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01. Simplest example phenomena

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There are three types of objects in play:

$$\left\{ \begin{array}{l} \text{primitive/primordial} \quad (\text{integers, primes, lattice points, ...}) \\ \text{zeta functions, } L\text{-functions} \\ \text{modular/automorphic forms} \end{array} \right.$$

The primitive objects are familiar, and accessible to everyone:

$$\left\{ \begin{array}{l} \mathbb{Z} = \text{integers} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} \\ \text{prime numbers} = \{2, 3, 5, 7, 11, 13, 17, \dots\} \end{array} \right.$$

[0.1] Questions about integers and primes It is easy to formulate questions about *integers* and *primes*, though not easy to distinguish those questions that have deeper meanings. More than 2000 years ago, contemporaries of Euclid knew that there are *infinitely-many* prime numbers.

Quantifying the distribution of primes is still essentially an open question! Chebyshev (1848-50), Riemann (1857-8), Hadamard (1896), and de la Vallée-Poussin (1896) touched various aspects of the basic **Prime Number Theorem**

$$\lim_{x \rightarrow \infty} \pi(x) / \frac{x}{\log x} = 1 \quad (\text{where } \pi(x) = \text{number of primes less than } x)$$

That is, in Landau's big-oh/little-oh notation, [1] there is an asymptotic

$$\pi(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right)$$

Riemann's 1857-8 memoir suggested that the main term can be adjusted and a much more precise error term given, contingent on properties of **the zeta function** (below). Very little progress has been made in this direction!

In 1837 Dirichlet used **L-functions** (below) to prove that there are infinitely-many primes in arithmetic progressions $a, a + m, a + 2m, a + 3m, \dots$, whenever the obviously-necessary condition $\gcd(a, m) = 1$ holds.

It is not known whether there are infinitely-many primes of the form $n^2 + 1$. It is not known whether there are infinitely-many *twin primes* p , that is, primes p such that $p + 2$ is also prime, although around 1915 Viggo Brun proved that

$$\sum_{\text{twin primes } p} \frac{1}{p} < \infty$$

while Euler had proven

$$\sum_{\text{all primes } p} \frac{1}{p} = \infty$$

[1] For two functions f, g , say $f = o(g)$ as $x \rightarrow x_o$ when $\lim_{x \rightarrow x_o} f(x)/g(x) = 0$. Say $f = O(g)$ as $x \rightarrow x_o$ when $\limsup_{x \rightarrow x_o} |f(x)/g(x)| < \infty$.

More to the point, we do not know *connections* of many such questions to anything else.

[0.2] Zeta and L -functions: simplest example The simplest zeta function is the (Euler-) **Riemann zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\text{complex variable } s)$$

Convergence for $\text{Re}(s) > 1$ follows immediately by comparison with the integral

$$\int_1^{\infty} \frac{dx}{x^s} = \lim_{T \rightarrow \infty} \left[\frac{-1}{(s-1)x^{s-1}} \right]_1^T = \lim_T \frac{1}{s-1} \cdot (1 - T^{1-s})$$

The first non-trivial assertion about $\zeta(s)$ is that it has an **Euler product** expansion

$$\sum_n \frac{1}{n^s} = \prod_{\text{primes } p} \frac{1}{1 - \frac{1}{p^s}}$$

This follows from expanding the geometric series

$$\frac{1}{1 - \frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots$$

and using *unique factorization* of positive integers into primes. This factorization produces an expression involving *just primes* equated to an expression *not* overtly involving primes. This hints at the relevance of the zeta function to prime numbers.

The Euler product for $\zeta(s)$ is the entry to non-elementary study of prime numbers by *analysis*. Whether or not we care greatly about prime numbers, the *connection* between primes and behavior of $\zeta(s)$ as a function of a complex variable will be striking: see Riemann's *Explicit Formula* a bit later.

