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Transition exercise on Eisenstein series

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

A simple Eisenstein series, for $z \in \mathfrak{H}$, $\Gamma = SL_2(\mathbb{Z})$, is

$$E_s(z) \;=\; \sum_{\Gamma_\infty \backslash \Gamma} \mathrm{Im}\,(\gamma \cdot z)^s \;=\; \tfrac{1}{2} \sum_{c,d \text{ coprime}} \frac{y^s}{|cz+d|^{2s}} \qquad (\text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \Gamma_\infty = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma\})$$

We *rewrite* these Eisenstein series to exhibit the role of p-adic groups $GL_2(\mathbb{Q}_p)$, exhibiting Hecke operators as *integral operators*, parallel to the relevance of the representation theory of $GL_2(\mathbb{R})$ to invariant differential operators. [1] The explicit construction of Eisenstein series allows vivid illustration of some basic mechanisms.

For $GL_2(\mathbb{Q})$, one can survive without this viewpoint. However, for GL_2 over number fields with non-trivial class groups, for GL_n with $n > 2$, and for more general groups, the neo-classical elementary ideas sufficient for $GL_2(\mathbb{Q})$ fall short.

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1. *Rewriting the GL_2 Eisenstein series*

Let v be an index for *places* of $\mathbb{Q}$, with $\mathbb{Q}_v$ the v^{th} completion, and $\mathbb{Z}_v$ the v-adic integers for v finite.

[1.1] The localized rewrite For each place v, finite and infinite, define a function φ_v on $G_v = GL_2(\mathbb{Q}_v)$ by

$$\varphi_v(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot k) \;=\; \Big|\frac{a}{d}\Big|_v^s \qquad \begin{cases} \text{for } k \in GL_2(\mathbb{Z}_v) & (\text{for } v \text{ finite}) \\ \text{for } k = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in O(2) & (\text{for } v \text{ real}) \end{cases}$$

where in all cases $a, d \in \mathbb{Q}_v^\times$ and $b \in \mathbb{Q}_v$. By design, each φ_v is invariant under the center Z_v of G_v). Let

$$\varphi \;=\; \bigotimes_v \varphi_v \qquad (\text{meaning } \varphi(\{g_v\}) = \textstyle\prod_v \varphi_v(g_v), \text{ where } g_v \in G_v)$$

Let P be upper triangular matrices in $G = GL_2$. The *product formula* shows that φ is left $P_\mathbb{Q}$-invariant. The first main claim of this note is that, for $g_\infty \in GL_2(\mathbb{R})$, acting as usual on $i \in \mathfrak{H}$,

$$E_s(g_\infty \cdot i) \;=\; \sum_{\gamma \in P_\mathbb{Q} \backslash G_\mathbb{Q}} \varphi(\gamma \cdot g_\infty)$$

Further, this expression gives a left $G_\mathbb{Q}$-invariant, $Z_\mathbb{A}$-invariant function (still denoted E_s) on the adele group $G_\mathbb{A}$:

$$E_s(g) \;=\; \sum_{\gamma \in P_\mathbb{Q} \backslash G_\mathbb{Q}} \varphi(\gamma \cdot g)$$

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[1] Despite occasional contrary assertions in the literature, rewriting Eisenstein series, as opposed to more general automorphic forms, to make sense on adele groups is *not* about *Strong Approximation*. Strong Approximation *does* make precise the relation between *general* automorphic forms on adele groups and automorphic forms on SL_n, but rewriting these Eisenstein series does not need this comparison. Indeed, Strong Approximation is not valid for general semi-simple or reductive groups, but this does not impede developments.

[1.2] Disambiguation The immediate question arises of evaluation of $\varphi(\gamma \cdot g_\infty)$. One point is that $GL_2(\mathbb{Q})$ should *not* be considered as *only* a subgroup of $GL_2(\mathbb{R})$, but also potentially a subgroup of *every* $GL_2(\mathbb{Q}_v)$. This discussion will *eventually* be clarified by looking at functions on adele groups. However, Eisenstein series *directly* illustrate the usefulness of the viewpoint, without thinking in terms of automorphic forms on adele groups. Adding a temporary notational burden for clarity, for each place v let $j_v : GL_2(\mathbb{Q}) \to GL_2(\mathbb{Q}_v)$ be the natural injective map, and let

$$j \;=\; \prod_v j_v \;:\; GL_2(\mathbb{Q}) \longrightarrow \prod_v GL_2(\mathbb{Q}_v)$$

be the natural *diagonal map* to the product. Then

$$\varphi(\gamma \cdot g_\infty) \;=\; \varphi_\infty(j_\infty(\gamma)\cdot g_\infty)\cdot \prod_{v<\infty} \varphi_\infty(j_v(\gamma))$$

That is, in the definition of φ, the archimedean $g_\infty \in GL_2(\mathbb{R})$ does not interact with the finite-prime groups $GL_2(\mathbb{Q}_v)$.

