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Primes in arithmetic progressions

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Dirichlet's 1837 theorem combines ideas from Euclid's argument for the infinitude of primes with harmonic analysis on finite abelian groups, and some subtler things, to show that there are infinitely many primes $p = a \bmod N$ for fixed a invertible modulo fixed N.

The most intelligible proof of this result uses a bit of analysis, in addition to some interesting algebraic ideas. The analytic idea already arose with Euler's proof of the infinitude of primes, which we give below. New algebraic ideas due to Dirichlet allowed isolation of primes in different congruence classes modulo N.

Dirichlet's result introduces the **dual group**, or **group of characters**, of a finite abelian group. This idea was one impetus to the development of a more abstract notion of *group*, and also of *group representations* studied by Schur and Frobenius.

The subtle element is *non-vanishing of L-functions*, explained below. For expediency, we first give an ugly proof that explains little, does not apply in many interesting situations, but has few prerequisites.

There are two better lines of argument, both of which give *mechanisms* whereby L-functions ought not vanish. A form of the simpler one was used by Dirichlet, expressing products of Dirichlet L-functions as *zeta functions of number fields*. The less simple one is at most 50 years old, and uses *Eisenstein series*. Both better viewpoints will be explained subsequently.

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1. *Dirichlet's theorem*

In addition to Euler's observation that the analytic behavior [1] of $\zeta(s)$ at $s = 1$ implied the existence of infinitely-many primes, Dirichlet found an algebraic device to focus attention on single congruence classes modulo N.

This section gives the central argument, and in doing so *uncovers* several issues taken up in following sections.

[1.0.1] Theorem: *(Dirichlet)* Given an integer $N > 1$ and an integer a such that $\gcd(a, N) = 1$, there are infinitely many primes p with

$$p \;=\; a \bmod N$$

[1.0.2] Remark: If $\gcd(a, N) > 1$, then there is at most one prime p meeting the condition $p = a \bmod n$, since any such p would be divisible by the *gcd*. Thus, the *gcd* condition is *necessary*. The point is that this *obvious necessary* condition is also *sufficient.*

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[1] Euler's proof uses only simple properties of $\zeta(s)$, and only of $\zeta(s)$ as a function of a *real*, rather than *complex*, variable. Given the status of complex number and complex analysis in Euler's time, this is not surprising. It is slightly more surprising that Dirichlet's original argument also was a real-variable argument, since by that time, a hundred years later, complex analysis was well-established. Still, until Riemann's memoir of 1857-8 there was little reason to believe that the behavior of $\zeta(s)$ off the real line played a critical role.

[1.0.3] **Remark:** For $a = 1$, there is a simple purely algebraic argument using cyclotomic polynomials. For general a the intelligible argument involves a little analysis.

Proof: A **character** modulo N is a group homomorphism

$$\chi : (\mathbb{Z}/N)^\times \longrightarrow \mathbb{C}^\times$$

Given such a character, *extend it by* 0 to all of $\mathbb{Z}/N$, by defining $\chi(a) = 0$ for a not invertible modulo N. Then compose χ with the reduction-mod-N map $\mathbb{Z} \to \mathbb{Z}/N$ and consider χ as a function on $\mathbb{Z}$. Even when extended by 0 the function χ is still *multiplicative* in the sense that

$$\chi(mn) \;=\; \chi(m) \cdot \chi(n)$$

whether or not either of the values is 0. The pulled-back-to-$\mathbb{Z}$ version of χ, with the extension by 0, is a **Dirichlet character**. The **trivial** Dirichlet character χ_o modulo N is the character which takes only the value 1 (and 0).

Recall the standard **cancellation trick**, that applies more generally to arbitrary finite groups:

$$\sum_{a \bmod N} \chi(a) \;=\; \begin{cases} \varphi(N) & (\text{for } \chi = \chi_o) \\ 0 & (\text{otherwise}) \end{cases}$$

where φ is Euler's *totient* function. Dirichlet's *dual* trick is to sum over characters χ mod N evaluated at fixed a in $(\mathbb{Z}/N)^\times$: we *claim* that

$$\sum_{\chi} \chi(a) \;=\; \begin{cases} \varphi(N) & (\text{for } a = 1 \bmod N) \\ 0 & (\text{otherwise}) \end{cases}$$

We will prove this in the next section.

