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Continuous spectrum for $SL_2(\mathbb{Z}) \backslash \mathfrak{H}$

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We recall the notion of *cusppform* for $\Gamma = SL_2(\mathbb{Z})$ acting on the upper half-plane, and prove that the orthogonal complement in $L^2(\Gamma \backslash \mathfrak{H})$ to cuspforms is spanned by *pseudo-Eisenstein series*, which in turn are expressible as *wave packets* of *Eisenstein series* E_s . Further we have a Plancherel theorem for this space. The non-trivial harmonic analysis resides in the application of the *Fourier transform* on the real line, in coordinates in which it is called a *Mellin transform*. That is, this part of the harmonic analysis of $SL_2(\mathbb{Z}) \backslash \mathfrak{H}$ reduces to harmonic analysis on a related, lower-dimensional group.

I first saw this argument in [Godement 1966].

1. Pseudo-Eisenstein series adjunction to constant term

Let N be the subgroup of G consisting of upper-triangular unipotent matrices, A^+ the subgroup of diagonal matrices with positive diagonal entries, and P the parabolic subgroup consisting of all upper-triangular matrices.

[1.1] Constant terms

The **constant term** along P of a reasonable^[1] function f on $\Gamma \backslash G$ is

$$(\text{constant term of } f)(g) = c_P f(g) = \int_{N \cap \Gamma \backslash N} f(ng) \, dn$$

By changing variables, $c_P f$ is left N -invariant. Since P normalizes N , the constant term also retains the left $P \cap \Gamma$ -invariance of f .

[1.2] Cuspforms

A reasonable function f on $\Gamma \backslash G$ is a **cusppform** when its constant term essentially^[2] vanishes:

$$f \text{ cusppform} \iff c_P f = 0$$

It is better to recast the cusppform condition in the form

$$f \text{ cusppform} \iff \int_{N \backslash G} c_P f(g) \cdot \varphi(g) \, dg = 0 \quad (\text{for all } \varphi \in C_c^\infty((P \cap \Gamma)N \backslash G))$$

[1] Rather than worry about conditions guaranteeing convergence of the integral for $c_P f$, or about the nature of the function $c_P f$, it suffices to consider *distributions* f on $\Gamma \backslash G$, and consider $c_P f$ as a *distribution* on $(P \cap \Gamma)N \backslash G$. Elaboration of this viewpoint is implicit in the subsequent discussion, and *explicitly* produces pseudo-Eisenstein series in a natural fashion.

[2] Again, the constant term $c_P f$ is best treated as a distribution, as we implicitly do in the sequel. It is possible, as in many sources, to treat $c_P f$ as a measurable function. In that context, *essentially* vanishing means vanishing *almost everywhere*.

That is, the cuspform condition is that the constant term vanishes *as a distribution* on $(P \cap \Gamma)N \backslash G$.

[1.3] Pseudo-Eisenstein series

Pseudo-Eisenstein series appear naturally as the solution to an *adjunction* problem, described as follows. Use the notational convention

$$\langle f, F \rangle_{H \backslash G} = \int_{H \backslash G} f \cdot F dg \quad (\mathbb{C}\text{-bilinear, not hermitian})$$

even when f, F need not lie in the same space of functions, and use the same notation for extensions of integral pairings to distributions. The problem is, given φ in $C_c^\infty((P \cap \Gamma)N \backslash G)$, find $\Psi_\varphi \in C_c^\infty(\Gamma \backslash G)$ such that

$$\langle c_P f, \varphi \rangle_{(P \cap \Gamma)N \backslash G} = \langle f, \Psi_\varphi \rangle_{\Gamma \backslash G} \quad (\text{for } f \text{ on } \Gamma \backslash G)$$

Incidentally, exhibition of such Ψ_φ will completely legitimize treatment of c_P as a map from distributions on $\Gamma \backslash G$ to distributions on $(P \cap \Gamma)N \backslash G$, even though we do not need this degree of generality.