[1.3] Comparison to classical formulation The familiar Eisenstein series $E_s(z)$ can be obtained from the above by *reverting* to a form that does not refer to anything p-adic or adelic. That is, we claim that

$$\varphi_\infty(g_\infty) \;=\; \big(\mathrm{Im}\,(g_\infty \cdot i)\big)^s \qquad\qquad (\text{for } g_\infty \in SL_2(\mathbb{R}))$$

with $SL_2(\mathbb{R})$ acting on the upper half-plane $\mathfrak{H}$ as usual. The argument is about the *Iwasawa decomposition*, namely, that any element of $SL_2(\mathbb{R})$ can be written as a product of upper-triangular and orthogonal matrices. Indeed, given a matrix in $SL_2(\mathbb{R})$, right multiplication by an orthogonal matrix can be viewed as *rotating* the bottom row. This suggests the appropriate orthogonal group element: formulaically,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & \frac{c}{\sqrt{c^2+d^2}} \\ \frac{-c}{\sqrt{c^2+d^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{pmatrix} \;=\; \begin{pmatrix} \frac{ad-bc}{\sqrt{c^2+d^2}} & \frac{ac+bd}{\sqrt{c^2+d^2}} \\ 0 & \frac{c^2+d^2}{\sqrt{c^2+d^2}} \end{pmatrix} \;=\; \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & * \\ 0 & \sqrt{c^2+d^2} \end{pmatrix}$$

Thus,

$$\varphi_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \varphi_\infty \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & * \\ 0 & \sqrt{c^2+d^2} \end{pmatrix} \;=\; \left|\frac{1/\sqrt{c^2+d^2}}{\sqrt{c^2+d^2}}\right|^s \;=\; \left|\frac{1}{c^2+d^2}\right|^s$$

On the other hand, a more familiar computation gives

$$\mathrm{Im} \begin{pmatrix} a & b \\ c & d \end{pmatrix}(i) \;=\; \frac{1}{2i}\Big(\frac{ai+b}{ci+d} - \frac{-ai+b}{-ci+d}\Big) \;=\; \frac{ad-bc}{c^2+d^2} \;=\; \frac{1}{c^2+d^2}$$

Since $\gamma \in SL_2(\mathbb{Z})$ maps to $GL_2(\mathbb{Z}_v)$ at all finite places v,

$$\varphi_v(\gamma) \;=\; 1 \qquad\qquad (\text{for } \gamma \in SL_2(\mathbb{Z}) \text{ and finite place } v)$$

Thus,

$$\sum_{\gamma \in P_{\mathbb{Z}}\backslash GL_2(\mathbb{Z})} \varphi(\gamma \cdot g_\infty) \;=\; \sum_{\gamma \in P_{\mathbb{Z}}\backslash GL_2(\mathbb{Z})} \varphi_\infty(\gamma \cdot g_\infty)\cdot 1 \;=\; \sum_{\gamma \in P_{\mathbb{Z}}\backslash GL_2(\mathbb{Z})} \Big(\mathrm{Im}\,\gamma g_\infty \cdot i\Big)^s$$

Taking the popular choice

$$g_\infty \;=\; \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \qquad\qquad (\text{with } x \in \mathbb{R} \text{ and } y > 0)$$

produces $E_s(x+iy)$ as claimed.

[1.4] Well-definedness on $P_\mathbb{Q}\backslash G_\mathbb{Q}$ We should show that $\varphi(\gamma g_\infty)$ depends only upon the coset $P_\mathbb{Q}\gamma$.

First, any $\gamma \in GL_2(\mathbb{Q})$ is in $GL_2(\mathbb{Z}_v)$ for almost all v, since the entries are in $\mathbb{Z}_v$ for almost all v, and the determinant is a v-adic unit for almost all v, so the inverse is v-integral also. Thus, in an infinite product $\prod_{v<\infty} \varphi_v(\gamma)$, all but finitely-many factors are 1.

Let χ_v be the character on upper-triangular v-adic matrices P_v given by

$$\chi_v \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \left|\frac{a}{d}\right|_v^s \qquad \text{(with } a, d \in \mathbb{Q}_v^\times \text{ and } b \in \mathbb{Q}_v\text{)}$$

The usual maximal compact subgroups K_v of the groups $GL_2(\mathbb{Q}_v)$ are

$$K_v = \begin{cases} GL_2(\mathbb{Q}_v) & \text{(for } v \text{ finite)} \\ O(2) & \text{(for } v \text{ real)} \end{cases}$$

The description of φ_v can be rewritten more succinctly as

$$\varphi_v(pk) = \chi_v(p) \qquad \text{(for } p \in P_v \text{ and } k \in K_v\text{)}$$

For $g_\infty \in G_\mathbb{R}$, $\gamma \in G_\mathbb{Q}$, and $\beta \in P_\mathbb{Q}$, keeping in mind that $G_\mathbb{Q}$ maps to *all* groups G_v, not just to G_∞,

$$\varphi(\beta\cdot\gamma\cdot g_\infty) = \varphi_\infty(\beta\cdot\gamma\cdot g_\infty)\cdot \prod_{v<\infty} \varphi_v(\beta\cdot\gamma)$$

