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# Exceptional regular singular points of second-order ODEs

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1. Solving second-order ODEs
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Frobenius' method for solving

$$u'' + \frac{b(x)}{x} u' + \frac{c(x)}{x^2} u = 0 \quad (\text{with } b, c \text{ analytic near } 0)$$

is slightly more complicated when the indicial equation

$$\alpha(\alpha - 1) + b(0)\alpha + c(0) = 0$$

has *repeated roots* or *roots differing by an integer*.<sup>[1]</sup> An important example in which this occurs is the differential equation in radial coordinates for spherical functions on hyperbolic  $n$ -space:

$$u'' + (n - 1) \coth r \cdot u' - \lambda u = 0$$

The point 0 is a regular singular point for this equation. The indicial equation is

$$\alpha(\alpha - 1) + (n - 1)\alpha = 0$$

When  $n = 2$ , the indicial equation has a double root 0. When  $n \geq 3$ , the roots are 0 and  $2 - n$ , which differ by an integer.

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## 1. Solving second-order ODEs

### [1.1] Frobenius' method

Let

$$E = \frac{d^2}{dx^2} + \frac{b(x)}{x} \frac{d}{dx} + \frac{c(x)}{x^2} \quad (\text{with } b, c \in \mathbb{C}[[x]])$$

be the differential operator and let

$$p(\alpha) = \alpha(\alpha - 1) + b(0)\alpha + c(0) \in \mathbb{C}[\alpha]$$

Consider

$$f(x, \alpha) = x^\alpha \cdot \sum_{n \geq 0} a_n x^n \quad (\text{on } x > 0, \text{ with } a_0 \neq 0)$$

with  $a_n$  depending on  $\alpha$ . We can solve

$$Ef(x, \alpha) = p(\alpha) \cdot a_0 \cdot x^\alpha$$

by recursively solving for the coefficients  $a_n$ , as follows. Expand

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<sup>[1]</sup> For example, see [Coddington 1961], chapter 4, section 6.

$$Ef(x, \alpha) = p(\alpha) \cdot a_0 \cdot x^\alpha + \left[ p(\alpha + 1) \cdot a_1 + (\text{something involving } a_0) \right] \cdot x^{\alpha+1} \\ + \left[ p(\alpha + 2) \cdot a_2 + (\text{something involving } a_0, a_1) \right] \cdot x^{\alpha+2} + \dots$$

That is, the coefficient of  $x^{\alpha+n-2}$  in  $Ef$  is

$$\left[ (\alpha + n)(\alpha + n - 1) + b(0)(\alpha + n) + c(0) \right] \cdot a_n + (\text{linear combination of } a_\ell \text{'s with } \ell < n)$$

The coefficient of  $x^\alpha$  is  $p(\alpha) \cdot a_0$ . For  $n \geq 1$ , as long as the coefficient  $p(\alpha + n) \in \mathbb{C}$  of  $a_n$  is non-zero, the condition that the coefficient of  $x^{\alpha+n}$  *vanish* determines  $a_n$  in terms of  $a_\ell$  with  $\ell < n$ .

Thus, when the two roots of  $p(\alpha) = 0$  do *not* differ by an integer, the differential equation has a solution of the form  $x^\alpha \sum_{n \geq 0} a_n x^n$  with *either* root  $\alpha$ , since the recursion for the coefficients succeeds. It is not difficult to prove that the series has positive radius of convergence: we prove this later.

[1.1.1] **Remark:** In any case, a *first solution* can be obtained by taking  $\alpha$  to be the solution of the indicial equation with larger real part, thus avoiding failure of the recursion. The problem is to obtain a *second solution* when the roots of the indicial equation are equal or differ by an integer.