[1.3.1] Remark: For explicit computations, it is important to understand the measure on $N \backslash G/K$ or $N \backslash G$. The measure on $\mathfrak{H} \approx G/K$ is the usual left G -invariant measure $dx dy/y^2$ inherited from Haar measure on $G = NA^+K$. This measure descends to dy/y^2 on $N \backslash \mathfrak{H} \approx N \backslash G/K$, *not* the measure dy/y from the Haar measure on A^+ . For $\Gamma = SL_2(\mathbb{Z})$, by chance $(P \cap \Gamma) \subset NK$, so $(P \cap \Gamma)N \backslash \mathfrak{H} = N \backslash \mathfrak{H}$, and the measure is dy/y^2 .

A canonical *expression* for the desired Ψ_φ is found by direct computation, using the left $(P \cap \Gamma)N$ -invariance of φ and the left Γ -invariance of f :

$$\begin{aligned} \langle c_P f, \varphi \rangle_{(P \cap \Gamma)N \backslash G} &= \int_{(P \cap \Gamma)N \backslash G} c_P f(g) \varphi(g) dg = \int_{(P \cap \Gamma)N \backslash G} \left(\int_{N \cap \Gamma \backslash N} f(n g) dn \right) \varphi(g) dg \\ &= \int_{(P \cap \Gamma)N \backslash G} \left(\int_{N \cap \Gamma \backslash N} f(n g) \varphi(n g) dn \right) dg = \int_{P \cap \Gamma \backslash G} f(g) \varphi(g) dg \end{aligned}$$

Winding up, this is

$$\int_{P \cap \Gamma \backslash G} f(g) \varphi(g) dg = \int_{\Gamma \backslash G} \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} f(\gamma g) \varphi(\gamma g) dg = \int_{\Gamma \backslash G} f(g) \left(\sum_{\gamma \in P \cap \Gamma \backslash \Gamma} \varphi(\gamma g) \right) dg$$

Thus, the pseudo-Eisenstein series^[3] attached to φ is

$$\Psi_\varphi(g) = \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} \varphi(\gamma g)$$

While cuspforms are mysterious, these pseudo-Eisenstein series are completely *not*-mysterious.

[1.3.2] Lemma: The series for a pseudo-Eisenstein series Ψ_φ is *locally finite*, meaning that for g in a fixed compact in G , there are only finitely-many non-zero summands in $\sum_\gamma \varphi(\gamma g)$. Thus, $\Psi_\varphi \in C_c^\infty(\Gamma \backslash G)$.

Proof: Given $\varphi \in C_c^\infty((P \cap \Gamma)N \backslash G)$, let C be a compact set in G so that $(P \cap \Gamma)N \cdot C$ contains the support of φ . Fix a compact subset C_o of G in which $g \in G$ is constrained to lie. Then a summand $\varphi(\gamma g)$ is non-zero only if $\gamma g \in (P \cap \Gamma)N \cdot C$, which is to say

$$\gamma \in (P \cap \Gamma) \cdot C \cdot g^{-1}$$

[3] In 1966 Godement called these *incomplete theta series*, but more recently Moeglin-Waldspurger strengthened the precedent of calling them *pseudo-Eisenstein series*

so

$$\gamma \in \Gamma \cap (P \cap \Gamma) \cdot C \cdot C_o^{-1}$$

In the quotient $G \rightarrow (P \cap \Gamma) \backslash G$, the image of Γ is discrete. The image of the compact set $(P \cap \Gamma)N \cdot C \cdot C_o^{-1}$ under the continuous quotient map is *compact*, since $(P \cap \Gamma) \backslash (P \cap \Gamma)N$ is compact, and continuous images of compacts are compact. Thus, left modulo $P \cap \Gamma$, that intersection is the intersection of a discrete set and a compact set, so *finite*. Therefore, the series is *locally finite*, and defines a smooth function on $\Gamma \backslash G$.

To show that Ψ_φ has compact support in $\Gamma \backslash G$, proceed similarly. That is, for a summand $\varphi(\gamma g)$ to be non-zero, it must be that

$$g \in \Gamma \cdot C$$

The image $\Gamma \backslash (\Gamma \cdot C)$ is compact, being the continuous image of the compact set C under the continuous map $G \rightarrow \Gamma \backslash G$, proving the compact support. ///

[1.3.3] **Corollary:** The square-integrable cuspforms are the orthogonal complement of the (closed) space spanned by the pseudo-Eisenstein series in $L^2(\Gamma \backslash G)$. ///

2. Decomposition of pseudo-Eisenstein series: beginning

Spectral decomposition of the data φ in Ψ_φ induces a spectral decomposition of Ψ_φ itself, in a natural manner.