At the archimedean place, let $\gamma g_\infty = pk$ be an Iwasawa decomposition in G_v, with $p \in P_v$ and $k \in K_v$. We see the left equivariance of φ_v by χ_v, namely,

$$\varphi_v(\beta\gamma g_\infty) = \varphi_v(\beta pk) = \chi_v(\beta\cdot p) = \chi_v(\beta)\cdot\chi_v(p) = \chi_v(\beta)\cdot\varphi_v(pk) = \chi_v(\beta)\cdot\varphi_v(\gamma g_\infty)$$

Similarly, but now without g_∞ playing any role, at a finite place v, let $\gamma = pk$ be an Iwasawa decomposition in G_v, with $p \in P_v$ and $k \in K_v$. We see the left equivariance of φ_v by χ_v:

$$\varphi_v(\beta\gamma) = \varphi_v(\beta pk) = \chi_v(\beta\cdot p) = \chi_v(\beta)\cdot\chi_v(p) = \chi_v(\beta)\cdot\varphi_v(\gamma)$$

Putting all these local equivariances together,

$$\varphi(\beta\cdot\gamma\cdot g_\infty) = \prod_v \chi_v(\beta)\cdot\varphi_v(\gamma\cdot g_\infty) \qquad \text{(for } \beta \in P_\mathbb{Q},\ \gamma \in G_\mathbb{Q}, \text{ and } g_\infty \in G_\infty\text{)}$$

By the product formula,

$$\prod_v \chi_v \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \prod_v \left|\frac{a}{d}\right|_v^s = 1 \qquad \text{(for } a/d \in \mathbb{Q}^\times\text{)}$$

That is, we have the left invariance

$$\varphi(\beta\cdot\gamma\cdot g_\infty) = \varphi(\gamma\cdot g_\infty) \qquad \text{(for } \beta \in P_\mathbb{Q},\ \gamma \in G_\mathbb{Q}, \text{ and } g_\infty \in G_\infty\text{)}$$

[1.5] Bijection of cosets

We claim that

$$P_\mathbb{Q}\backslash G_\mathbb{Q} \approx P_\mathbb{Z}\backslash G_\mathbb{Z}$$

that is, that every coset $P_\mathbb{Q}h$ with $h \in G_\mathbb{Q}$ has a representative in $G_\mathbb{Z} = GL_2(\mathbb{Z})$. The argument attaches meaning to both these coset spaces, and will thereby give the bijection.

The coset space $P_\mathbb{Q}\backslash G_\mathbb{Q}$ is in bijection with the set of lines in $\mathbb{Q}^2$, respecting the right multiplication by $G_\mathbb{Q}$, because $G_\mathbb{Q}$ is transitive on these lines, and $P_\mathbb{Q}$ is the stabilizer of the line $\{(0\ *)\}$. Next, each line in $\mathbb{Q}^2$ meets $\mathbb{Z}^2$ in a free rank-one $\mathbb{Z}$-module generated by a primitive vector (x, y), meaning that $\gcd(x, y) = 1$. Call such a $\mathbb{Z}$-module a *primitive* $\mathbb{Z}$*-line* in $\mathbb{Q}^2$. The collection of lines in $\mathbb{Q}^2$ is thus in bijection with primitive $\mathbb{Z}$-lines in $\mathbb{Z}^2$, by sending a line to its intersection with $\mathbb{Z}^2$. The group $SL_2(\mathbb{Z}) \subset GL_2(\mathbb{Z})$ is transitive on primitive $\mathbb{Z}$-lines: for $\gcd(x, y) = 1$, let $b, d \in \mathbb{Z}$ be such that

$$bx + dy \;=\; \gcd(x, y) \;=\; 1$$

Then

$$(x\ \ y) \cdot \begin{pmatrix} y & b \\ -x & d \end{pmatrix} \;=\; (0\ \ 1)$$

That is, any primitive vector can be mapped to $(0\ 1)$, so the action of $SL_2(\mathbb{Z})$ is transitive on primitive *vectors*, hence on primitive $\mathbb{Z}$*-lines*. Thus, certainly the slightly larger group $GL_2(\mathbb{Z})$ is transitive on primitive vectors. The stabilizer subgroup of the primitive $\mathbb{Z}$-line spanned by $(0, 1)$ in $GL_2(\mathbb{Z})$ is

$$P_\mathbb{Z} \;=\; \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;:\; a, b \in \mathbb{Z}^\times,\ b \in \mathbb{Z}\}$$

This proves the bijection of coset spaces.