## [1.2] Simpler exceptional case

The simpler *exceptional case* is that  $p(\alpha) = (\alpha - \alpha_o)^2$ . Then the recursion for the coefficients  $a_n$  succeeds, but produces just one solution  $f(x, \alpha_o)$ . To obtain a second solution, apply  $\partial/\partial\alpha$  to

$$Ef(x, \alpha) = p(\alpha) \cdot a_0 \cdot x^\alpha$$

and evaluate at  $\alpha = \alpha_o$ :

$$E \frac{\partial f}{\partial \alpha}(x, \alpha_o) = p'(\alpha_o) a_0 x^{\alpha_o} + p(\alpha_o) a_0 x^{\alpha_o} \log x = 0$$

Given the expansion  $f(x, \alpha) = x^\alpha \sum_n a_n x^n$ , the expansion of the second solution is

$$\frac{\partial f}{\partial \alpha}(x, \alpha_o) = x^{\alpha_o} \sum_n \frac{\partial a_n}{\partial \alpha} x^n + \log x \cdot x^{\alpha_o} \sum_n a_n x^n = x^{\alpha_o} \sum_n \frac{\partial a_n}{\partial \alpha} x^n + \log x \cdot f(x, \alpha_o)$$

Since  $a_0$  does not depend on  $\alpha$ , the differentiation in  $\alpha$  annihilates this term. Thus, the leading term of the second solution is  $\log x \cdot x^{\alpha_o} \cdot a_0$ : taking  $a_0 = 1$ ,

$$\begin{cases} \text{(first solution)} & = x^{\alpha_o} + \dots \\ \text{(second solution)} & = \log x \cdot x^{\alpha_o} + \dots \end{cases} \quad (\text{for } p(\alpha) = (\alpha - \alpha_o)^2)$$

We prove convergence later.

## [1.3] Less-simple exceptional case

In the less-simple exceptional case, the roots of the indicial equation are  $\alpha_o$  and  $\alpha_o - n_o$  with a positive integer  $n_o$ . The recursion for the coefficients of a solution of the form  $x^\alpha \sum_{n \geq 0} a_n x^n$  may fail at  $n_o$ , because it apparently requires division by

$$p((\alpha_o - n_o) + n_o) = p(\alpha_o) = 0$$

That is,  $f(x, \alpha)$  is of the form

$$f(x, \alpha) = A(x, \alpha) + \frac{B(x, \alpha)}{\alpha - (\alpha_o - n_o)}$$

where  $A(x, \alpha)$  is the polynomial in  $x$  consisting of terms of degree below  $n_o$ , and  $B(x, \alpha)$  is the part above-or-equal that degree. Application of the differential operator  $E$  to  $(\alpha - (\alpha_o - n_o)) \cdot f(x, \alpha)$  gives

$$E\left(\frac{\partial}{\partial \alpha}\Big|_{\alpha=\alpha_o-n_o}\left((\alpha - (\alpha_o - n_o)) \cdot f(x, \alpha)\right)\right) = \frac{\partial}{\partial \alpha}\Big|_{\alpha=\alpha_o-n_o}\left((\alpha - (\alpha_o - n_o)) \cdot p(\alpha) a_0 x^\alpha\right) = 0$$

That is, a *second solution* is

$$\frac{\partial}{\partial \alpha}\Big|_{\alpha=\alpha_o-n_o}\left((\alpha - (\alpha_o - n_o)) \cdot f(x, \alpha)\right) = A(x, \alpha_o - n_o) + \frac{\partial B}{\partial \alpha}(x, \alpha_o - n_o)$$

Term-wise, writing

$$f(x, \alpha) = x^\alpha \sum_{n < n_o} a_n x^n + \frac{x^\alpha}{\alpha - (\alpha_o - n_o)} \sum_{n \geq n_o} b_n x^n$$

a second solution is

$$\frac{\partial}{\partial \alpha}\Big|_{\alpha=\alpha_o-n_o}\left((\alpha - (\alpha_o - n_o)) \cdot f(x, \alpha)\right) = x^{\alpha_o-n_o} \sum_{n < n_o} a_n x^n + \log x \cdot x^{\alpha_o-n_o} \sum_{n \geq n_o} b_n x^n + x^{\alpha_o-n_o} \sum_{n \geq n_o} \frac{\partial b_n}{\partial \alpha} x^n$$

In particular, the log terms only appear in the higher-order terms: taking  $a_0 = 1$ ,

$$\begin{cases} \text{(first solution)} & = x^{\alpha_o} + \dots \\ \text{(second solution)} & = x^{\alpha_o-n_o} + \dots \end{cases} \quad (\text{for } p(\alpha) = (\alpha - \alpha_o)(\alpha - (\alpha_o - n_o)) \text{ and } 0 < n_o \in \mathbb{Z})$$

We prove convergence later.

