(*December 21, 2010*)

Abelian topological groups and $(\mathbb{A}/k)\hat{} \approx k$

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

The specific goal here is to prove that the unitary dual of the adele quotient $\mathbb{A}/k$ is k, for any number field k. Other than the compactness of $\mathbb{A}/k$ and the self-duality of $\mathbb{A}$, the argument rests on general principles.

The specific fact that the unitary dual of the adele quotient $\mathbb{A}/k$ is isomorphic to k is essential to the Iwasawa-Tate treatment of automorphic zeta-functions and L-functions for GL_1: [Iwasawa 1950/52], [Tate 1950/67], [Iwasawa 1952/92].

See [Weil 1940/1965] for a bibliography of the development of the theory of topological groups prior to 1938.

---

1. *Compact-discrete duality*

Consider *abelian* topological groups G. Let S^1 be the unit circle in $\mathbb{C}$. The unitary dual of G is

$$\widehat{G} \;=\; \mathrm{Hom}^o(G, S^1) \;=\; \{\text{continuous group homs } G \to S^1\}$$

Pointwise multiplication makes $\widehat{G}$ an abelian group. A reasonable topology [1] on $\widehat{G}$ is the *compact-open* topology, with a sub-basis of opens

$$U \;=\; U_{C,E} \;=\; \{f \in \widehat{G} \;:\; f(C) \subset E\} \qquad\qquad (\text{for compact } C \text{ in } G, \text{ open } E \text{ in } S^1)$$

Granting that the compact-open topology makes $\widehat{G}$ a abelian (locally-compact, Hausdorf) topological group,

[1.0.1] Theorem: The unitary dual of a *compact* abelian group is *discrete*. The unitary dual of a *discrete* abelian group is *compact*.

Proof: Let G be compact. Let E be a small-enough open in S^1 so that E contains no non-trivial subgroups of G. Noting that G itself is *compact*, let $U \subset \widehat{G}$ be the open

$$U \;=\; \{f \in \widehat{G} \;:\; f(G) \subset E\}$$

Since E is *small*, $f(G) = \{1\}$. That is, f is the trivial homomorphism. This proves discreteness of $\widehat{G}$.

For G discrete, *every* group homomorphism to S^1 is continuous. The space of *all* functions $G \to S^1$ is the cartesian product of copies of S^1 indexed by G. By Tychonoff's theorem, with the product topology, this product is *compact*. Indeed, for *discrete* X, the compact-open topology on the space $C^o(X, Y)$ of continuous functions from $X \to Y$ *is* the product topology on copies of Y indexed by X.

---

[1] The reasonable-ness of the compact-open topology is in its function. First, on a compact topological space X, the space $C^o(X)$ of continuous $\mathbb{C}$-valued functions with the *sup-norm* (of absolute value) is a *Banach space* . On non-compact X, the semi-norms given by sups of absolute values on compacts make $C^o(X)$ a *Fréchet space*. The compact-open topology accommodates spaces of continuous functions $C^o(X, Y)$ where the target space Y is not a subset of a normed real or complex vector space, and is most interesting when Y is a *topological group*. In the latter case, when the source X is also a topological group, the subset of all continuous functions $f : X \to Y$ consisting of *group homomorphisms* is a (locally compact, Hausdorff) topological group. This is proven in an appendix.

The subset of functions f satisfying the group homomorphism condition

$$f(gh) \;=\; f(g)\cdot f(h) \qquad\qquad (\text{for } g,h\in G)$$

is *closed*, since the group multiplication $f(g)\times f(h)\to f(g)\cdot f(h)$ in S^1 is continuous. Since the product is also *Hausdorff*, $\widehat{G}$ is also compact. ///

---

2. $(\mathbb{A}/k)\widehat{}\approx k$

Without other details, for a (discretely topologized) field k with adeles $\mathbb{A}$, we grant that $\mathbb{A}/k$ is *compact*, and that $\mathbb{A}$ is *self-dual*. In fact, the same argument succeeds for any field k sitting discretely inside a *self-dual* abelian topological group $\mathbb{A}$, such that the quotient $\mathbb{A}/k$ is compact.

[2.0.1] Theorem: The unitary dual of the compact quotient $\mathbb{A}/k$ is isomorphic to k. In particular, given any non-trivial character ψ on $\mathbb{A}/k$, *all* characters on $\mathbb{A}/k$ are of the form $x\to\psi(\alpha\cdot x)$ for some $\alpha\in k$.