[2.1] Fourier transform

Fourier transform \hat{f} of f on the real line is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$$

Fourier inversion for Schwartz functions is

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

For Schwartz functions f the integral for Fourier transform and for its inversion both converge very well. Replacing ξ by $\xi/(2\pi)$ gives the variant

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-it\xi} dt \right) e^{i\xi x} d\xi$$

[2.2] Multiplicative coordinates

Fourier transforms on \mathbb{R} put into multiplicative coordinates are Mellin transforms: for $F \in C_c^\infty(0, +\infty)$, take

$$f(x) = F(e^x)$$

Let $y = e^x$ (and r the exponentiated variable in the implied inner integral) and rewrite Fourier inversion as

$$F(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_0^{\infty} F(r) r^{-i\xi} \frac{dr}{r} \right) y^{i\xi} d\xi$$

The Fourier transform in these coordinates is called a *Mellin* or *Laplace* transform $\mathcal{M}F$, defined by

$$\mathcal{M}F(i\xi) = \int_0^{\infty} F(r) r^{-i\xi} \frac{dr}{r}$$

For compactly-supported F , the integral definition extends to complex s :

$$\mathcal{M}F(s) = \int_0^\infty F(r) r^{-s} \frac{dr}{r}$$

The variant Fourier inversion identity gives the Mellin inversion formula

$$F(y) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{M}F(i\xi) y^{i\xi} d\xi$$

View ξ as the imaginary part of a complex variable s , rewrite the latter integral as a complex path integral, so it becomes (since $d\xi = -i ds$)

$$F(y) = \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} \mathcal{M}F(s) y^s ds$$

[2.2.1] **Remark:** It is important that for $f \in C_c^\infty(\mathbb{R})$ the Fourier transform $\hat{f}(\xi)$ extends to an *entire* function in ξ , of rapid decay on horizontal lines. [4]

Then certainly the same is true for the transform $\mathcal{M}F$ of $F \in C_c^\infty(0, +\infty)$. In this case, for *any* real σ , Mellin inversion is

$$F(y) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}F(s) y^s ds$$

For $\varphi \in C_c^\infty(0, \infty)$, the Mellin inversion formula gives

$$\varphi(y) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) y^s ds$$

Identifying $N \backslash G / K \approx A^+$, this is

$$\varphi(g) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) \chi_s(a(g)) ds$$

Thus, the pseudo-Eisenstein series is expressible as

$$\Psi_\varphi(g) = \sum_{\gamma \in (\Gamma \cap N) \backslash \Gamma} \varphi(\gamma g) = \frac{1}{2\pi i} \sum_{\gamma \in (\Gamma \cap N) \backslash \Gamma} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) \cdot \chi_s(a(\gamma g)) ds$$

Taking $\sigma = 0$ would be natural, but, with $\sigma = 0$, the double integral (sum and integral) is not absolutely convergent, and the two integrals cannot be interchanged. Subsequently, we will see that the best line is $\sigma = 1/2$, but this is not in the region of convergence. For $\sigma > 1$, elementary estimates show that the double integral *is* absolutely convergent, and (by Fubini's theorem) the two integrals can be interchanged:

$$\Psi_\varphi(g) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) \sum_{\gamma \in (\Gamma \cap N) \backslash \Gamma} \chi_s(a(\gamma g)) ds \quad (\text{with } \sigma > 1)$$

The inner sum defines the (spherical) *Eisenstein series*

$$E_s(g) = \sum_{\gamma \in (\Gamma \cap P) \backslash \Gamma} \chi_s(a(\gamma g)) = \sum_{\gamma \in (\Gamma \cap P) \backslash \Gamma} \text{Im}(\gamma z)^s$$

[4] This is the easy half of the Paley-Wiener theorem.

where we let Γ act as usual on the upper half-plane, and use

$$\operatorname{Im}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \frac{y}{|cz + d|^2} \quad (\text{with } z = x + iy)$$

That is,

$$\Psi_\varphi(g) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) \cdot E_s(g) ds \quad (\sigma > 1)$$

[2.2.2] **Remark:** The decomposition should refer to $\mathcal{M}_{c_P}(\Psi_\varphi)$, not $\mathcal{M}\varphi$, to have whatever integral formulas expressed in terms of the automorphic forms Ψ_φ themselves, not in terms of the auxiliary functions φ from which they're made.