## 2. Examples

### [2.1] Euclidean $n$ -space: radial eigenfunctions near 0

In radial coordinates, the Laplacian on Euclidean  $n$ -space is

$$u'' + \frac{n-1}{r} u'$$

The corresponding eigenvalue equation is

$$u'' + \frac{n-1}{r} u' - \lambda u = 0$$

The equation for a *fundamental solution* is also of interest:

$$u'' + \frac{n-1}{r} u' - \lambda u = \delta \quad (\text{with Dirac delta})$$

Rewritten to appraise the regular singular point at 0, the operator is

$$u'' + \frac{n-1}{r} u' - \frac{r^2 \lambda}{r^2} u$$

also showing that the indicial equation is independent of  $\lambda$ : it is

$$\alpha(\alpha - 1) + (n - 1)\alpha = 0$$

with roots  $0, 2-n$ , notably differing by an integer. For  $n = 2$ , the root is double-0. With  $\lambda = 0$ , the equation

$$u'' + \frac{n-1}{r}u' = 0$$

is of Euler-Cauchy type, explicitly solvable: the two solutions are

$$\text{first and second solutions} = \begin{cases} 1 & \text{and } r^{2-n} & (\text{for } n \neq 2) \\ 1 & \text{and } \log r & (\text{for } n = 2) \end{cases} \quad (\text{for } \lambda = 0)$$

In this example, the *first solution* 1 is *analytic* at 0. The *second solution* is *not* analytic at 0.

For general  $\lambda$ , the first solution is analytic at 0, the second solution is not analytic at 0, and should be expected to have logarithmic terms even for  $n > 2$ , although these will not appear in the leading term:

$$\text{first and second solutions} = \begin{cases} 1 + \dots & \text{and } r^{2-n} + \dots & (\text{for } n \neq 2) \\ 1 + \dots & \text{and } \log r + \dots & (\text{for } n = 2) \end{cases} \quad (\text{for } \lambda \neq 0)$$

[2.1.1] **Remark:** Computation of asymptotics at the *irregular* singular point *at infinity* reveals that for *odd*  $n$  the two solutions of the homogeneous equation have elementary expressions of the form

$$e^{\pm r\sqrt{\lambda}} \cdot (\text{polynomial in } \frac{1}{r})$$

For example,

$$\begin{cases} e^{\pm r\sqrt{\lambda}} & (\text{on } \mathbb{R}^1) \\ \frac{e^{\pm r\sqrt{\lambda}}}{r} & (\text{on } \mathbb{R}^3) \\ e^{\pm r\sqrt{\lambda}} \left( \frac{1}{r^2} - \frac{1}{\pm r^3 \sqrt{\lambda}} \right) & (\text{on } \mathbb{R}^5) \\ e^{\pm r\sqrt{\lambda}} \left( \frac{1}{r^3} - \frac{3}{\pm r^5 \sqrt{\lambda}} + \frac{3}{r^7 \lambda} \right) & (\text{on } \mathbb{R}^7) \end{cases}$$

Thus, in fact, logarithmic terms *do not appear* in the asymptotic near 0 for  $n$  odd, although this is not obvious from the expansion at 0.

[2.1.2] **Remark:** Further, in the elementary expressions for odd  $n$ , for the blow-ups at 0 to cancel, up to a constant the *first solution*, analytic at 0, must be the *sum* of the two expressions. Thus, the first solution can never have exponential decay at infinity. Unless  $\sqrt{\lambda}$  is purely imaginary, it will have exponential blow-up. For  $\sqrt{\lambda}$  purely imaginary, it will have polynomial decay.

[2.1.3] **Remark:** Since the support of  $\delta$  is just  $\{0\}$ , a solution of

$$u'' + \frac{n-1}{r}u' - \lambda u = \delta$$

satisfies the *homogeneous* form of the equation *away from* 0, so is a linear combination of the first and second solution, with the second solution being necessary to produce  $\delta$  at 0. For  $\text{Re}(\lambda) > 0$ , Fourier transform methods on  $\mathbb{R}^n$  produce an exponentially decreasing fundamental solution.