The first *qualitative* result about primes was Euler's of about 1750, using the Euler product expansion of $\zeta(s)$, proving

$$\sum_{\text{all primes } p} \frac{1}{p} = \infty$$

The proof is by calculus: first, comparison with $\int_1^{\infty} x^{-s} dx$ for real $s > 1$ proves that $\zeta(s) \rightarrow +\infty$ as $s \rightarrow 1^+$. On the other hand, from $-\log(1-x) = x + x^2/2 + x^3/3 + \dots$,

$$\log \zeta(s) = \sum_p -\log \left(1 - \frac{1}{p^s}\right) = \sum_p \left(\frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \dots\right)$$

The terms after the initial $1/p^s$ do not contribute to any blow-up as $s \rightarrow 1^+$, as is seen by crudely estimating primes by positive integers:

$$\begin{aligned} \sum_p \sum_{\ell \geq 2} \frac{1}{\ell p^{\ell s}} &\leq \sum_{n \geq 2} \sum_{\ell \geq 2} \frac{1}{\ell n^{\ell s}} \leq \sum_{\ell \geq 2} \frac{1}{\ell} \int_1^{\infty} \frac{dx}{x^{\ell s}} \\ &= \sum_{\ell \geq 2} \frac{1}{\ell} \cdot \frac{1}{\ell s - 1} \leq \sum_{\ell \geq 2} \frac{1}{\ell} \cdot \frac{1}{\ell - 1} < \infty \quad (\text{for } s > 1) \end{aligned}$$

Thus, for some finite constant C ,

$$\log \zeta(s) \leq C + \sum_p \frac{1}{p^s} \quad (\text{for } s > 1)$$

Thus, the blow-up of $\log \zeta(s)$ as $s \rightarrow 1^+$ forces the divergence^[2] of $\sum_p 1/p$.

Dirichlet's proof of the infinitude of primes in arithmetic progressions $a, a + N, a + 2N, \dots$ is a refinement of Euler's argument, with a new ingredient, *characters of abelian groups*. A Dirichlet **L-function** mod N is

$$L(s, \chi) = \sum_n \frac{\chi(n)}{n^s}$$

where χ is a \mathbb{C} -valued function defined modulo N , that is,

$$\chi(n + N) = \chi(n) \quad (\text{for all } n = 1, 2, 3, \dots)$$

and χ is *multiplicative* in the sense that

$$\chi(ab) = \chi(a) \cdot \chi(b)$$

For fixed positive integer n , **Hurwitz' zeta function** is defined on positive-definite symmetric real matrices Y by

$$\zeta_n(Y, s) = \sum_{0 \neq \ell \in \mathbb{Z}^n} \frac{1}{(\ell^\top Y \ell)^s}$$

In fact, in terms of its *function*, this turns out to be a degenerate *Eisenstein series*, so is really an *automorphic form*.

[0.3] Evaluation of $\zeta(2)$, $\zeta(4)$, \dots About 1750, the state of the art was that no one could understand the number

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \zeta(2)$$

or other values of $\zeta(s)$. In that context, the following heuristic of Euler's was a sensation: imagine that $\sin x$ has a product expansion analogous to the product expansion of polynomials, given their zeros,

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(\pi n)^2}\right) = x - x^3 \sum_n \frac{1}{\pi^2 n^2} + \dots = x - x^3 \frac{\zeta(2)}{\pi^2} + \dots$$

On the other hand, we know the power series expansion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Thus, equating coefficients, apparently

$$\frac{1}{3!} = \frac{\zeta(2)}{\pi^2}$$

Slightly messier manipulations yield all values $\zeta(2k)$.

Nowadays, many things are known about **special values of L-functions**, but many mysteries remain. For example, in 1978 Apéry proved that $\zeta(3)$ is *irrational*, but very little more is known about *bad* values of *L-functions*, such as $\zeta(3)$.

Altogether, despite more than 150 years of study, there are many unresolved questions about $\zeta(s)$, and about every other reasonable zeta and *L-function*!

[2] For real $s > 1$, certainly $\sum_p p^{-s} \leq \sum_p 1/p$, so the blow-up of the left-hand side as $s \rightarrow 1^+$ implies divergence of the right.