[1.6] The essential conclusion

From

$$E_s(g_\infty \cdot i) \;=\; \sum_{\gamma \in P_\mathbb{Z}\backslash G_\mathbb{Z}} \varphi_\infty(\gamma \cdot g_\infty) \qquad\qquad (\text{with } g_\infty \in GL_2(\mathbb{R}))$$

from bijection of coset spaces, and from the well-definedness of φ left modulo $P_\mathbb{Q}$, we have the desired re-expression of the Eisenstein series

$$E_s(g_\infty \cdot i) \;=\; \sum_{\gamma \in P_\mathbb{Q}\backslash G_\mathbb{Q}} \varphi(\gamma \cdot g_\infty) \qquad\qquad (\text{with } g_\infty \in GL_2(\mathbb{R}))$$

This is the first main point, and there are further advantages to the viewpoint. Computation of the **constant term** is the first illustration, in the next section.

[1.7] Bruhat decomposition

Expressing the coset space $P_\mathbb{Z}\backslash G_\mathbb{Z}$ in terms of *rational* matrices $P_\mathbb{Q}\backslash G_\mathbb{Q}$, rather than *integral*, allows application of the Bruhat decomposition

$$G_\mathbb{Q} \;=\; P_\mathbb{Q} \,\sqcup\, P_\mathbb{Q} w N_\mathbb{Q} \qquad\qquad (\text{with } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix})$$

This purely algebraic fact holds over any field in place of $\mathbb{Q}$, and has natural extensions to GL_n and the classical groups. For $G = GL_2$, the Bruhat decomposition is easy to prove, upon noting that the *big cell* $P_\mathbb{Q} w N_\mathbb{Q}$ is

$$P_\mathbb{Q} w N_\mathbb{Q} \;=\; \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}) \;:\; c \neq 0\}$$

Of course, the *little cell* $P_\mathbb{Q}$ consists of matrices with $c = 0$. The main benefit of this expression for $G_\mathbb{Q}$ is that the coset space $P_\mathbb{Q}\backslash G_\mathbb{Q}$ has a remarkably simple set of representatives:

$$P_\mathbb{Q}\backslash G_\mathbb{Q} \;=\; P_\mathbb{Q}\backslash P_\mathbb{Q} \cup P_\mathbb{Q}\backslash\big(P_\mathbb{Q} w N_\mathbb{Q}\big) \;\approx\; \{1\} \cup w \cdot \big(w^{-1} P_\mathbb{Q} w \cap N_\mathbb{Q}\big)\backslash N_\mathbb{Q} \;\approx\; \{1\} \cup w N_\mathbb{Q}$$

Using the Bruhat decomposition, the sum defining the Eisenstein series can be written as

$$E_s(g_\infty) \;=\; \sum_{\gamma\in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}} \varphi(\gamma\cdot g_\infty) \;=\; \varphi(g_\infty) \;+\; \sum_{\gamma\in wN_{\mathbb{Q}}} \varphi(\gamma\cdot g_\infty)$$

That is, the summands can be parametrized by $\{1\}$ and $N_{\mathbb{Q}} \approx \mathbb{Q}$. This is convenient in computing the **constant term** below.

[1.8] Another comparison

We can do another computation to verify our rewrite of the Eisenstein series, paying attention to the Bruhat-cell parametrization. In principle, this computation is unnecessary after the discussion above, but it is informative.

As φ_∞ is right $O(2)$-invariant and center-invariant, by an Iwasawa decomposition for $G_{\mathbb{R}}$ we can take

$$g_\infty \;=\; \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad (\text{with } x\in\mathbb{R} \text{ and } y>0)$$

With this g_∞, the term $\gamma = 1$ is

$$\varphi(g_\infty) \;=\; \varphi_\infty(g_\infty)\cdot \prod_{v<\infty} \varphi_v(1) \;=\; |y|^s\cdot 1 \;=\; |y|^s$$

For the big cell contribution to the sum, we need to compute the archimedean part

$$\varphi_\infty(w\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}) \qquad\qquad (\text{for } t\in\mathbb{Q})$$

and the non-archimedean

$$\varphi_v(w\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}) \qquad\qquad (\text{for finite } v, \text{ with } t\in\mathbb{Q})$$

In the archimedean case,

$$w\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} 0 & -1 \\ y & x+t \end{pmatrix}$$

Right multiplying by a suitable orthogonal matrix rotates the bottom row to put the result in $P_{\mathbb{R}}$, namely,

$$\begin{pmatrix} 0 & -1 \\ y & x+t \end{pmatrix}\begin{pmatrix} \frac{x+t}{\sqrt{(x+t)^2+y^2}} & \frac{y}{\sqrt{(x+t)^2+y^2}} \\ \frac{-y}{\sqrt{(x+t)^2+y^2}} & \frac{x+t}{\sqrt{(x+t)^2+y^2}} \end{pmatrix} \;=\; \begin{pmatrix} \frac{y}{\sqrt{(x+t)^2+y^2}} & * \\ 0 & \sqrt{(x+t)^2+y^2} \end{pmatrix}$$

Thus,

$$\varphi_\infty(w\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}) \;=\; \Big|\frac{y}{(x+t)^2+y^2}\Big|_\infty^s$$