Granting this, for b invertible modulo N,

$$\sum_{\chi} \chi(a)\chi(b)^{-1} \;=\; \sum_{\chi} \chi(ab^{-1}) \;=\; \begin{cases} \varphi(N) & (\text{for } a = b \bmod N) \\ 0 & (\text{otherwise}) \end{cases}$$

Given a Dirichlet character χ modulo N, the corresponding **Dirichlet L-function** is

$$L(s, \chi) \;=\; \sum_{n \geq 1} \frac{\chi(n)}{n^s}$$

By the multiplicative property $\chi(mn) = \chi(m)\chi(n)$, each such L-function has an **Euler product** expansion

$$L(s, \chi) \;=\; \prod_{p \text{ prime},\, p \nmid N} \frac{1}{1 - \chi(p)\, p^{-s}}$$

proven as for $\zeta(s)$, by expanding geometric series. Take a logarithmic derivative, as with zeta:

$$\frac{d}{ds} \log L(s, \chi) \;=\; \sum_{p \nmid N \;\; m \geq 1} \frac{\chi(p)^m \, \log p}{p^{ms}} \;=\; \sum_{p \nmid N} \frac{\chi(p) \, \log p}{p^s} \;+\; \sum_{p \nmid N,\, m \geq 2} \frac{\chi(p)^m \, \log p}{p^{ms}}$$

The second sum on the right will turn out to be subordinate to the first, so we aim our attention at the first sum, where $m = 1$.

To pick out the primes p with $p = a \bmod N$, use Dirichlet's sum-over-χ trick to obtain

$$\sum_{\chi \bmod N} \chi^{-1}(a) \cdot \frac{\chi(p)\log p}{p^s} \;=\; \begin{cases} \varphi(N) \cdot \dfrac{\log p}{p^s} & (\text{for } p = a \bmod N) \\ 0 & (\text{otherwise}) \end{cases}$$

Thus,

$$\sum_{\chi \bmod N} \chi^{-1}(a) \frac{d}{ds} \log L(s,\chi) \;=\; \sum_{\chi \bmod N} \chi^{-1}(a) \sum_{p \nmid N,\, m \geq 1} \frac{\chi(p)^m \log p}{p^{ms}}$$

$$=\; \varphi(N) \sum_{p = a \bmod N} \frac{\log p}{p^s} \;+\; \sum_{\chi \bmod N} \chi^{-1}(a) \sum_{p \nmid N,\, m \geq 2} \frac{\chi(p)^m \log p}{p^{ms}}$$

We do not care about cancellation in the second sum. All that we need is its absolute convergence for $\mathrm{Re}(s) > \frac{1}{2}$, needing no subtle information about primes. Dominate the sum over primes by the corresponding sum over integers ≥ 2. Namely,

$$\sum_{p \nmid N,\, m \geq 2} \left| \frac{\chi(p)^m \log p}{p^{ms}} \right| \leq \sum_{n \geq 2,\, m \geq 2} \frac{\log n}{n^{m\sigma}} \;=\; \sum_{n \geq 2} \frac{(\log n)/n^{2\sigma}}{1 - n^{-\sigma}} \leq \frac{1}{1 - 2^{-\sigma}} \sum_{n \geq 2} \frac{\log n}{n^{2\sigma}}$$

where $\sigma = \mathrm{Re}(s)$. This converges for $\mathrm{Re}(s) > \frac{1}{2}$. That is, for $s \to 1^+$,

$$\sum_{\chi \bmod N} \chi^{-1}(a) \frac{d}{ds} \log L(s,\chi) \;=\; \varphi(N) \sum_{p = a \bmod N} \frac{\log p}{p^s} \;+\; (\text{something continuous at } s = 1)$$

We have isolated the primes $p = a \bmod N$. Thus, as Dirichlet saw, to prove the infinitude of primes $p = a \bmod N$ it would suffice to show that the left-hand side of the last inequality blows up at $s = 1$. In particular, for the **trivial** character $\chi_o \bmod N$, with values

$$\chi(b) \;=\; \begin{cases} 1 & (\text{for } \gcd(b,N) = 1) \\ 0 & (\text{for } \gcd(b,N) > 1) \end{cases}$$

the associated L-function is essentially the zeta function, namely

$$L(s,\chi_o) \;=\; \zeta(s) \cdot \prod_{p|N} \left(1 - \frac{1}{p^s}\right)$$

Since none of those finitely-many factors for primes dividing N is 0 at $s = 1$, $L(s,\chi_o)$ still blows up at $s = 1$, like a non-zero constant multiple of $1/(s-1)$.