To accomplish this, several features of the Eisenstein series E_s are necessary. We recall these before proceeding.

3. Recollection of facts about Eisenstein series

We review basic features of the spherical Eisenstein series for $\Gamma = SL_2(\mathbb{Z})$. In the Iwasawa decomposition $G = N \cdot A^+ \cdot K$, use coordinates on A^+

$$a_y = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{1/y} \end{pmatrix}$$

Let $g \rightarrow a(g)$ be the A^+ -component of $g \in G$. The characters

$$\chi_s : a_y \longrightarrow y^s \quad (\text{for } s \in \mathbb{C})$$

exhaust the continuous group homomorphisms $A^+ \rightarrow \mathbb{C}^\times$.

[3.1] Adjunction

Consider only right K -invariant automorphic forms f . Because $NA^+ \cap K = \{1\}$ and $A^+ \cap \Gamma = \{1\}$, we can identify $(P \cap \Gamma)N \backslash G / K \approx A^+$, and use the y -coordinate on A^+ .

The Eisenstein series E_s should be a function on $\Gamma \backslash \mathfrak{H}$ fitting into the adjunction

$$\langle E_s, f \rangle_{\Gamma \backslash \mathfrak{H}} = \langle y^s, c_P f \rangle_{(P \cap \Gamma)N \backslash \mathfrak{H}} \quad (\text{for } f \text{ on } \Gamma \backslash \mathfrak{H})$$

whenever the implied integrals converge. The Eisenstein series is *determined* by this relation: with the measure of K normalized to 1,

$$\begin{aligned} \langle y^s, c_P f \rangle_{A^+} &= \int_{(P \cap \Gamma)N \backslash G / K} c_P f(g) \cdot \chi_s(a(g)) dg = \int_{(P \cap \Gamma)N \backslash G} c_P f(g) \cdot \chi_s(a(g)) dg \\ &= \int_{(P \cap \Gamma)N \backslash G} \left(\int_{N \cap \Gamma \backslash N} f(n g) dn \right) \cdot \chi_s(a(g)) dg \\ &= \int_{(P \cap \Gamma)N \backslash G} \left(\int_{N \cap \Gamma \backslash N} f(n g) \cdot \chi_s(a(n g)) dn \right) dg = \int_{P \cap \Gamma \backslash G} f(g) \cdot \chi_s(a(g)) dg \end{aligned}$$

At this point, the notation becomes more transparent in upper half-plane notation. Identifying $G/K \approx \mathfrak{H}$ since everything is right K -invariant, and using $\chi_s(a(n a_y k)) = \chi_s(a_y) = y^s$,

$$\langle y^s, c_P f \rangle_{A^+} = \int_{P \cap \Gamma \backslash G} f(g) \cdot \chi_s(a(g)) dg = \int_{P \cap \Gamma \backslash \mathfrak{H}} f(z) \cdot y^s \frac{dx dy}{y^2}$$

Now wind this up:

$$\int_{P \cap \Gamma \backslash \mathfrak{H}} f(z) \cdot y^s \frac{dx dy}{y^2} = \int_{\Gamma \backslash \mathfrak{H}} \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} f(\gamma z) \cdot \text{Im}(\gamma z)^s \frac{dx dy}{y^2} = \int_{\Gamma \backslash \mathfrak{H}} f(z) \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} \text{Im}(\gamma z)^s \frac{dx dy}{y^2}$$

Thus, the adjunction gives

$$E_s(z) = \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} \text{Im}(\gamma z)^s = \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} \chi_s(a(\gamma g))$$

However, this series converges only for $\text{Re}(s) > 1$, and meromorphic continuation is necessary.

