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Riemann's Explicit/Exact formula

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Even more interesting than a Prime Number Theorem is the *precise* relationship between primes and zeros of zeta found by Riemann. The idea applies to any zeta or L-function for which we know an analytic continuation and other reasonable properties.

It took 40 years for [Hadamard 1893], [vonMangoldt 1895], and others to complete Riemann's 1857-8 sketch of the *Explicit Formula* relating primes to zeros of the Euler-Riemann zeta function. Even then, lacking a zero-free strip inside the critical strip, the Explicit Formula does *not* yield a Prime Number Theorem, despite giving a precise relationship between primes and zeros of zeta.

The *idea* is that the equality of the Euler product and Riemann-Hadamard product for zeta allows extraction of an *exact formula* for a suitably-weighted counting of primes, a sum over zeros of zeta, via a contour integration of the logarithmic derivatives. As observed by Weil, the classical formulas are instances of evaluations of a certain *distribution*, in the sense of *generalized functions*.

Various lower-level analytical difficulties arise, but by now have been resolved. The main point is the relationship of primes to zeros of zeta.

An essential supporting point is the *integral representation* of $\zeta(s)$ in terms of a *theta function*, the latter a simple example of *automorphic forms*. This integral representation is the source of the basic non-obvious analytical properties of $\zeta(s)$, such as *analytic continuation* and *functional equation*, and also *growth estimates*. The growth estimates are critical for invocation of Hadamard's theorem on product expansions of entire functions. This situation is an archetype: automorphic forms are the principal device for proof of non-trivial essential facts about zeta functions and L-functions generally.

A key step in proof that the theta series *is* an automorphic form is the *Poisson summation formula*, itself a corollary of the representability of nice functions by their *Fourier series*.

Also, the *Gamma function* and some of its *asymptotics* are needed to use the functional equation of $\zeta(s)$ to prove the growth bound on $\zeta(s)$ necessary to apply the Hadamard result. It is noteworthy that what is needed is something simpler than Stirling's formula, and admitting a more straightforward proof.

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1. *Riemann's explicit formula*

Riemann's dramatic relation between primes and zeros of the zeta function depends on many ideas undeveloped in Riemann's time. Thus, the following sketch, roughly following Riemann, is not a proof. Rather, the sketch tells which supporting ideas need development to produce a proof. This example is an archetype, and the supporting ideas are applicable to a broad class of zeta functions and L-functions.

If we believe, as Riemann did, and as Hadamard and others later *proved*, that $\zeta(s)$ has both its *Euler product* expansion in a half-plane

$$\zeta(s) \;=\; \sum_{n\geq 1} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \frac{1}{1-\frac{1}{p^s}} \qquad\qquad (\text{for } \operatorname{Re} s > 1)$$

and has a *Riemann-Hadamard product* expansion of its *meromorphic continuation* throughout $\mathbb{C}$

$$(s-1)\,\zeta(s) \;=\; e^{a+bs}\prod_\rho \Big(1-\frac{s}{\rho}\Big)\,e^{s/\rho}\cdot \prod_{n=1}^\infty \Big(1+\frac{s}{2n}\Big)\,e^{-s/2n} \qquad (\rho \text{ non-trivial zero of } \zeta, \text{ for all } s\in\mathbb{C})$$

then we can follow Riemann to extract more tangible information from the equality of the two products

$$(s-1)\prod_p \frac{1}{1-\frac{1}{p^s}} \;=\; e^{a+bs}\prod_\rho \Big(1-\frac{s}{\rho}\Big)\,e^{s/\rho}\cdot \prod_{n=1}^\infty \Big(1+\frac{s}{2n}\Big)\,e^{-s/2n} \qquad (\mathrm{Re}\,s>1)$$

First, take logarithmic derivatives of both sides, using $-\log(1-x) = x + x^2/2 + x^3/3 + \ldots$ on the left-hand side:

$$\frac{1}{s-1} - \sum_{m\ge 1,\;p} \frac{\log p}{p^{ms}} \;=\; b \;+\; \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) \;+\; \sum_n \Big(\frac{1}{s+2n} - \frac{1}{2n}\Big)$$