## [2.2] Hyperbolic $n$ -space: radial eigenfunctions near 0

The differential equation in radial coordinates for zonal spherical functions on hyperbolic  $n$ -space is

$$u'' + (n-1) \coth r \cdot u' - \lambda u = 0$$

Rewritten to highlight the regular singular point at  $r = 0$ , it is

$$u'' + \frac{(n-1)r \coth r}{r} \cdot u' - \frac{r^2 \lambda}{r^2} u = 0$$

The indicial equation is the same as that for Euclidean space, and is independent of  $\lambda$ :

$$\alpha(\alpha-1) + (n-1)\alpha = 0$$

When  $n = 2$ , the indicial equation has a double root 0. When  $n \geq 3$ , the roots are 0 and  $2 - n$ , which differ by an integer.

Since the solution of the indicial equation with larger real part is 0, the *first solution* is *regular* at  $r = 0$ . This is the *zonal spherical function* for that eigenvalue.

When  $n = 2$ , the *second solution* has a logarithmic leading term. For  $n > 2$ , the second solution has logarithmic terms further out in the expansion, but not in the leading term:

$$\text{first and second solutions} = \begin{cases} 1 + \dots & \text{and } r^{2-n} + \dots & (\text{for } n \neq 2) \\ 1 + \dots & \text{and } \log r + \dots & (\text{for } n = 2) \end{cases} \quad (\text{near } 0)$$

[2.2.1] **Remark:** Away from 0, a solution to

$$u'' + (n-1) \coth r \cdot u' - \lambda u = \delta \quad (\text{Dirac delta at base point})$$

satisfies the corresponding homogeneous equation. Being a linear combination of the first and second solutions, it must always non-trivially include the second solution to obtain  $\delta$  at 0. In particular, the full asymptotics of a fundamental solution near 0 should be expected to include logarithmic terms for  $n > 2$ , although not in the leading term.

## [2.3] Hyperbolic $n$ -space: special radial eigenfunctions near $+\infty$

Unlike the Euclidean case, in hyperbolic  $n$ -space the radial (spherically symmetric) Laplacian has a *regular* singular point at infinity, in suitable coordinates. With  $y = e^r$ , it suffices to look at  $y \rightarrow 0^+$  or, equivalently  $y \rightarrow +\infty$ . The  $y$ -coordinate is already adapted to the case  $y \rightarrow 0^+$ , so we consider this. The equation becomes

$$y^2 u'' + y(1 + (n-1) \frac{y + 1/y}{y - 1/y}) u' - \lambda u = 0 \quad (\text{as } y \rightarrow 0^+)$$

The singular point at 0 is visibly regular, with indicial equation

$$\alpha(\alpha - (n-1)) - \lambda = 0$$

Parametrizing the eigenvalue by  $\lambda = s(s - (n-1))$ , the roots are

$$\alpha = \begin{cases} s \\ (n-1) - s \end{cases} \quad (\text{with eigenvalue } \lambda = s(s - (n-1)))$$

The special case  $s = (n - 1)/2$  plays a central role in HarishChandra's treatment of Schwartz functions, and in that case  $(n - 1)/2$  is a *double* root. Thus, the two solutions are

$$\begin{cases} \text{(first solution)} & = y^{\frac{n-1}{2}} + \dots \\ \text{(second solution)} & = \log y \cdot y^{\frac{n-1}{2}} + \dots \end{cases} \quad (\text{as } y \longrightarrow 0^+)$$

The spherical symmetry gives symmetry under  $y \rightarrow 1/y$ , so

$$\begin{cases} \text{(first solution)} & = y^{-\frac{n-1}{2}} + \dots \\ \text{(second solution)} & = \log y \cdot y^{-\frac{n-1}{2}} + \dots \end{cases} \quad (\text{as } y \longrightarrow +\infty)$$

In terms of the original radius  $r$ ,

$$\begin{cases} \text{(first solution)} & = e^{-\frac{n-1}{2}|r|} + \dots \\ \text{(second solution)} & = |r| \cdot e^{-\frac{n-1}{2}|r|} + \dots \end{cases} \quad (\text{as } |r| \longrightarrow +\infty)$$

### 3. Convergence

#### [3.1] First solutions

We recall the relatively easy argument for convergence of the series for the *first solution*.