By contrast, we will show below that for **non-trivial** character $\chi \bmod N$, $\lim_{s \to 1^+} L(s,\chi)$ is *finite*, and

$$\lim_{s \to 1^+} L(s,\chi) \;\neq\; 0$$

Thus, for non-trivial character, the logarithmic derivative is finite and non-zero at $s = 1$. Putting this all together, we will have

$$\lim_{s \to 1^+} \sum_{\chi \bmod N} \chi(a) \frac{d}{d} \log L(s,\chi) \;=\; +\infty$$

Then necessarily

$$\lim_{s \to 1^+} \sum_{p = a \bmod N} \frac{\log p}{p^s} \;=\; +\infty$$

and there must be infinitely many primes $p = a \bmod N$. ///

[1.1] What remains to be done? The non-vanishing of the non-trivial L-functions at 1, which we prove a bit further below, is the crucial technical point. We prove Dirichlet's dual cancellation trick in the next section: this is an immediate consequence of Fourier analysis on finite abelian groups. We will also check that the L-functions $L(s, \chi)$ have analytic continuations to regions including $s = 1$.

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2. *Dual groups of abelian groups*

Dirichlet's use of group characters to isolate primes in a specified congruence class modulo N was a big innovation in 1837. These ideas were predecessors of the group theory work of Frobenious and Schur 50 years later, and one of the ancestors of *representation theory* of groups.

The **dual group** or **group of characters** $\widehat{G}$ of a finite abelian group G is by definition

$$\widehat{G} \;=\; \{\text{group homomorphisms } \chi : G \to \mathbb{C}^\times\}$$

This $\widehat{G}$ is itself an abelian group under the operation on characters defined for $g \in G$ by

$$(\chi_1 \cdot \chi_2)(g) \;=\; \chi_1(g) \cdot \chi_2(g)$$

Recall the basic result on Fourier expansions on finite abelian groups:

[2.0.1] Theorem: For a finite abelian group G with dual group $\widehat{G}$, any complex-valued function f on G has a Fourier expansion

$$f(g) \;=\; \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \widehat{f}(\chi)\, \chi(g) \qquad\qquad (\text{for all } g \in G)$$

where the Fourier coefficients $\widehat{f}(\chi)$ are

$$\widehat{f}(\chi) \;=\; \sum_{g \in G} f(g)\, \overline{\chi}(g)$$

The characters are an orthogonal basis for $L^2(G)$. In particular, Fourier coefficients are unique. (This is really about *commuting unitary operators* on finite-dimensional complex vector spaces, and the main point is the spectral theorem for unitary operators.) ///

[2.0.2] Corollary: Let G be a finite abelian group. For $g \neq e$ in G, there is a character $\chi \in \widehat{G}$ such that $\chi(g) \neq 1$. [2]

Proof: Suppose that $\chi(g) = 1$ for all $\chi \in \widehat{G}$. That is, $\chi(g) = \chi(e)$ for all χ. Then, for any coefficients c_χ,

$$\sum_\chi c_\chi\, \chi(e) \;=\; \sum_\chi c_\chi\, \chi(g)$$

Since every function on the group has such a Fourier expansion, this says that every function on G has the same value at g as at e. Thus, $g = e$. ///

[2.0.3] Corollary: For a finite abelian group G,

$$|G| \;=\; |\widehat{G}|$$

[2] This idea that characters can distinguish group elements from each other is just the tip of an iceberg.

Proof: The characters form an orthogonal basis for $L^2(G)$, so the number of characters is the dimension of $L^2(G)$, which is $|G|$. ///

[2.0.4] Remark: In fact, using the structure theorem for finite abelian groups, one can show that G and its dual are *isomorphic*, but this isomorphism is *not canonical*.

[2.0.5] Corollary: *(Dual version of cancellation trick)* For g in a finite abelian group,

$$\sum_{\chi \in \widehat{G}} \chi(g) \;=\; \begin{cases} |\widehat{G}| & (\text{for } g = e) \\ 0 & (\text{otherwise}) \end{cases}$$