A slight rearrangement:

$$\sum_{m\ge 1,\;p} \frac{\log p}{p^{ms}} \;=\; \frac{1}{s-1} - b \;-\; \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) \;-\; \sum_n \Big(\frac{1}{s+2n} - \frac{1}{2n}\Big) \qquad (\text{for } \mathrm{Re}\,s>1)$$

The left-hand side needs $\mathrm{Re}\,s>1$ for convergence, while the right-hand side converges for all $s\in\mathbb{C}$ apart from the visible poles at 1, the non-trivial zeros ρ, and the trivial zeros $2,4,6,\ldots$. Second, *if* we can apply the Perron integral operator

$$f \;\longrightarrow\; \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} f(s)\cdot \frac{X^s}{s}\,ds \qquad (\text{with } \sigma>1)$$

to both sides of the equality, *and* apply the identity

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{Y^s}{s}\,ds \;=\; \begin{cases} 1 & (\text{for } Y>1) \\ 0 & (\text{for } 0<Y<1)\end{cases} \qquad (\text{for } \sigma>0)$$

term-wise to the left-hand side, *and* use residues *term*-wise to evaluate the right-hand side, we would have

$$\sum_{p^m<X} \log p \;=\; (X-1) \;-\; b \;-\; \sum_\rho \Big(\frac{X^\rho}{\rho} + \frac{1}{-\rho} + \frac{1}{\rho}\Big) \;-\; \sum_n \Big(\frac{X^{-2n}}{-2n} + \frac{1}{2n} - \frac{1}{2n}\Big)$$

which simplifies to von Mangoldt's reformulation of **Riemann's Explicit Formula:**

$$\sum_{p^m<X} \log p \;=\; X \;-\; (b+1) \;-\; \sum_\rho \frac{X^\rho}{\rho} \;+\; \sum_{n\ge 1} \frac{X^{-2n}}{2n}$$

Slightly more precisely, because of the way the Perron integral transform is applied, and the fragility of the convergence, we should say

$$\sum_{p^m<X} \log p \;=\; X \;-\; (b+1) \;-\; \lim_{T\to\infty} \sum_{|\mathrm{Im}\,(\rho)|<T} \frac{X^\rho}{\rho} \;+\; \sum_{n\ge 1} \frac{X^{-2n}}{2n}$$

[1.0.1] Remark: Immediately, it must be said that the above sketch has many potential problems, despite having a clear intention. The same is true of Riemann's original.

[1.0.2] Remark: The existence of the Riemann-Hadamard product needs both *generalities* about Weierstraß-Hadamard product expressions for entire functions of prescribed growth, and *specifics* about the growth of the *analytic continuation* of $\zeta(s)$. The analytic continuation of $\zeta(s)$ is discussed in the next section, and growth properties thereafter.

[1.0.3] Remark: The two sides of the equality of logarithmic derivatives are very different. The logarithmic derivative of the Euler product converges well in right half-planes, and converges all the better farther to the right. The logarithmic derivative of the Riemann-Hadamard product does not converge powerfully, but is not restricted to a half-plane, and its poles are exhibited explicitly by the expression.

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2. *Analytic continuation and functional equation of* $\zeta(s)$

The following ideas gained publicity and importance from Riemann's 1857-8 paper, but were apparently known before Riemann's time.

The key is that the completed zeta function has an *integral representation* in terms of an *automorphic form*, namely, the simplest theta function. Both the analytic continuation and the functional equation of zeta follow from this integral representation using a parallel functional equation of the theta function, the latter demonstrated by *Poisson summation*.

[2.1] Elementary-but-doomed argument It is worthwhile to see that simple calculus can extend the domain of $\zeta(s)$ a little. The idea is to pay attention to *quantitative* aspects of the integral test. That is,

$$\zeta(s) - \frac{1}{s-1} \;=\; \zeta(s) - \int_1^\infty \frac{dx}{x^s} \;=\; \sum_n \Big(\frac{1}{n^s} - \int_n^{n+1} \frac{dx}{x^s}\Big) \;=\; \sum_n \Big(\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big]\Big)$$