[3.2] Eisenstein series and Mellin transforms

Via the adjunction characterizing E_s , integrals against Eisenstein series are Mellin transforms of constant terms:

$$\langle E_s, f \rangle_{\Gamma \backslash \mathfrak{H}} = \int_0^\infty c_P f(iy) y^s \frac{dy}{y^2} = \int_0^\infty c_P f(iy) y^{-(1-s)} \frac{dy}{y} = \mathcal{M}(c_P f)(1-s)$$

[3.3] Meromorphic continuation and functional equation

With the usual G -invariant Laplacian

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

on $\mathfrak{H} = G/K$, from

$$\Delta y^s = s(s-1) \cdot y^s$$

in the region of convergence

$$\Delta E_s = s(s-1) \cdot E_s$$

Since Δ commutes with the map $f \rightarrow c_P f$, we see that $c_P E_s$ is a function $u(y)$ of y satisfying the Eulerian equation

$$y^2 \frac{\partial^2}{\partial y^2} u(y) = s(s-1) u(y)$$

For $s \neq 1/2$ this has the two linearly independent elementary solutions y^s and y^{1-s} , so, for some meromorphic functions a_s and c_s ,

$$c_P E_s = a_s y^s + c_s y^{1-s}$$

In fact, a direct computation shows $a_s = 1$. That is,

$$c_P E_s = y^s + c_s y^{1-s}$$

We grant that the Eisenstein series E_s has a *meromorphic continuation*. In particular, the modified function

$$\tilde{E}_s = s(1-s) \pi^{-s} \Gamma(s) \zeta(2s) \cdot E_s$$

is *entire* and has the *functional equation*

$$\tilde{E}_s = \tilde{E}_{1-s}$$

Another characterization of E_s is as the unique solution to the equations

$$\Delta w = s(s-1) \cdot w \quad \left(y \frac{\partial}{\partial y} - (1-s) \right) w = (2s-1) \cdot y^s \quad (\text{on } \Gamma \backslash \mathfrak{H})$$

This characterization yields the *functional equation*

$$E_{1-s} = c_{1-s} E_s$$

as follows. One can check directly that E_{1-s}/c_{1-s} satisfies those equations as well, so by *uniqueness*

$$\frac{E_{1-s}}{c_{1-s}} = E_s$$

Applying the functional equation twice gives

$$c_s c_{1-s} = 1$$

Since $E_{\bar{s}} = \overline{E_s}$, also $\overline{c_s} = c_{\bar{s}}$ and

$$|c_{\frac{1}{2}+it}|^2 = 1 \quad (\text{for } t \in \mathbb{R})$$

In particular, c_s does not vanish on the line $\text{Re}(s) = \frac{1}{2}$.

[3.3.1] Remark: Just below we will see that, from the expression of pseudo-Eisenstein series in terms of Eisenstein series, poles of E_s sufficiently far to the right play a role in the decomposition of $L^2(\Gamma \backslash G)$.

4. Decomposition of pseudo-Eisenstein series: conclusion

So far, we have a spectral decomposition of Ψ_φ in terms of φ :

$$\Psi_\varphi(g) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) \cdot E_s(g) ds \quad (\sigma > 1)$$

We want to rewrite this to refer only to Ψ_φ , not φ .

Granting the meromorphic continuation of the Eisenstein series, move the vertical line of integration to the left, say to the line $\sigma = 1/2$ stabilized by the functional equation of E_s :

$$\Psi_\varphi(g) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s) E_s(g) ds + \sum_{s_o} \text{res}_{s=s_o}(E_s \cdot \mathcal{M}\varphi(s))$$

The $1/2\pi i$ from the inversion formula cancels the $2\pi i$ in the residue formula.

[4.0.1] Remark: We also need to know that E_s has no pole on the line $\text{Re}(s) = 1/2$.