Proof: If $g = e$, then the sum counts the characters in $\widehat{G}$. On the other hand, given $g \neq e$ in G, let χ_1 be in $\widehat{G}$ such that $\chi_1(g) \neq 1$, from a previous corollary. The map on $\widehat{G}$

$$\chi \to \chi_1 \cdot \chi$$

is a bijection of $\widehat{G}$ to itself, so

$$\sum_{\chi \in \widehat{G}} \chi(g) \;=\; \sum_{\chi \in \widehat{G}} (\chi \cdot \chi_1)(g) \;=\; \chi_1(g) \cdot \sum_{\chi \in \widehat{G}} \chi(g)$$

which gives

$$(1 - \chi_1(g)) \cdot \sum_{\chi \in \widehat{G}} \chi(g) \;=\; 0$$

Since $1 - \chi_1(g) \neq 0$, it must be that the sum is 0. ///

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3. *Ugly proof of non-vanishing on* $\mathrm{Re}(s) = 1$

Dirichlet's argument for the infinitude of primes $p = a \bmod N$ (for $\gcd(a, N) = 1$) requires that $L(1, \chi) \neq 0$ for all $\chi \bmod N$. We prove this now, *granting* that these functions have meromorphic extensions to some neighborhood of $s = 1$. We also need to know that for the trivial character $\chi_o \bmod N$ the L-function $L(s, \chi_o)$ has a *simple* pole at $s = 1$. These analytical facts are proven in the next section.

The argument here is unilluminating, but has low prerequisites.

[3.0.1] Theorem: For a Dirichlet character $\chi \bmod N$ other than the trivial character $\chi_o \bmod N$,

$$L(1, \chi) \;\neq\; 0$$

Proof: To prove that the L-functions $L(s, \chi)$ do not vanish at $s = 1$, and in fact do not vanish on the whole line [3] $\mathrm{Re}(s) = 1$, direct arguments involve tricks similar to what we do here.

[3] Non-vanishing of $\zeta(s)$ on the whole line $\mathrm{Re}(s) = 1$ yields the Prime Number Theorem: let $\pi(x)$ be the number of primes less than x. Then $\pi(x) \sim x/\ln x$, meaning that the limit of the ratio of the two sides as $x \to \infty$ is 1. This was first proven in 1896, separately, by Hadamard and de la Vallée Poussin. The same sort of argument also gives an analogous *asymptotic* statement about primes in each congruence class modulo N, namely that $\pi_{a,N}(x) \sim x/[\varphi(N) \cdot \ln x]$, where $\gcd(a, N) = 1$ and φ is Euler's totient function.

First, for χ whose square is not the trivial character χ_o modulo N, the standard trick is to consider

$$\lambda(s) \;=\; L(s,\chi_o)^3 \cdot L(s,\chi)^4 \cdot L(s,\chi^2)$$

Then, letting $\sigma = \mathrm{Re}\,(s)$, from the Euler product expressions for the L-functions noted earlier, in the region of convergence,

$$|\lambda(s)| \;=\; |\exp\left(\sum_{m,p} \frac{3+4\chi(p^m)+\chi^2(p^m)}{mp^{ms}}\right)| \;=\; \exp\left|\sum_{m,p} \frac{3+4\cos\theta_{m,p}+\cos 2\theta_{m,p}}{mp^{m\sigma}}\right|$$

where for each m and p we let

$$\theta_{m,p} \;=\; (\text{the argument of } \chi(p^m)) \in \mathbb{R}$$

The trick [4] is that for *any* real θ

$$3+4\cos\theta+\cos 2\theta \;=\; 3+4\cos\theta+2\cos^2\theta-1 \;=\; 2+4\cos\theta+2\cos^2\theta \;=\; 2(1+\cos\theta)^2 \ge 0$$

Therefore, all the terms inside the large sum being exponentiated are non-negative, and, [5]

$$|\lambda(s)| \ge e^0 \;=\; 1$$

In particular, if $L(1,\chi)=0$ were to be 0, then, since $L(s,\chi_o)$ has a simple pole at $s=1$ and since $L(s,\chi^2)$ does *not* have a pole (since $\chi^2 \neq \chi_o$), the multiplicity ≥ 4 of the 0 in the product of L-functions would overwhelm the three-fold pole, and $\lambda(1)=0$. This would contradict the inequality just obtained.