For finite v, adjust the given matrix by right multiplication by $GL_2(\mathbb{Z}_v)$ to make the result upper-triangular. For $t\in\mathbb{Z}_v$, the matrix is already in $GL_2(\mathbb{Z}_v)$, so

$$\varphi_v(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}) \;=\; 1 \qquad\qquad (\text{for finite } v, \text{ with } t\in\mathbb{Q}\cap\mathbb{Z}_v)$$

For $t\notin\mathbb{Z}_v$, necessarily $t^{-1}\in\mathbb{Z}_v$. Thus, the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix}$$

can be multiplied by $\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}$ in $GL_2(\mathbb{Z}_v)$ to obtain

$$\begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} = \begin{pmatrix} t^{-1} & * \\ 0 & t \end{pmatrix}$$

Thus,

$$\varphi_v\Big(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\Big) = |t|_v^{-2s} \qquad \text{(for finite } v\text{, with } t \notin \mathbb{Q} \cap \mathbb{Z}_v)$$

That is,

$$\varphi_v\Big(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\Big) = \begin{cases} 1 & (\text{for } |t|_v \le 1) \\ |t|_v^{-2s} & (\text{for } |t|_v \ge 1) \end{cases}$$

Combining the archimedean and non-archimedean,

$$\varphi\Big(w\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\Big) = \Big|\frac{y}{(x+t)^2+y^2}\Big|_\infty^s \cdot \prod_{v<\infty} \begin{cases} 1 & (\text{for } |t|_v \le 1) \\ |t|_v^{-2s} & (\text{for } |t|_v > 1) \end{cases}$$

Since $\mathfrak{o}$ is a principal ideal domain with units ± 1, we can easily parametrize $t \in \mathbb{Q}$ in a fashion conforming to evaluation of the displayed expression. Namely, write $t = d/c$ with c, d relatively prime, modulo ± 1. Note that for relatively prime integers c, d

$$\begin{cases} 1 & (\text{for } |d/c|_v \le 1) \\ |d/c|_v^{-2s} & (\text{for } |d/c|_v > 1) \end{cases} = \begin{cases} 1 & (\text{for } p_v \text{ not dividing } c) \\ |d/c|_v^{-2s} & (\text{for } p_v \text{ dividing } c) \end{cases} = \begin{cases} 1 & (\text{for } p_v \text{ not dividing } c) \\ |c|_v^{2s} & (\text{for } p_v \text{ dividing } c) \end{cases}$$

That is, by the product formula,

$$\prod_{v<\infty} \begin{cases} 1 & (\text{for } |d/c|_v \le 1) \\ |d/c|_v^{-2s} & (\text{for } |d/c|_v > 1) \end{cases} = \prod_{v<\infty} |c|_v^{2s} = \frac{1}{|c|_\infty^{2s}}$$

Then, once again, we recover the expected:

$$\Big|\frac{y}{(x+\frac{d}{c})^2+y^2}\Big|_\infty^s \cdot \frac{1}{|c|_\infty^{2s}} = \Big|\frac{y}{(cx+d)^2+(cy)^2}\Big|_\infty^s = \frac{y^s}{|cz+d|^{2s}}$$

Again, in principle the above computation is unnecessary, but it is informative to see the details of the reversion to a classical form.

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2. *Application: constant term of GL_2 Eisenstein series*

An immediate use of the localized rewrite of the Eisenstein series is to computation of the constant term presenting each Bruhat cell's contribution as an Euler product, by *unwinding* the integral defining the constant term.

[2.1] Unwinding With

$$E_s(g_\infty) = \sum_{\gamma \in P_\mathbb{Q}\backslash G_\mathbb{Q}} \varphi(\gamma \cdot g_\infty)$$

the **constant term** of E_s along P is by definition the adelic integral

$$c_P E_s(g) = \int_{N_\mathbb{Q}\backslash N_\mathbb{A}} E_s(ng)\, dn$$

Parametrizing $P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}$ via the Bruhat decomposition makes computation nearly trivial:

$$\int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} E_s(ng)\,dn \;=\; \int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} \sum_{\gamma\in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}} \varphi(\gamma ng)\,dn \;=\; \sum_{w\in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/N_{\mathbb{Q}}} \int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} \sum_{\gamma\in P_{\mathbb{Q}}\backslash P_{\mathbb{Q}}wN_{\mathbb{Q}}} \varphi(\gamma ng)\,dn$$

By the Bruhat decomposition, $P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/N_{\mathbb{Q}}$ has exactly two representatives, $1, w$, and the constant term becomes

$$\int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} \varphi(ng)\,dn \;+\; \int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} \sum_{\gamma\in N_{\mathbb{Q}}} \varphi(w\gamma ng)\,dn \;=\; \int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} \varphi(ng)\,dn \;+\; \int_{N_{\mathbb{A}}} \varphi(wng)\,dn$$

Because φ is left $N_{\mathbb{A}}$-invariant, the first of the two summands is

$$\int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} \varphi(ng)\,dn \;=\; \varphi(g)\cdot \mathrm{vol}\,(N_{\mathbb{Q}}\backslash N_{\mathbb{A}}) \qquad\qquad \text{(the small Bruhat cell contribution)}$$