To rewrite this in terms of Ψ_φ , on one hand, from above,

$$\langle E_s, \Psi_\varphi \rangle_{\Gamma \backslash H} = \mathcal{M}(c_P \Psi_\varphi)(1-s)$$

On the other hand, by the adjunction/unwinding property of Ψ_φ ,

$$\langle E_s, \Psi_\varphi \rangle = \langle c_P E_s, \varphi \rangle_{(P \cap \Gamma) \backslash N \backslash \mathfrak{H}}$$

This is

$$\begin{aligned} \langle c_P E_s, \varphi \rangle_{(P \cap \Gamma) \backslash N \backslash \mathfrak{H}} &= \langle y^s + c_s y^{1-s}, \varphi \rangle_{(P \cap \Gamma) \backslash N \backslash \mathfrak{H}} = \int_0^\infty (y^s + c_s y^{1-s}) \cdot \varphi(y) \frac{dy}{y^2} \\ &= \int_0^\infty (y^{-(1-s)} + c_s y^{-s}) \cdot \varphi(y) \frac{dy}{y} = \mathcal{M}\varphi(1-s) + c_s \mathcal{M}\varphi(s) \end{aligned}$$

Thus,

$$\mathcal{M}(c_P \Psi_\varphi)(1-s) = \langle E_s, \Psi_\varphi \rangle_{\Gamma \backslash \mathfrak{H}} = \mathcal{M}\varphi(1-s) + c_s \mathcal{M}\varphi(s)$$

Replacing s by $1-s$, this is

$$\mathcal{M}(c_P \Psi_\varphi)(s) = \langle E_{1-s}, \Psi_\varphi \rangle_{\Gamma \backslash \mathfrak{H}} = \mathcal{M}\varphi(s) + c_{1-s} \mathcal{M}\varphi(1-s)$$

The right-hand side of the latter will appear just below.

The integral part of the expression of Ψ_φ in terms of Eisenstein series can be folded in half, integrating from $\frac{1}{2} + i0$ to $\frac{1}{2} + i\infty$ rather than from $\frac{1}{2} - i\infty$ to $\frac{1}{2} + i\infty$:

$$\Psi_\varphi - (\text{residual part}) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s) \cdot E_s(g) ds = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s) E_s + \mathcal{M}\varphi(1-s) E_{1-s} ds$$

Using the functional equation $E_{1-s} = c_{1-s} E_s$, and recognizing $\mathcal{M}(c_P \Psi_\varphi)$ as expressed above in terms of $\mathcal{M}\varphi$, this becomes

$$\frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} (\mathcal{M}\varphi(s) + c_{1-s} \mathcal{M}\varphi(1-s)) \cdot E_s ds = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}c_P \Psi_\varphi(s) E_s ds$$

by the expression above for the Mellin transform of the constant term of Ψ_φ . That is, a pseudo-Eisenstein series is expressible as an integral of Eisenstein series E_s on the line $\text{Re}(s) = 1/2$, plus a sum of residues:

$$\Psi_\varphi - (\text{residual part}) = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}c_P \Psi_\varphi(s) \cdot E_s ds = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \langle E_{1-s}, \Psi_\varphi \rangle_{\Gamma \backslash \mathfrak{H}} \cdot E_s ds$$

This is the decomposition of pseudo-Eisenstein series Ψ_φ in terms of Eisenstein series E_s . ///

5. Residue of Eisenstein series: the constant function

For $\Gamma = SL_2(\mathbb{Z})$, we know by various means that there is a single pole of E_s in the half-plane $\text{Re}(s) \geq 1/2$, at $s = 1$, and it is simple. Indeed, we know that the residue is a *constant* function. Thus,

$$\Psi_\varphi = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \langle E_{1-s}, \Psi_\varphi \rangle_{\Gamma \backslash \mathfrak{H}} \cdot E_s ds + \mathcal{M}\varphi(1) \cdot \text{res}_{s=1} E_s$$

The coefficient $\mathcal{M}\varphi(1)$ is

$$\mathcal{M}\varphi(1) = \int_0^{+\infty} \varphi(a_y) y^{-1} \frac{dy}{y} = \int_0^{+\infty} \varphi(a_y) \frac{dy}{y^2} = \int_{N \backslash G} \varphi(g) dg$$

Rearranging,

$$\begin{aligned} \mathcal{M}\varphi(1) &= \int_{N \backslash G} \varphi(g) dg = \int_{N \backslash G} \int_{N \cap \Gamma \backslash N} \varphi(ng) dn dg \\ &= \int_{N \backslash G} \varphi(ng) \left(\int_{N \cap \Gamma \backslash N} 1 dn \right) dg = \int_{N \cap \Gamma \backslash G} \varphi(g) dg \end{aligned}$$

since the natural volume of $(N \cap \Gamma) \backslash N$ is 1 and φ is left N -invariant. Winding up,

$$\mathcal{M}\varphi(1) = \int_{\Gamma \backslash G} \sum_{\gamma \in N \cap \Gamma \backslash \Gamma} \varphi(g) dg = \int_{\Gamma \backslash G} \Psi_\varphi(g) dg = \langle 1, \Psi_\varphi \rangle$$