For $\chi^2 = \chi_o$, instead consider

$$\lambda(s) \;=\; L(s,\chi)\cdot L(s,\chi_o) \;=\; \exp\left(\sum_{p,m} \frac{1+\chi(p^m)}{mp^{ms}}\right)$$

If $L(1,\chi)=0$, then this would cancel the simple pole of $L(s,\chi_o)$ at 1, giving a non-zero finite value at $s=1$. The series inside the exponentiation is a *Dirichlet series with non-negative coefficients*, and for real s

$$\sum_{p,m} \frac{1+\chi(p^m)}{mp^{ms}} \ge \sum_{p,\,m\text{ even}} \frac{1+1}{mp^{ms}} \;=\; \sum_{p,\,m} \frac{1+1}{2mp^{2ms}} \;=\; \sum_{p,\,m} \frac{1}{mp^{2ms}} \;=\; \log\zeta(2s)$$

Since $\zeta(2s)$ has a simple pole at $s=\frac{1}{2}$ the series

$$\log\left(L(s,\chi)\cdot L(s,\chi_o)\right) \;=\; \sum_{p,m} \frac{1+\chi(p^m)}{mp^{ms}} \ge \log\zeta(2s)$$

necessarily blows up as $s \to \frac{1}{2}^+$. But by **Landau's Lemma** below, a Dirichlet series with non-negative coefficients cannot blow up as $s \to s_o$ along the real line unless the function represented by the series fails to be holomorphic at s_o. Since the function given by $\lambda(s)$ is holomorphic at $s=1/2$, this gives a contradiction to the supposition that $\lambda(s)$ is holomorphic at $s=1$ (which had allowed this discussion at $s=1/2$). That is, $L(1,\chi)\neq 0$. ///

[3.0.2] Remark: Again, the above argument is quick, but unilluminating. We will give better proofs later.

[4] Presumably found after considerable fooling around.

[5] Miraculously...

4. *Analytic continuations*

Dirichlet's original argument did not emphasize holomorphic functions, but by now we know that discussion of vanishing and blowing-up of functions is most clearly and simply accomplished if the functions are *meromorphic* when viewed as functions of a complex variable.

For the purposes of Dirichlet's theorem, it suffices to meromorphically continue [6] the L-functions to $\mathrm{Re}(s) > 0$. [7] Since we need only this *slight* analytic continuation, we can give a simpler argument than would be needed to analytically continue these L-functions to the entire plane.

[4.0.1] Theorem: The Dirichlet L-functions

$$L(s,\chi) \;=\; \sum_n \frac{\chi(n)}{n^s} \;=\; \prod_p \frac{1}{1-\chi(p)\,p^{-s}}$$

have meromorphic continuations to $\mathrm{Re}(s) > 0$. For χ non-trivial, $L(s,\chi)$ is *holomorphic* on that half-plane. For χ trivial, $L(s,\chi_o)$ has a *simple* pole at $s=1$ and is holomorphic otherwise.

Proof: First, to treat the trivial character χ_o mod N, recall, as already observed, that the corresponding L-function differs in an elementary way from $\zeta(s)$, namely

$$L(s,\chi_o) \;=\; \zeta(s)\cdot \prod_{p|N}\left(1-\frac{1}{p^s}\right)$$

Thus, we analytically continue $\zeta(s)$ instead of $L(s,\chi_o)$. To analytically continue $\zeta(s)$ to $\mathrm{Re}(s) > 0$ observe that the sum for $\zeta(s)$ is fairly well approximated by a more elementary function

$$\zeta(s)-\frac{1}{s-1} \;=\; \sum_{n=1}^\infty \frac{1}{n^s} - \int_1^\infty \frac{dx}{x^s} \;=\; \sum_{n=1}^\infty \left[\frac{1}{n^s} - \frac{\left(\frac{1}{n^{s-1}}-\frac{1}{(n+1)^{s-1}}\right)}{1-s}\right]$$

Since

$$\frac{\left(\frac{1}{n^{s-1}}-\frac{1}{(n+1)^{s-1}}\right)}{1-s} \;=\; \frac{1}{n^s}+O(\frac{1}{n^{s+1}})$$

with a uniform O-term, we obtain

$$\zeta(s)-\frac{1}{s-1} \;=\; \sum_n O(\frac{1}{n^{s+1}}) \;=\; \text{holomorphic for } \mathrm{Re}(s)>0$$

[6] An extension of a holomorphic function to a larger region, on which it may have some poles, is called a **meromorphic continuation**. There is *no* general methodology for proving that functions have meromorphic continuations, due in part to the fact that, generically, functions *do not* have continuations beyond some natural region where they're defined by a convergent series or integral. Indeed, to be able to prove a meromorphic continuation result for a given function is tantamount to proving that it has some deeper significance.

[7] Already prior to Riemann's 1859 paper, it was known that the Euler-Riemann zeta function and all the L-functions we need here did indeed have meromorphic continuations to the whole complex plane, have no poles unless the character χ is trivial, and have functional equations similar to that of zeta, namely that $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is invariant under $s\to 1-s$.