Since the integral in the second summand unwound, it factors over primes

$$\int_{N_{\mathbb{A}}} \varphi(wng)\,dn \;=\; \prod_{v\le\infty} \int_{N_v} \varphi_v(wng)\,dn$$

[2.2] Evaluation of local factors: non-archimedean case

For $g\in G_\infty$, so that $g_v = 1$, the finite-prime local factors in the Euler product for the big Bruhat cell are readily evaluated, as follows. Above, we computed

$$\varphi_v(w\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix})\;=\;\begin{cases} 1 & (\text{for } |t|_v\le 1)\\ |t|_v^{-2s} & (\text{for } |t|_v > 1)\end{cases}$$

With the v-adic factor corresponding to prime p, the v-adic local factor is

$$\int_{|t|_v\le 1} 1\,dt \;+\; \int_{|t|_v>1} |t|_v^{-2s}\,dt \;=\; 1+\sum_{\ell=1}^{\infty} |p^{-\ell}|_v^{-2s}\cdot\int_{p^{-\ell}\mathbb{Z}_p^\times} 1\,dt \;=\; 1+\sum_{\ell=1}^{\infty}(p^\ell)^{-2s}\cdot p^{\ell-1}(p-1)$$

$$=\;1+\big(1-\frac{1}{p}\big)\frac{p^{1-2s}}{1-p^{1-2s}} \;=\; \frac{1-p^{1-2s}+p^{1-2s}-p^{-2s}}{1-p^{1-2s}} \;=\; \frac{1-p^{-2s}}{1-p^{1-2s}} \;=\; \frac{\zeta_v(2s-1)}{\zeta_v(2s)}$$

where $\zeta_v(s)$ is the v^{th} Euler factor of the zeta function. Thus, the finite-prime part of the big-cell summand is $\zeta(2s-1)/\zeta(2s)$.

[2.3] Evaluation of local factors: archimedean case

The archimedean factor of the big-cell summand of the constant term is

$$\int_{\mathbb{R}} \Big|\frac{y}{(x+t)^2+y^2}\Big|_\infty^s\,dt \;=\; \int_{\mathbb{R}} \frac{y^s}{\big((x+t)^2+y^2\big)^s}\,dt \;=\; y^s\cdot\int_{\mathbb{R}} \frac{1}{\big(t^2+y^2\big)^s}\,dt \;=\; y^{1+s}\cdot\int_{\mathbb{R}} \frac{1}{\big((ty)^2+y^2\big)^s}\,dt$$

$$=\;y^{1-s}\cdot\int_{\mathbb{R}} \frac{1}{(t^2+1)^s}\,dt \;=\; \frac{y^{1-s}}{\Gamma(s)}\cdot\int_{\mathbb{R}}\int_0^\infty e^{u(1+t^2)}\,u^s\,\frac{du}{u}\,dt \;=\; \frac{y^{1-s}}{\Gamma(s)}\cdot\int_{\mathbb{R}}\int_0^\infty e^{u+t^2}\,u^{s-\frac{1}{2}}\,\frac{du}{u}\,dt$$

$$=\;\frac{y^{1-s}\,\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} \;=\; y^{1-s}\cdot\frac{\pi^{-(s-\frac{1}{2})}\,\Gamma(s-\frac{1}{2})}{\pi^{-s}\,\Gamma(s)} \;=\; y^{1-s}\cdot\frac{\zeta_\infty(2s-1)}{\zeta_\infty(2s)} \qquad\qquad (\text{with } \zeta_\infty(s)=\pi^{-s/2}\,\Gamma(s/2))$$

[2.4] Conclusion of constant-term computations Thus, with $\xi(s)$ the completed zeta function $\xi(s) = \zeta_\infty(s) \cdot \zeta(s)$, the constant term of E_s is

$$c_P E_s(x+iy) \;=\; y^s + \frac{\xi(2s-1)}{\xi(2s)} \cdot y^{1-s}$$

The present point is that rewriting the Eisenstein series as an automorphization of a *product of local data* makes the computation of the constant term far more natural, and more genuinely representative of the corresponding computation for larger groups.

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3. *Application: Hecke operators on GL_2 Eisenstein series*

The rewritten Eisenstein series will show that the Hecke operators are not *global* things, but are *local*, just acting on the local components φ_v. Indeed, the local components φ_v are eigenfunctions for the *local* version of Hecke operators, with eigenvalues depending on the parameter s.

[3.1] Classical description of Hecke operators The p^{th} Hecke operator T_p on weight-0 automorphic forms f for $\Gamma = SL_2(\mathbb{Z})$ is

$$T_p f(z) \;=\; \sum_{\gamma \in \Gamma\backslash\Theta_p} f(\gamma \cdot z) \qquad\qquad \text{(where } \Theta_p = \text{integer matrices with } \det = p\text{)}$$

with action [2] by linear fractional transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \dfrac{az+b}{cz+d}$.