[0.4] **Modular forms** Modular forms are also known as **automorphic forms**, and by many other similar names. They are hard to explain or characterize, but have been crucial in many developments in modern mathematics, especially number theory. A few examples can be written down directly, but their significance is not clear from their expressions.

The simplest modular forms are functions on the complex upper half-plane \mathfrak{H} . Three of the simplest types are

$$\left\{ \begin{array}{ll} \text{(holomorphic) Eisenstein series} & = E_{2k}(z) = \sum_{\substack{c,d \in \mathbb{Z}, c \geq 0 \\ \text{coprime } c,d}} \frac{1}{(cz+d)^{2k}} \quad (z \in \mathfrak{H}, 2 < 2k \in 2\mathbb{Z}) \\ \text{theta series} & = \theta_k(z) = \sum_{\ell \in \mathbb{Z}^k} e^{\pi i z (\ell_1^2 + \dots + \ell_k^2)} \quad (z \in \mathfrak{H}, 1 \leq k \in \mathbb{Z}) \\ \text{(waveform) Eisenstein series} & = E_s(z) = \sum_{\substack{c,d \in \mathbb{Z}, c \geq 0 \\ \text{coprime } c,d}} \frac{\text{Im}(z)^s}{|cz+d|^{2s}} \quad (z \in \mathfrak{H}, s \in \mathbb{C}) \\ \text{Ramanujan's } \Delta & = \Delta(z) = e^{2\pi i z} \prod_n (1 - e^{2\pi i n z})^{24} \quad (z \in \mathfrak{H}) \end{array} \right.$$

Apart from being functions on the upper half-plane, it is not clear what these have in common.

Theta series have immediate relevance to more tangible things, since they can be re-arranged to be *generating functions* counting the expressions of non-negative integers as sums of k squares. Specifically, let

$$\nu_k(n) = \text{number of } (\ell_1, \dots, \ell_k) \in \mathbb{Z}^k \text{ such that } n = \ell_1^2 + \dots + \ell_k^2$$

Then

$$\theta_k(z) = \sum_{n=0}^{\infty} \nu_k(n) e^{\pi i z \cdot \nu_k(n)}$$

One might have observed that the k^{th} theta series in this family is the k^{th} power of the first one:

$$\sum_{\ell \in \mathbb{Z}^k} e^{\pi i z (\ell_1^2 + \dots + \ell_k^2)} = \left(\sum_{\ell \in \mathbb{Z}} e^{\pi i z \ell^2} \right)^k$$

An **archetype** for the relation of modular/automorphic forms to zeta functions and L -functions is the following connection between the simplest theta series $\theta(z) = \sum_{\ell \in \mathbb{Z}} e^{\pi i z \ell^2}$ and the zeta function. Recall [3] Euler's integral for the **gamma function**

$$\Gamma(s) = \int_0^{\infty} y^s e^{-y} \frac{dy}{y} \quad (\text{interpolating factorials, since } \Gamma(n) = (n-1)!)$$

The relation of interest, known to Riemann and probably much earlier, is the **integral representation** of $\zeta(s)$ in terms of $\theta(z)$:

$$2\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s) = \int_0^{\infty} y^{s/2} (\theta(iy) - 1) \frac{dy}{y}$$

This relation is essential in proof of basic properties of $\zeta(s)$, such as *analytic continuation*, *functional equation*, and *vertical growth estimates*. This archetypical example is one starting point for applications of automorphic forms to zeta functions, and then to prime numbers.