The obvious analytic continuation of $1/(s-1)$ allows analytic continuation of $\zeta(s)$.

A relatively elementary analytic continuation argument for *non-trivial* characters uses **partial summation**. That is, let $\{a_n\}$ and $\{b_n\}$ be sequences of complex numbers such that the partial sums $A_n = \sum_{i=1}^n a_i$ are *bounded*, and $b_n \to 0$. Then it is useful to rearrange (taking $A_0 = 0$ for notational convenience)

$$\sum_{n=1}^\infty a_n b_n \;=\; \sum_{n=1}^\infty (A_n - A_{n-1}) b_n \;=\; \sum_{n=0}^\infty A_n b_n - \sum_{n=0}^\infty A_n b_{n+1} \;=\; \sum_{n=0}^\infty A_n (b_n - b_{n+1})$$

Taking $a_n = \chi(n)$ and $b_n = 1/n^s$ gives

$$L(s,\chi) \;=\; \sum_{n=0}^\infty \left(\sum_{\ell=1}^n \chi(\ell) \right) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right)$$

The difference $1/n^s - 1/(n+1)^s$ is s/n^{s+1} up to higher-order terms, so this expression gives a holomorphic function for $\mathrm{Re}(s) > 0$. ///

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5. *Dirichlet series with positive coefficients*

Now we prove Landau's result on Dirichlet series with positive coefficients. (More precisely, the coefficients are *non-negative*.)

[5.0.1] Theorem: *(Landau)* Let

$$f(s) \;=\; \sum_{n=1}^\infty \frac{a_n}{n^s}$$

be a Dirichlet series with real coefficients $a_n \ge 0$. Suppose that the series defining $f(s)$ converges for $\mathrm{Re}(s) > \sigma_o$. Suppose further that the function f extends to a function holomorphic in a neighborhood of $s = \sigma_o$. Then, in fact, the series defining $f(s)$ converges for $\mathrm{Re}(s) > \sigma_o - \varepsilon$ for some $\varepsilon > 0$.

Proof: First, by replacing s by $s - \sigma_o$ we lighten the notation by reducing to the case that $\sigma_o = 0$. Since the function $f(s)$ given by the series is holomorphic on $\mathrm{Re}(s) > 0$ and on a neighborhood of 0, there is $\varepsilon > 0$ such that $f(s)$ is holomorphic on $|s-1| < 1 + 2\varepsilon$, and the power series for the function converges nicely on this open disk. Differentiating the original series termwise (Abel's theorem), we evaluate the derivatives of $f(s)$ at $s = 1$ as

$$f^{(i)}(1) \;=\; \sum_n \frac{(-\log n)^i\, a_n}{n} \;=\; (-1)^i \sum_n \frac{(\log n)^i\, a_n}{n}$$

and Cauchy's formulas yield, for $|s-1| < 1 + 2\varepsilon$,

$$f(s) \;=\; \sum_{i \ge 0} \frac{f^{(i)}(1)}{i!}\,(s-1)^i$$

In particular, for $s = -\varepsilon$, we are assured of the convergence to $f(-\varepsilon)$ of

$$f(-\varepsilon) \;=\; \sum_{i \ge 0} \frac{f^{(i)}(1)}{i!}\,(-\varepsilon - 1)^i$$

Note that $(-1)^i f^{(i)}(1)$ is a positive Dirichlet series, so we move the powers of -1 a little to obtain

$$f(-\varepsilon) \;=\; \sum_{i \ge 0} \frac{(-1)^i f^{(i)}(1)}{i!}\,(\varepsilon + 1)^i$$

The series

$$(-1)^i f^{(i)}(1) \;=\; \sum_n (\log n)^i \frac{a_n}{n}$$

has positive terms, so the double series (convergent, with positive terms)

$$f(-\varepsilon) \;=\; \sum_{n,i} \frac{a_n\,(\log n)^i}{i!}(1+\varepsilon)^i \frac{1}{n}$$

can be rearranged to

$$f(-\varepsilon) \;=\; \sum_n \frac{a_n}{n}\left(\sum_i \frac{(\log n)^i(1+\varepsilon)^i}{i!}\right) \;=\; \sum_n \frac{a_n}{n} n^{(1+\varepsilon)} \;=\; \sum_n \frac{a_n}{n^{-\varepsilon}}$$

That is, the latter series converges (absolutely). ///