Even for complex s, we have a Taylor-Maclaurin expansion with error term

$$(n+1)^{1-s} \;=\; \big(n\cdot(1+\frac{1}{n})\big)^{1-s} \;=\; n^{1-s}\cdot\big(1+\frac{1-s}{n} + O(\tfrac{1}{n^2})\big) \;=\; \frac{1}{n^{s-1}} - \frac{s-1}{n^s} + O(\frac{s-1}{n^{s+1}})$$

The constant in the big-O term is *uniform* in n for fixed s. Thus,

$$\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big] \;=\; \frac{1}{n^s} - \frac{1}{n^s} + \frac{1}{s-1}O\big(\tfrac{1}{n^{s+1}}\big) \;=\; O\big(\tfrac{1}{n^{s+1}}\big)$$

That is, for fixed [1] $\mathrm{Re}(s) > 0$, we have *absolute convergence* of

$$\sum_n \Big(\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big]\Big) \qquad\qquad (\text{for } \mathrm{Re}(s) > 0)$$

in the larger region $\mathrm{Re}(s) > 0$.

[2.1.1] Remark: A similar but increasingly complicated device produces a meromorphic continuation to half-planes $\mathrm{Re}(s) > \ell$. However, this approach is under-powered to discuss things like Riemann's Explicit Formula, for example.

[2.2] More serious argument Euler's integral for the **gamma function** [2] is

$$\Gamma(s) \;=\; \int_0^\infty e^{-t}\, t^s \, \frac{dt}{t}$$

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[1] In fact, the big-O constant is also *uniform* for s in compacts inside $\mathrm{Re}(s) > 0$. Thus, the series converges *locally uniformly on compacts*, so does give a *holomorphic* function.

[2] Among other roles, the gamma function $\Gamma(s)$ interpolates the *factorial function*: integration by parts yields $\Gamma(n) = (n-1)!$ for positive integer n.

[2.2.1] Theorem: The **completed** zeta function

$$\xi(s) \;=\; \pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})\,\zeta(s)$$

has an analytic continuation to $s \in \mathbb{C}$, except for simple poles at $s = 0, 1$, and has the *functional equation*

$$\xi(1-s) \;=\; \xi(s)$$

[2.2.2] Remark: The following *proof* is itself an archetype.

Proof: The simplest *theta function* is

$$\theta(z) \;=\; \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} \qquad \text{(with } z \text{ in the complex upper half-plane } \mathfrak{H}\text{)}$$

By Riemann's time, Jacobi's functional equation of $\theta(z)$ was well-known:

$$\theta(z) \;=\; \frac{1}{\sqrt{-iz}}\cdot\theta(-1/z)$$

This is proven below. The connection to $\zeta(s)$ is the *integral representation*: [3]

[2.2.3] Claim: For $\mathrm{Re}\,(s) > 1$

$$\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s) \;=\; \int_0^\infty \frac{\theta(iy)-1}{2}\, y^{s/2}\,\frac{dy}{y}$$

Proof: (of claim) Starting from the integral, for $\mathrm{Re}\,(s) > 1$, compute directly

$$\int_0^\infty \frac{\theta(iy)-1}{2}\, y^{s/2}\,\frac{dy}{y} \;=\; \int_0^\infty \sum_{n\geq 1} e^{-\pi n^2 y}\, y^{s/2}\,\frac{dy}{y} \;=\; \sum_{n\geq 1}\int_0^\infty e^{-\pi n^2 y}\, y^{s/2}\,\frac{dy}{y}$$

$$=\; \pi^{-s/2}\sum_{n\geq 1}\frac{1}{n^{2s}}\int_0^\infty e^{-y}\, y^{s/2}\,\frac{dy}{y} \qquad \text{(by replacing } y \text{ by } y/(\pi n^2)\text{)}$$

$$=\; \pi^{-s/2}\Gamma(\frac{s}{2})\sum_{n\geq 1}\frac{1}{n^{2s}} \;=\; \pi^{-s/2}\Gamma(\frac{s}{2})\,\zeta(s)$$

This establishes the integral representation. ///

As $y \to +\infty$ the function $\frac{\theta(iy)-1}{2}$ is of rapid decay, by an easy estimate such as

$$\frac{\theta(iy)-1}{2} \;=\; \sum_{n\geq 1} e^{-\pi n^2 y} \;\leq\; e^{-\pi y/2}\sum_{n\geq 1} e^{-\pi n^2/2} \;=\; \text{const}\cdot e^{-\pi y/2} \qquad \text{(for } y \geq 1\text{)}$$