[0.5] **The Δ -function** It is a little unfortunate that the symbol Δ is over-used, but here the explanation is that this Δ arose historically as essentially the *discriminant* of Weierstraß' cubic equation

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3 \quad (\text{for } u \in \mathbb{C})$$

[3] When these notes say *recall* it is not to imply dereliction on the part of the reader if the discussion is unfamiliar.

for Weierstraß' elliptic function attached to the lattice $\Lambda = \mathbb{Z} + \mathbb{Z} \cdot z$ depending on $z \in \mathfrak{H}$

$$\wp(u) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z + \lambda)^2} - \frac{1}{\lambda^2} \right) \quad (\text{for lattice } \Lambda = \mathbb{Z} + \mathbb{Z} \cdot z \text{ in } \mathbb{C})$$

The symbols g_2, g_3 are traditional, and slightly mask the relation to the holomorphic Eisenstein series listed above:

$$g_2 = g_2(z) = 60 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^4} = 60 \cdot 2\zeta(4) \cdot E_4(z)$$

$$g_3 = g_3(z) = 140 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^3} = 140 \cdot 2\zeta(6) \cdot E_4(z)$$

Then, up to choices of normalization, and not at all obviously, $\Delta(z)$ really is the discriminant of the cubic, namely

$$\Delta(z) = \frac{1}{1728} (E_4(z)^3 - E_6(z)^2)$$

The product expansion of $\Delta(z)$ is a non-trivial assertion, and needs a special argument. The Δ -function was studied for many decades before Ramanujan, but he found many, many unexpected connections to special functions and combinatorics, so his name is often associated to this example.

[0.6] Siegel-Weil formulas The very different appearance of theta series and Eisenstein series gives no hint that there is any relation between them. To give the simplest example of such a relationship, a slightly adjusted theta series is needed. Instead of sums-of-squares, a different quadratic form is needed. Some other possibilities are suggested by rewriting the sum of k squares as

$$\ell_1^2 + \dots + \ell_k^2 = \ell^\top \ell = \ell^\top \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \ell \quad (\text{viewing } \ell \text{ as a column vector})$$

Generalizing the identity matrix to a general (symmetric) k -by- k matrix Q , the family of functions

$$q(\ell_1, \dots, \ell_k) = \ell^\top \cdot Q \cdot \ell$$

includes the sums-of-squares as a special case.

Because of some technical requirements we won't discuss just yet, for our example, we take^[4]

$$Q = \begin{pmatrix} 8 & 3 & & & & & & \\ 3 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & 1 \\ & & & & & & 1 & 2 \end{pmatrix} \quad (\text{unmarked entries are 0})$$

^[4] Elaborating a few details, although still not explaining *why*: we need an 8-by-8 symmetric positive-definite integer matrix Q with 2's on the diagonal and with determinant 1. Admittedly, it is not obvious why we need this, or that it is possible, but one should try to view these requirements as innocent variations on the sum-of-squares theme. The rationale for the specifics will be clear later. It's not obvious that there exists such a thing, or how to find it, but the example given is one. It's not trivial to verify that it satisfies these conditions, either!

and instead of counting expressions of n as a sum of 8 squares, we count expressions

$$n = \ell^\top \cdot Q \cdot \ell = 8\ell_1^2 + 6\ell_1\ell_2 + 2\ell_2^2 + 2\ell_2\ell_3 + 2\ell_3^2 + 2\ell_3\ell_4 + 2\ell_4^2 + \dots + 2\ell_7\ell_8 + 2\ell_8^2 \quad (\ell_j \in \mathbb{Z})$$

Not as appealing or simple as the sum of eight squares would be. But, with this adaptation, form the analogous theta series

$$\theta_Q(z) = \sum_{\ell \in \mathbb{Z}^8} e^{\pi i z \cdot \ell^\top Q \ell} \quad (\text{with } z \in \mathfrak{H})$$

Amazingly, this theta series is exactly an Eisenstein series:

$$\theta_Q(z) = E_4(z)$$

This is crazy! This is the simplest **Siegel-Weil** relation. We will see that this relation has a more down-to-earth corollary, giving another proof of

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \zeta(4) = \frac{\pi^4}{90}$$

[0.7] **Chaos?** The above sample phenomena merely touch the surface of many deep relationships. Without knowing more, these examples are admittedly chaotic and of unclear import. Subsequent discussion will assimilate them into a larger, coherent picture.