[3.2] Hecke operators on rewritten Eisenstein series Directly computing, with $g_\infty \in G_\infty$,

$$T_p E_s(g_\infty) \;=\; \sum_{\delta \in \Gamma\backslash\Theta_p} E_s(j_\infty(\delta) \cdot g_\infty) \;=\; \sum_{\delta \in \Gamma\backslash\Theta_p} \; \sum_{\gamma \in P_\mathbb{Q}\backslash G_\mathbb{Q}} \varphi\Big(\gamma \cdot j_\infty(\delta) \cdot g_\infty\Big)$$

Let $j_o = \prod_{v<\infty} j_v$. Replace γ by $\gamma \cdot \delta^{-1}$ in $G_\mathbb{Q}$, to obtain

$$\sum_{\delta \in \Gamma\backslash\Theta_p} \; \sum_{\gamma \in P_\mathbb{Q}\backslash G_\mathbb{Q}} \varphi\Big(\gamma \cdot j_o(\delta^{-1}) \cdot g_\infty\Big)$$

The finite-prime factor $j_o(\delta^{-1})$ commutes with the archimedean-prime factor g_∞, so this is

$$\sum_{\delta \in \Gamma\backslash\Theta_p} \; \sum_{\gamma \in P_\mathbb{Q}\backslash G_\mathbb{Q}} \varphi\Big(\gamma \cdot g_\infty \cdot j_o(\delta^{-1})\Big) \;=\; \sum_{\delta \in \Gamma\backslash\Theta_p} \; \sum_{\gamma \in P_\mathbb{Q}\backslash G_\mathbb{Q}} \varphi_\infty(j_\infty(\gamma) \cdot g_\infty) \cdot \prod_{v<\infty} \varphi_v(j_v(\gamma \cdot \delta^{-1}))$$

At all finite places v' but $v \sim p$, δ^{-1} is in the local maximal compact $K_{v'} = GL_{v'}(\mathbb{Z}_{v'})$, so $\varphi_{v'}(\gamma \cdot \delta^{-1}) = \varphi_{v'}(\gamma)$ for $v' \neq v$. Thus, suppressing j_v in the notation,

$$T_p E_s(g_\infty) \;=\; \sum_{\gamma \in P_\mathbb{Q}\backslash G_\mathbb{Q}} \varphi(\gamma \cdot g_\infty) \cdot \frac{\sum_{\delta \in \Gamma\backslash\Theta_p} \varphi_v(\gamma \cdot \delta^{-1})}{\varphi_v(\gamma)}$$

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[2] That is, the weight-0 situation allows us to avoid worry over what to do with the *determinant* in GL_2. In the classical holomorphic case, the so-called *slash operator* is a normalization that accommodates this, usually without explanation or motivation.

Thus, the p^{th} Hecke operator's effect is *local* at $v \sim p$. Further, this situation correctly suggests that we should hope that φ_v is an *eigenfunction* for the effect of T_p, of course with eigenvalue depending on the complex parameter s.

[3.3] Hecke operators as integral operators Continue to let v correspond to prime p. The function φ_v on $G_v = GL_2(\mathbb{Q}_v)$ is left P_v-equivariant by χ_v, and right K_v-invariant. Each of the right translates $g \to \varphi_v(g \cdot \delta^{-1})$ with $g \in G_v$ retains the left P_v, χ_v-equivariance, but cannot be expected to retain right K_v-invariance.

Nevertheless, we claim that the sum over $\delta \in \Gamma\backslash\Theta_p$ recovers the right K_v-invariance. That is, apparently, $\Gamma\backslash\Theta_p$, or its image projected to G_v, is somehow stable under right multiplication by K_v. As it stands, this doesn't make sense, since Θ_p itself (projected to G_v) is not literally stable under right multiplication by K_v.

As on other occasions, the necessary claim suggests itself: let $\widetilde{\Theta}_v$ be the v-adic analogue of Θ_p, namely, elements of $G_v = GL_2(\mathbb{Q}_v)$ with entries in $\mathbb{Z}_v$ and determinant of p-adic ord 1. Then we *must* claim that the natural map $\Theta_p^{-1} \to \widetilde{\Theta}_v^{-1}/K_v$ induces a *bijection*

$$\Theta_p^{-1}/\Gamma \longrightarrow \widetilde{\Theta}_v^{-1}/K_v$$

Equivalently, inverting, $\Theta_p \to K_v\backslash\widetilde{\Theta}_v$ induces a bijection

$$\Gamma\backslash\Theta_p \longrightarrow K_v\backslash\widetilde{\Theta}_v$$

Indeed, $K_v\backslash\widetilde{\Theta}_v$ has the same representatives as $\Gamma\backslash\Theta_p$, namely,[3]

$$\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \qquad \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \qquad (\text{with } b \in \mathbb{Z},\ 0 \le b < p)$$

Thus, giving K_v total measure 1, using the right K_v-invariance of φ_v,

$$\sum_{\delta \in \Gamma\backslash\Theta_p} \varphi_v(g \cdot \delta^{-1}) = \sum_{h \in \widetilde{\Theta}^{-1}/K_v} \varphi_v(g \cdot h) = \int_{\widetilde{\Theta}^{-1}} \varphi_v(g \cdot h)\, dh$$

