n \).

**3.2 The polynomial representation**

Let \( G \times G \) act on holomorphic polynomials \( \mathbb{C}[E] \) on the collection \( E \) of complex \( n \)-by-\( n \) matrices, by

\[
(g \times h)f(x) = f(g^\top \cdot x \cdot h)
\]

Note that the left action has transpose in it, rather than inverse. We do this to have the special polynomial

\[
f_\lambda(x) = (\det x^{(1)})^{\lambda_1 - \lambda_2} (\det x^{(2)})^{\lambda_2 - \lambda_3} (\det x^{(3)})^{\lambda_3 - \lambda_2} \ldots (\det x^{(n-1)})^{\lambda_{n-1} - \lambda_n} (\det x^{(n)})^{\lambda_n}
\]

be acted upon on the left by \( P \) by

\[
f_\lambda(p^\top x) = p_1^{\lambda_1} p_2^{\lambda_2} \ldots p_n^{\lambda_n} \cdot f_\lambda(x)
\]

in addition to the corresponding action of \( P \) on the right noted above. Thus, the representation of \( G \times G \) generated by \( f_\lambda \) restricts to the irreducible representation of \( K \times K \) with highest weight \( \lambda \oplus \lambda \).

In particular, the representation \( \pi_\lambda \otimes \pi_\lambda \) occurs in \( \mathbb{C}[E] \) at least once. This is a lower bound on the decomposition of \( \mathbb{C}[E] \) into irreducibles.

To obtain an upper bound on \( \mathbb{C}[E] \), observe that the restriction of functions from \( G \) to \( K \) is injective on holomorphic functions. Thus, under restriction of functions, \( \mathbb{C}[E] \) injects to smooth functions on \( K \). Further, since \( \mathbb{C}[E] \) consists of polynomials, every vector in it is \( G \times G \)-finite\(^{[12]} \) and, hence, \( K \times K \)-finite. In the biregular representation of \( K \times K \) on functions on \( K \), the collection of \( K \times K \)-finite vectors is exactly the algebraic direct sum

\[
\sum_{\lambda} \tau_\lambda \otimes \tau_\lambda^\vee \quad \text{(summed over dominant integral highest weights)}
\]

where \( \tau_\lambda \) is the irreducible of \( K \) with highest weight \( \lambda \).

Since the left action of \( G \) on \( \mathbb{C}[E] \) is not \( (g \cdot f)(x) = f(g^{-1}x) \), but \( (g \cdot f)(x) = f(g^{-1}x) \), we might want to reconcile this with the convention for the biregular representation of \( K \times K \). We claim that for an irreducible \( \tau \) of \( K \), replacing \( k \cdot v = \tau(g)(v) \) by \( \tau((k^\top)^{-1})(v) \) yields the contragredient \( \tau^\vee \) to \( \tau \). To see this, use the classification by highest weights. Indeed, taking transpose converts a highest weight to a lowest weight, and vice-versa. The theory of highest weights shows that finite-dimensional highest-weight representations have lowest weight equal to the negative of the highest weight of the contragredient. Thus, taking inverse (after taking transpose) gives a representation of \( K \) with the same highest weight as the contragredient \( \tau^\vee \), hence, isomorphic to \( \tau^\vee \).

Thus, with inverse replaced by transpose for the left action, the restriction of the representation \( \mathbb{C}[E] \) of \( G \times G \) to a representation of \( K \times K \) necessarily gives a subrepresentation of

\[
\sum_{\lambda} \tau_\lambda \otimes \tau_\lambda
\]

\(^{[11]} \) This use of highest weight is a potentially confusing slight abuse of language, since the true weights for \( GL_n(\mathbb{C}) \) would necessarily include all characters of the Cartan subgroup (diagonal matrices), not merely holomorphic ones. But restriction to \( U(n) \) sends irreducibles to irreducibles, and it is literally correct to say that the restriction of \( \pi_\lambda \) to \( U(n) \) has highest weight \( \lambda \).

\(^{[12]} \) As usual, a vector \( v \) in a representation \( \pi \) of a group \( G \) is \( G \)-finite when the span of the images \( g \cdot v \) is finite-dimensional.
Since $\pi_\lambda$ restricts to $\tau_\lambda$, this implies that $\pi_\lambda \otimes \pi_\lambda$ occurs at most once in $\mathbb{C}[E]$, giving the desired upper bound on multiplicities.

Finally, since the representation of $G \times G$ on $\mathbb{C}[E]$ is polynomial, it certainly extends to $E \times E$, as do coefficient functions of it irreducible subrepresentations. From the earlier discussion of extendability, the only representations $\pi_\lambda \otimes \pi_\lambda$ appearing must have $\lambda_n \geq 0$. We have already observed that for $\lambda_n \geq 0$, the special polynomial $f_\lambda$ certainly is in $\mathbb{C}[E]$, and generates a copy of $\pi_\lambda \otimes \pi_\lambda$. Thus,

$$\mathbb{C}[E] \approx \sum_{\lambda : \lambda_n \geq 0} \pi_\lambda \otimes \pi_\lambda$$

### [3.3] Taking trace: the Cauchy identity

Though $\mathbb{C}[E]$ is an infinite-dimensional representation of $G \times G$, it is an ascending union, or colimit, of finite-dimensional representations, so has a trace.

On one hand, expressing the trace in terms of the irreducible summands $\pi_\lambda \otimes \pi_\lambda$ gives

$$\text{trace } \mathbb{C}[E] = \sum_{\lambda : \lambda_n \geq 0} \text{tr}(\pi_\lambda \otimes \pi_\lambda) = \sum_{\lambda : \lambda_n \geq 0} \text{tr}\pi_\lambda \cdot \text{tr}\pi_\lambda = \sum_{\lambda : \lambda_n \geq 0} s_\lambda \cdot s_\lambda$$

On the other hand, the monomials in $\mathbb{C}[E]$ are weight vectors for the subgroup $M \times M$ of diagonal matrices in $G \times G$. Thus, evaluating the trace on diagonal elements

$$a \times b = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \times \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$(\text{trace } \mathbb{C}[E])(a \times b) = \sum_{n_{ij} \geq 0} \prod_{i,j=1}^n (a_i b_j)^{n_{ij}} = \prod_{i,j=1}^n \frac{1}{1 - a_i b_j} = \frac{1}{\det (1_{n^2} - a \otimes b)}$$

Thus, we have Cauchy’s identity for diagonal elements:

$$\sum_{\lambda : \lambda_n \geq 0} s_\lambda(a) \cdot s_\lambda(b) = \frac{1}{\det (1_{n^2} - a \otimes b)}$$

either as a formal power series identity or for convergent power series with $a, b$ small. Both sides are conjugation-invariant, so the identity holds for diagonalizable elements of $G \times G$. These are dense, so the identity holds throughout:

$$\sum_{\lambda : \lambda_n \geq 0} s_\lambda(g) \cdot s_\lambda(h) = (\text{tr}\mathbb{C}[E])(g \times h) = \frac{1}{\det (1_{n^2} - g \otimes h)} \quad \text{for } g, h \in G$$