Proof: Because $\mathbb{A}/k$ is compact, $(\mathbb{A}/k)\widehat{}$ is *discrete*. Since multiplication by elements of k respects cosets $x+k$ in $\mathbb{A}/k$, the unitary dual has a k-vectorspace structure given by

$$(\alpha\cdot\psi)(x) \;=\; \psi(\alpha\cdot x) \qquad\qquad (\text{for } \alpha\in k,\, x\in\mathbb{A}/k)$$

There is no topological issue in this k-vectorspace structure, because $(\mathbb{A}/k)\widehat{}$ is discrete. The quotient map $\mathbb{A}\to\mathbb{A}/k$ gives a natural *injection* $(\mathbb{A}/k)\widehat{}\to\widehat{\mathbb{A}}$.

Given non-trivial $\psi\in(\mathbb{A}/k)\widehat{}$, the k-vectorspace $k\cdot\psi$ inside $(\mathbb{A}/k)\widehat{}$ injects to a copy of $k\cdot\psi$ inside $\widehat{\mathbb{A}}\approx\mathbb{A}$. *Assuming* for a moment that the image in $\mathbb{A}$ is essentially the same as the diagonal copy of k, the quotient $(\mathbb{A}/k)\widehat{}/k$ injects to the compact $\mathbb{A}/k$. The topology of $(\mathbb{A}/k)\widehat{}$ is discrete, and the quotient $(\mathbb{A}/k)\widehat{}/k$ is still discrete. Since all these maps are continuous group homomorphisms, the image of $(\mathbb{A}/k)\widehat{}/k$ in $\mathbb{A}/k$ is a discrete subgroup of a compact group, so is *finite*. Since $(\mathbb{A}/k)\widehat{}$ is a k-vectorspace, the quotient $(\mathbb{A}/k)\widehat{}/k$ must be a singleton. This proves that $(\mathbb{A}/k)\widehat{}\approx k$, granting that the image of $k\cdot\psi$ in $\mathbb{A}\approx\widehat{\mathbb{A}}$ is the usual diagonal copy.

To see how $k\cdot\psi$ is imbedded in $\mathbb{A}\approx\widehat{\mathbb{A}}$, fix non-trivial ψ on $\mathbb{A}/k$, and let ψ be the induced character on $\mathbb{A}$. The self-duality of $\mathbb{A}$ is that the action of $\mathbb{A}$ on $\widehat{\mathbb{A}}$ by $(x\cdot\psi)(y)=\psi(xy)$ gives an *isomorphism*. The subgroup $x\cdot\psi$ with $x\in k$ is certainly the usual diagonal copy. ///

---

3. *Appendix: compact-open topology*

Again, the space $C^o(X,Y)$ of continuous functions from topological space X to topological space Y is most often given the *compact-open* topology, which has a sub-basis of opens consisting of sets

$$U(C,E) \;=\; \{f\in\widehat{G} \;:\; f(C)\subset E\} \qquad\qquad (\text{for compact } C \text{ in } X,\text{ open } E \text{ in } Y)$$

Because the topologies in G and S^1 are group-invariant, any compact is of the form gC for a compact C containing $e\in G$, and any open neighborhood of h is of the form hE for an open E containing e.

[3.1] The main point The unitary dual $\widehat{G}$ of an abelian (locally compact, Hausdorff) topological group is given the subset topology from the compact-open topology on the collection $C^o(G,S^1)$ of all continuous maps from G to the circle group S^1. The elements of $\widehat{G}$ are *characters* of G. Our goal is to prove:

[3.1.1] Theorem: The unitary dual $\widehat{G}$ of an abelian (locally compact, Hausdorff) topological group is an abelian (locally compact, Hausdorff) topological group.

[3.1.2] Remark: We do not prove the *local compactness* in general. The important special cases considered earlier, that the dual of discrete is compact, and *vice-versa*, give the local compactness of the duals in those cases.

