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Closed topological subgroups of $\mathbb{R}^n$

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[0.0.1] Theorem: The closed topological subgroups H of $V \approx \mathbb{R}^n$ are the following: for a *vector subspace* W of V, and for a *discrete* subgroup Γ of V/W,

$$H \;=\; q^{-1}(\Gamma) \qquad\qquad \text{(with } q : V \to V/W \text{ the quotient map)}$$

The *discrete* subgroups Γ of $V \approx \mathbb{R}^n$ are free $\mathbb{Z}$-modules $\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_m$ on $\mathbb{R}$-linearly-independent vectors $v_j \in V$, with $m \le n$.

Proof: Unsurprisingly, part of the discussion does induction on $n = \dim_{\mathbb{R}} V$.

We treat $n = 1$ directly, to illustrate part of the mechanism. Let H be a non-trivial closed subgroup of $\mathbb{R}$. We need only consider *proper* closed subgroups H. We claim that H is a free $\mathbb{Z}$-module on a single generator. Since H is not 0, and is closed under additive inverses, H contains *positive* elements. In the case that there is a *least* positive element h_o, claim that $H = \mathbb{Z} \cdot h_o$. Indeed, given $0 < h \in H$, by the archimedean property of $\mathbb{R}$, there is an integer ℓ such that $\ell \cdot h_o \le h < (\ell + 1) \cdot h_o$. Either $h = \ell \cdot h_o$ and $h \in \mathbb{Z} \cdot h_o$, or else $0 < h - \ell \cdot h_o < h_o$, contradiction. Thus, when H has a smallest positive element h_o, $H = \mathbb{Z} \cdot h_o$. Now suppose that there are $h_1 > h_2 > \ldots > 0$ in H, and show that $H = \mathbb{R}$. Since H is closed, the infimum h_o of the h_j is in H. Since H is a group, $0 < h_j - h_o \in H$. Replacing h_j by $h_j - h_o$, we can suppose that $h_j \to 0$. Thus, $H \supset \mathbb{Z} \cdot h_j$. The collection of integer multiples of $h_j > 0$ contains elements within distance h_j of any real number, by the archimedean property of $\mathbb{R}$. Since $h_j \to 0$, every real number is in the closure of H. Since H is closed, $H = \mathbb{R}$. This completes the argument that proper, closed subgroups of $\mathbb{R}$ are free $\mathbb{Z}$-modules on a single generator.

Now the general case:

If H contains a *line* W, we reduce to a lower-dimensional question, as follows. Let $q : V \to V/W$ be the quotient map. Then $H = q^{-1}(q(H))$. With $H' = q(H)$, by induction on dimension of the ambient vectorspace, there is a vector subspace W' of V/W and discrete subgroup Γ' of $(V/W)/W'$ such that

$$H' \;=\; q'^{-1}(q'(\Gamma')) \qquad\qquad \text{(with } q' : V/W \to (V/W)/W' \text{ the quotient map)}$$

Then

$$H \;=\; q^{-1}\Big(q(H)\Big) \;=\; q^{-1}\Big(q'^{-1}(\Gamma')\Big) \;=\; (q' \circ q)^{-1}(\Gamma')$$

The kernel of $q' \circ q$ is the vector subspace $W = q^{-1}(W')$ of V. It is necessary to check that $q(H) = H/W$ is a *closed* subgroup of V/W. It suffices to prove that $q^{-1}(V/W - q(H))$ is *open*. Since H contains W, $q^{-1}\big(q(H)\big) = H$, and

$$q^{-1}(V/W - q(H)) \;=\; V - q^{-1}\big(q(H)\big) \;=\; V - H \;=\; V - (\text{closed}) \;=\; \text{open}$$

This shows that $q(H)$ is closed, and completes the induction step when $\mathbb{R} \cdot h \subset H$.

Next show that H containing *no* lines is *discrete*. If not, then there are distinct h_i in H with an accumulation point h_o. Since H is closed, $h_o \in H$, and replace h_i by $h_i - h_o$ so that, without loss of generality, the accumulation point is 0. Without loss of generality, remove any 0s from the sequence. The sequence $h_i/|h_i|$ has an accumulation point e on the unit sphere, since the sphere is compact. Replace the sequence by a subsequence so that the $h_i/|h_i|$ converge to e. Given real $t \ne 0$, let $n \ne 0$ be an integer so that $|n - \frac{t}{|h_i|}| \le 1$. Then

$$|n \cdot h_i - te| \;\le\; \Big|(n - \frac{t}{|h_i|})h_i\Big| + \Big|\frac{th_i}{|h_i|} - te\Big| \;\le\; 1 \cdot |h_i| + |t| \cdot \Big|\frac{h_i}{|h_i|} - e\Big|$$

Since $|h_i| \to 0$ and $h_i/|h_i| \to e$, this goes to 0. Thus, te is in the closure of $\bigcup_i \mathbb{Z} \cdot h_i$. Thus, H contains the line $\mathbb{R} \cdot e$, contradiction. That is, H is discrete.

For H containing no lines, we just showed that H is *discrete*. We claim that discrete H is generated as a $\mathbb{Z}$-module by at most n elements, and that these are $\mathbb{R}$-linearly independent. For $h_1, \ldots, h_m$ in H linearly *dependent* over $\mathbb{R}$, there are real numbers r_i so that

$$r_1 h_1 + \ldots + r_m h_m \;=\; 0$$

Re-ordering if necessary, suppose that $r_1 \neq 0$. Given a large integer N, let $a_i^{(N)}$ be integers so that $|r_i - a_i^{(N)}/N| < 1/N$. Then

$$\sum_i a_i^{(N)} h_i \;=\; N \sum_i \Big(\frac{a_i^{(N)}}{N} - r_i\Big) h_i + N \sum_i r_i h_i \;=\; N \sum_i \Big(\frac{a_i^{(N)}}{N} - r_i\Big) h_i + 0$$

Then

$$\Big| \sum_i a_i^{(N)} h_i \Big| \;\leq\; N \sum_i \frac{1}{N}\, |h_i| \;\leq\; \sum_i |h_i|$$

That is, the $\mathbb{Z}$-linear combination $\sum_i a_i^{(N)} h_i \in H$ is inside the ball of radius $\sum_i |h_i|$ centered at 0. There is such a point for every N. Since H is discrete, there are only finitely-many *different* points of this form. Since $r_1 \neq 0$ and $|N r_1 - a_1^{(N)}| < 1$, for large varying N the corresponding integers $a_1^{(N)}$ are *distinct*. Thus, for some large $N < N'$,

$$\sum_i a_i^{(N)} h_i \;=\; \sum_i a_i^{(N')} h_i$$

Subtracting,

$$\sum_i \big(a_i^{(N)} - a_i^{(N')}\big) h_i \;=\; 0 \qquad\qquad (\text{with } a_1^{(N)} - a_1^{(N')} \neq 0)$$

This is a non-trivial $\mathbb{Z}$-linear dependence relation among the h_i. Thus, $\mathbb{R}$-linear dependence implies $\mathbb{Z}$-linear dependence of the h_i in a *discrete* subgroup H. ///