Let  $A, M \geq 1$  be large enough so that

$$b(x) = \sum_{n \geq 0} b_n x^n \quad (\text{with } |b_n| \leq A \cdot M^n)$$

$$c(x) = \sum_{n \geq 0} c_n x^n \quad (\text{with } |c_n| \leq A \cdot M^n)$$

Inductively, suppose that  $|a_\ell| \leq (CM)^\ell$ , with a constant  $C \geq 1$  to be determined in the following. Then

$$|n(n + \alpha - \alpha') \cdot a_n| \leq A \sum_{i=1}^n |n - i + \alpha| M^i \cdot (CM)^{n-i} + A \sum_{i=1}^n M^i \cdot (CM)^{n-i} \leq AM^n C^{n-1} \left( \frac{n(n+1)}{2} + n|\alpha| + n \right)$$

Dividing through by  $n|n + \alpha - \alpha'|$ , this is

$$|a_n| \leq AM^n \cdot C^{n-1} \frac{(n+1) + 2|\alpha| + 2}{2|n + \alpha - \alpha'|}$$

This motivates the choice

$$C \geq \sup_{1 \leq n \in \mathbb{Z}} \frac{(n+1) + 2|\alpha| + 2}{2|n + \alpha - \alpha'|}$$

which gives  $|a_n| \leq A(CM)^n$ , and a positive radius of convergence.

#### [3.2] Simpler second solution

Proof of convergence of the second solution when  $p(\alpha) = (\alpha - \alpha_o)^2$  requires finer attention to the dependence on  $\alpha$ . The recursion for the coefficients  $a_n$  of  $f(x, \alpha)$  is

$$-p(\alpha + n) \cdot a_n = \sum_{0 \leq \ell < n} (\alpha + \ell) a_\ell b_{n-\ell} + \sum_{0 \leq \ell < n} a_\ell c_{n-\ell}$$

Differentiating with respect to  $\alpha$ ,

$$-p'(\alpha + n) \cdot a_n - p(\alpha + n) \cdot \frac{\partial a_n}{\partial \alpha} = \sum_{0 \leq \ell < n} a_\ell b_{n-\ell} + \sum_{0 \leq \ell < n} (\alpha + \ell) \frac{\partial a_\ell}{\partial \alpha} b_{n-\ell} + \sum_{0 \leq \ell < n} \frac{\partial a_\ell}{\partial \alpha} c_{n-\ell}$$

Invoking the estimates from above, and inductively supposing  $|\partial a_n / \partial \alpha| \leq (DM)^n$  with  $D \geq C$ ,

$$\begin{aligned} |p(\alpha + n) \cdot \frac{\partial a_n}{\partial \alpha}| &\leq |p'(\alpha + n) \cdot (CM)^n| \\ &+ \sum_{0 \leq \ell < n} (CM)^\ell AM^{n-\ell} + \sum_{0 \leq \ell < n} |\alpha + \ell| (DM)^\ell AM^{n-\ell} + \sum_{0 \leq \ell < n} (DM)^\ell AM^{n-\ell} \end{aligned}$$

Thus,

$$\left| \frac{\partial a_n}{\partial \alpha} \right| \leq \left| \frac{p'(\alpha + n) \cdot (CM)^n}{p(\alpha + n)} \right| + \frac{A(n + n|\alpha| + \frac{1}{2}n(n+1) + n) D^{n-1} M^n}{|p(\alpha + n)|}$$

Taking

$$D \geq \sup_{n \geq 1} \left( \left| \frac{p'(\alpha + n) \cdot C}{p(\alpha + n)} \right| + \frac{A(3n + n^2 + n|\alpha|)}{|p(\alpha + n)|} \right)$$

gives the inductive step, proving that the second solution converges absolutely on some non-trivial interval  $0 < x < x_o$ .

### [3.3] Less-simple second solution

A similar but messier argument succeeds in the less-simple case, as well, proving convergence on some non-trivial interval.

## Bibliography

[Coddington 1961] E.A. Coddington, *An introduction to ordinary differential equations*, Dover, 1961.