Proof: That the unitary dual is *abelian* is immediate, since the multiplication is pointwise by values, and the target group S^1 is abelian. The proof has several components, which we separate:

[3.2] Invariance of the topology First, verify that the topology is *invariant*. That is, given a sub-basis open

$$U(C,E) \;=\; \{f \in \widehat{G} \;:\; f(c) \in E, \text{ for all } c \in C\} \qquad \text{(with } C \text{ compact in } G,\ E \text{ open in } S^1)$$

and given $f_o \in \widehat{G}$, show that $f_o \cdot U(C,E)$ is open. This is not completely trivial, as $f_o \cdot U(C,E)$ is not obviously of the form $U(C',E')$:

$$f_o \cdot U(C,E) \;=\; \{f \in \widehat{G} \;:\; f(c) \in f_o(c) \cdot E, \text{ for all } c \in C\}$$

To show that $f_o \cdot U(C,E)$ is open, we show that every point is contained in a finite intersection of the basic opens, with that intersection contained in $f_o \cdot U(C,E)$.

Fix $f \in f_o \cdot U(C,E)$. Since $f_o^{-1}(c)f(c) \in E$, each $c \in C$ has a neighborhood N_c such that $f_o^{-1}(N_c) \cdot f(N_c) \subset E$. Shrink each N_c to have compact closure $\overline{N}_c$, and so that $f_o^{-1}(\overline{N}_c) \cdot f(\overline{N}_c) \subset E$. By compactness of C, it has a finite subcover $N_i = N_{c_i}$. Thus,

$$f(\overline{N}_i) \;\subset\; f_o(c') \cdot E \qquad \text{(for all } i, \text{ for all } c' \in \overline{N}_i)$$

From the result of the following subsection, an intersection of a *compact* family of opens is open, so

$$E_i \;=\; \bigcap_{c' \in \overline{N}_i} f_o(c') \cdot E \;=\; \text{open}$$

This open E_i is non-empty, since it contains $f(\overline{N}_i)$. Thus,

$$f \;\in\; \bigcap_i U(\overline{N}_i, E_i) \qquad \text{(a finite intersection)}$$

On the other hand, with c_i and $\overline{N}_i$ determined by f, take

$$f' \;\in\; \bigcap_i U(\overline{N}_i, E_i)$$

Then

$$f'(\overline{N}_i) \;\subset\; f_o(c) \cdot E \qquad \text{(for all } c \in \overline{N}_i)$$

In particular,

$$f'(c) \;\in\; f_o(c) \cdot E \qquad \text{(for all } c \in \overline{N}_i)$$

Since the sets $\overline{N}_i$ cover C, we have $f' \in f_o \cdot U(C,E)$. That is,

$$\bigcap_i U(\overline{N}_i, E_i) \;\subset\; f_o \cdot U(C,E)$$

This proves that the translate $f_o \cdot U(C,E)$ is open, in the compact-open topology. That is, the compact-open topology is translation-invariant.

[3.3] Intersections of compact families of opens Now we prove the fact needed above, that *compact intersections of opens are open*, in the following sense. Let H be a topological group, Hausdorff, but not necessarily locally compact. We claim that

$$\bigcap_{k\in K} k\cdot U \;=\; \text{open} \qquad\qquad (\text{for } U\subset H \text{ open, and } K\subset H \text{ compact})$$

For $u \in k\cdot U$ for all k, by the continuity of inversion and the group operation, there are neighborhoods U_k of u and V_k of k such that

$$V_k^{-1}\cdot U_k \;\subset\; U$$

Let $V_i = V_{k_i}$ be a finite subcover of K, and put $U_i = U_{k_i}$. Thus, for $k\in V_i$,

$$k^{-1}\cdot U_i \;\subset\; U \qquad\qquad (\text{for } k\in V_i)$$

Thus,

$$k^{-1}\cdot \bigcap_i U_i \;\subset\; U \qquad\qquad (\text{for all } k\in K)$$

Since *finite* intersections of opens are open, the intersection of the U_i, each containing u, is an open neighborhood of u. That is, the intersection of the translates $k\cdot U$ is open. This proves the claim

[3.4] Continuity of multiplication Next, show that the pointwise multiplication operation

$$(f_1\cdot f_2)(x) \;=\; f_1(x)\cdot f_2(x) \qquad\qquad (\text{for } f_i\in \widehat{G} \text{ and } x\in G)$$

in $\widehat{G}$ is continuous in the compact-open topology. Given a sub-basis neighborhood $U(C,E)$ of $f_1\cdot f_2$, the already-demonstrated invariance of the topology implies that $(f_1f_2)^{-1}U(C,E)$ is open, and is a neighborhood of the trivial character. Thus, without loss of generality, take $f_1 = f$ and $f_2 = f^{-1}$. Given a sub-basic neighborhood $U(C,E)$ of the trivial character in $\widehat{G}$, show that there are neighborhoods U_1 of f and U_2 of f^{-1} such that $U_1\cdot U_2 \subset U(C,E)$. For $U(C,E)$ to be a neighborhood of the trivial character means exactly that $1\in E$ (and $C\neq\phi$).