Letting η be the characteristic function of $\widetilde{\Theta}_v^{-1}$, this integral is an integral operator attached to the right translation action of G_v on functions on G_v:

$$\sum_{\delta \in \Gamma\backslash\Theta_p} \varphi_v(g \cdot \delta^{-1}) = \int_{G_v} \eta(h)\, \varphi_v(g \cdot h)\, dh = (\eta \cdot \varphi_v)(g)$$

The integral expression makes right K_v-invariance clear, by changing variables in the integral:

$$\int_{G_v} \eta(h)\, \varphi_v(gk \cdot h)\, dh = \int_{G_v} \eta(k^{-1}h)\, \varphi_v(g \cdot h)\, dh = \int_{G_v} \eta(h)\, \varphi_v(g \cdot h)\, dh \qquad (\text{for } k \in K_v)$$

[3] The v-adic argument is easier than that over $\mathbb{Z}$: given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\widetilde{\Theta}_v$, if $\gcd(a,c) = 1$, then either a or c is in $\mathbb{Z}_v^\times$. Thus, left multiplication by either $\begin{pmatrix} 1 & 0 \\ -c/a & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & -a/c \end{pmatrix}$ puts g into the form $\begin{pmatrix} 1 & b' \\ 0 & d' \end{pmatrix}$. Necessarily $\mathrm{ord}_v d' = 1$, so further left multiplication gives the form $\begin{pmatrix} 1 & b'' \\ 0 & p \end{pmatrix}$. Since $\mathbb{Z}_v/p\mathbb{Z}_v \approx \mathbb{Z}/p\mathbb{Z}$, further left multiplication by $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in K_v$ gives the indicated representatives parametrized by b. When $\gcd(a,c) = p$, a similar argument gives the form $\begin{pmatrix} p & b' \\ 0 & 1 \end{pmatrix}$, and now the b' entry can be made 0.

since η is left and right K_v-invariant.

[3.4] Hecke eigenvalues Now we can prove that φ_v is an eigenvector for the localized version of T_p (with $v \sim p$), and compute its eigenvalue. The v-adic *Iwasawa decomposition* asserts that $G_v = P_v \cdot K_v$.[4] Thus, up to constant multiples, there is a *unique* left P_v, χ_v-equivariant, right K_v-invariant function on G_v. Thus, every such is a multiple of φ_v.

In particular, with η the characteristic function of $\widetilde{\Theta}_v^{-1}$, necessarily $\eta \cdot \varphi_v = \lambda_s \cdot \varphi_v$ for some $\lambda_s \in \mathbb{C}$. To determine λ_s, it suffices to evaluate at $g = 1$, using $\varphi_v(1) = 1$. Thus,

$$\lambda_s \;=\; \int_{G_v} \eta(h)\; \varphi_v(h)\; dh \;=\; \int_{\widetilde{\Theta}_v^{-1}/K_v} \varphi_v(h)\; dh \;=\; \sum_{\delta \in \Gamma \backslash \Theta_p} \varphi_v(\delta^{-1})$$

$$= \sum_b \chi_v\Big(\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}^{-1} \Big) \;+\; \chi_v\Big(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Big) \;=\; p \cdot \chi_v \begin{pmatrix} 1 & * \\ 0 & p^{-1} \end{pmatrix} \;+\; \chi_v \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \; p \cdot \Big| \frac{1}{p^{-1}} \Big|_v^s + \Big| \frac{p^{-1}}{1} \Big|_v^s \;=\; p^{1-s} + p^s$$

This is the p^{th} Hecke eigenvalue of E_s:

$$T_p E_s \;=\; (p^{1-s} + p^s) \cdot E_s$$

Bibliographic notes:

Only a few early examples are noted: [Tamagawa 1960] and [Tamagawa 1963] were among the earliest examples of treatment of automorphic forms on adele groups, giving evidence of the feasibility and utility of this viewpoint. [Weil 1961] effectively demonstrated that the extension of Iwasawa-Tate theory to reductive adele groups truly captured interesting number-theoretic features. [Godement 1952] and [Mautner 1958] had made the relevant harmonic analysis arguably feasible. [GGPS 1969], a translation of an earlier Russian edition, systematically rewrote the basic theory of automorphic forms in the context of adele groups and their representations.

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[4] Proof of v-adic Iwasawa decomposition: in $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, for $\mathrm{ord}_v(c) \geq \mathrm{ord}_v(d)$, right multiplication by $\begin{pmatrix} 1 & 0 \\ -c/d & 1 \end{pmatrix} \in K_v$ produces an upper-triangular matrix. For $\mathrm{ord}_v(c) \leq \mathrm{ord}_v(d)$, right multiplication by $\begin{pmatrix} -d/c & 1 \\ 1 & 0 \end{pmatrix} \in K_v$ produces an upper-triangular matrix.