That is, $\mathcal{M}\varphi(1)$ is the inner product of Ψ_φ with the constant function 1. Constants are square-integrable, since the measure of $\Gamma \backslash G$ is finite, but the sense of $\langle 1, \Psi_\varphi \rangle$ does not depend upon this.

[5.0.1] **Remark:** The condition $\mathcal{M}\varphi(1) \neq 0$ for $\langle 1, \Psi_\varphi \rangle \neq 0$ is easily met, since $\mathcal{M}\varphi$ is entire and of rapid decay on vertical lines: if $\mathcal{M}\varphi(1) \neq 0$, then for some $0 < \ell \in \mathbb{Z}$ $(s-1)^{-\ell} \mathcal{M}\varphi(s)$ is finite and non-zero at $s=1$. The integration by parts identity

$$\begin{aligned} \mathcal{M}\varphi(s-1) &= \int_0^\infty y^{-(s-1)} \frac{\varphi(y)}{y} \frac{dy}{y} = \frac{-1}{s-1} \int_0^\infty y \frac{\partial}{\partial y} y^{-(s-1)} \cdot \frac{\varphi(y)}{y} \frac{dy}{y} \\ &= \frac{1}{s-1} \int_0^\infty y^{-(s-1)} \cdot y \frac{\partial}{\partial y} \left(\frac{\varphi(y)}{y} \right) \frac{dy}{y} \end{aligned}$$

Thus, the needed division by powers of $s-1$ can be achieved by taking suitable derivatives of the data entering the pseudo-Eisenstein series. In particular, constant functions are *not* orthogonal to pseudo-Eisenstein series.

6. Toward a Plancherel theorem

For $\varphi \in C_c^\infty(N \backslash G)$, the pseudo-Eisenstein series Ψ_φ is expressed in terms of Eisenstein series, with complex-bilinear pairing $\langle \cdot, \cdot \rangle$,

$$\Psi_\varphi = \left\langle \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \langle E_{1-s}, \Psi_\varphi \rangle E_s ds + \langle 1, \Psi_\varphi \rangle \text{res}_{s=1} E_s \right\rangle$$

Then for $f \in C_c^\infty(\Gamma \backslash G)$, still with complex-bilinear pairing,

$$\langle \Psi_\varphi, f \rangle = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \langle E_{1-s}, \Psi_\varphi \rangle \langle E_s, f \rangle ds + \langle 1, \Psi_\varphi \rangle \langle \text{res}_{s=1} E_s, f \rangle$$

Replacing f by $\overline{\Psi}_\psi$ for another $\psi \in C_c^\infty((P \cap \Gamma)N \backslash \mathfrak{H})$, using $\overline{E_s} = E_{1-s}$, and noting that the residue of the Eisenstein series is constant, this gives part of a **Plancherel theorem** on the $L^2(\Gamma \backslash \mathfrak{H})$ -closure of the space of pseudo-Eisenstein series:

[6.0.1] **Theorem:**

$$\langle \Psi_\varphi, \overline{\Psi}_\psi \rangle = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \langle \Psi_\varphi, \overline{E_s} \rangle \langle E_s, \overline{\Psi}_\psi \rangle ds + \langle \Psi_\varphi, 1 \rangle \langle 1, \overline{\Psi}_\psi \rangle \cdot \text{res}_{s=1} E_s$$

[6.0.2] **Remark:** What is missing in this description is characterization of the functions $s \rightarrow \langle \Psi_\varphi, E_{1-s} \rangle$, or of the L^2 closure of this image.

Bibliography

[Godement 1966a] R. Godement, *Decomposition of $L^2(\Gamma \backslash G)$ for $\Gamma = SL(2, Z)$* , in Proc. Symp. Pure Math. **9** (1966), AMS, 211-24.