Let E' be an open neighborhood of 1 such that $E'\cdot E'\subset E$. For

$$f' \;\in\; \big(f\cdot U(C,E')\big)\cdot\big(f^{-1}\cdot U(C,E')\big)$$

we have

$$f'(c) \;\in\; \big(f(c)\cdot E'\big)\cdot\big(f^{-1}(c)\cdot E'\big) \;=\; E'\cdot E' \;\subset\; E \qquad\qquad (\text{for all } c\in C)$$

That is,

$$\big(f\cdot U(C,E')\big)\cdot\big(f^{-1}\cdot U(C,E')\big) \;\subset\; U(C,E)$$

This proves continuity of multiplication. Continuity of inversion is similar.

[3.5] Hausdorff-ness Take $f_1\neq f_2$ in $\widehat{G}$. For some $g\in G$, $f_1(g)\neq f_2(g)$. Since the target S^1 is Hausdorff, there are opens $E_1\ni f_1(g)$ and $E_2\ni f_2(g)$ with $E_1\cap E_2=\phi$. Since the source G is Hausdorff, the singleton $\{g\}$ is compact. Thus, $f_i\in U(\{g\},E_i)$, and these opens are disjoint.

[3.6] Summary Apart from proving the local compactness in the general case, the above discussion verifies several foundational aspects of unitary dual groups of abelian topological groups. This fully legitimizes the earlier argument that unitary duals of discrete groups are are compact, and *vice-versa*. ///

---

4. *Appendix: no small subgroups*

The circle group S^1 *has no small subgroups*, in the sense that there is a neighborhood U of the identity $1 \in S^1$ such that the only subgroup of S^1 inside U is the trivial group $\{1\}$.

We recall a proof. [2]

Use the copy of S^1 inside the complex plane. Specifically, we claim that taking U to be the open right half

$$U \;=\; S^1 \cap \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$$

suffices: the only subgroup G of S^1 inside this U is $G = \{1\}$. Indeed, suppose not. Let $1 \neq e^{i\theta} \in G \cap U$. We can take $0 < \theta < \pi/2$, since both $\pm\theta$ must appear. Using the archimedean property of $\mathbb{R}$, let $0 < \ell \in \mathbb{Z}$ be the smallest positive integer such that $\ell \cdot \theta > \pi/2$. Then, since $(\ell - 1)\cdot\theta < \pi/2$ and $0 < \theta < \pi/2$,

$$\frac{\pi}{2} \;<\; \ell\cdot\theta \;=\; (\ell-1)\cdot\theta + \theta \;<\; \frac{\pi}{2} + \frac{\pi}{2} \;=\; \pi$$

Thus, $\ell \cdot \theta$ falls outside U, contradiction. ///

[Iwasawa 1950/52] K. Iwasawa, [brief announcement], in Proceedings of the 1950 International Congress of Mathematicians, Vol. 1, Cambridge, MA, 1950, p. 322, Amer. Math. Soc., Providence, RI, 1952.

[Iwasawa 1952/92] K. Iwasawa, *Letter to J. Dieudonné*, dated April 8, 1952, in *Zeta Functions in Geometry*, editors N. Kurokawa and T. Sunada, Advanced Studies in Pure Mathematics **21** (1992), 445-450.

[Tate 1950/67] J. Tate, *Fourier analysis in number fields and Hecke's zeta functions*, Ph.D. thesis, Princeton (1950), in *Algebraic Number Theory*, J. Cassels and J. Frölich, editors, Thompson Book Co., 1967.

[Weil 1940/1965] A. Weil, *L'intégration dans les groupes topologiques, et ses applications*, Hermann, Paris, 1940, second edition 1965.

---

[2] In fact, this argument essentially proves the analogous result for *real Lie groups*.