Thus, the integral from 1 (not 0) to $+\infty$ is nicely convergent for *all* values of s, and

$$\int_1^\infty \frac{\theta(iy)-1}{2}\, y^{s/2}\,\frac{dy}{y} \;=\; \text{entire in } s$$

[3] This sort of integrals, against t^s, is a *Mellin transform*. It is a *Fourier transform* in different coordinates. The measure dt/t is invariant under *dilations*.

The main trick (known to Riemann and before) is to use Jacobi's functional equation for $\theta(z)$ to convert the other part of the integral, from 0 to 1, into a similar integral from 1 to $+\infty$, up to more elementary terms admitting direct analysis. [4] The minor book-keeping complication is that the function whose Mellin transform we have is not exactly $\theta(z)$. Thus, we must do a little computation

$$\frac{\theta(-1/iy)-1}{2} = \frac{y^{1/2}\theta(iy)-1}{2} = y^{1/2}\frac{\theta(iy)-1}{2} + \frac{y^{1/2}}{2} - \frac{1}{2}$$

Then

$$\int_0^1 \frac{\theta(iy)-1}{2}\, y^{s/2}\,\frac{dy}{y} = \int_1^\infty \frac{\theta(-1/iy)-1}{2}\, y^{-s/2}\,\frac{dy}{y} = \int_1^\infty \left(y^{1/2}\frac{\theta(iy)-1}{2} + \frac{y^{1/2}}{2} - \frac{1}{2}\right) y^{-s/2}\,\frac{dy}{y}$$

$$= \int_1^\infty \frac{\theta(iy)-1}{2}\, y^{-s/2}\,\frac{dy}{y} + \int_1^\infty \left(\frac{y^{(1-s)/2}}{2} - \frac{y^{-s/2}}{2}\right)\frac{dy}{y} = \int_1^\infty \frac{\theta(iy)-1}{2}\, y^{-s/2}\,\frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s}$$

$$= \text{(entire)} + \frac{1}{s-1} - \frac{1}{s}$$

The elementary expressions $1/(s-1)$ and $1/s$ certainly have meromorphic continuations to $\mathbb{C}$, with explicit poles. Thus, together with the first integral from 1 to ∞, we have

$$\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s) \;=\; \int_1^\infty \frac{\theta(iy)-1}{2}\,(y^{s/2} + y^{(1-s)/2})\,\frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s}$$

The integral on the right-hand side is entire, and the two elementary summands have obvious extensions. This gives the meromorphic continuation of $\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s)$. Further, the right-hand side is visibly symmetrical under $s \to 1-s$, which gives the functional equation. ///

[2.2.4] Remark: Attempting to avoid the gamma factor leads to an unsymmetrical and unenlightening form of the functional equation.

[2.2.5] Remark: The fact that $\Gamma(s/2)$ has no zeros assures that the meromorphic continuation of $\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s)$ masks no poles of $\zeta(s)$. The non-vanishing of $\Gamma(s)$ follows from the identity [5]

$$\Gamma(s)\cdot\Gamma(1-s) \;=\; \frac{\pi}{\sin \pi s}$$

Now we prove Jacobi's functional equation for $\theta(z)$

[2.2.6] Claim: $\theta(-1/iy) = \sqrt{y}\cdot\theta(iy)$

Proof: This underlying symmetry itself follows from a more fundamental fact, the **Poisson summation formula** [6]

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n) \qquad \text{(for suitable functions } f\text{, with Fourier transform } \widehat{f}\text{)}$$

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[4] It is not obvious that $\frac{\theta(iy)-1}{2}$ has any property that would ensure this. However, in the context of 1850 properties of functions akin to $\theta(z)$ had been much studied by Jacobi and others, so such possibilities would have been well known.

[5] This identity for $\Gamma(s)$ is proven in an appendix.

[6] There are at least two different-looking proofs for the Poisson summation formula. One reduces the assertion to Fourier series. The other is uses tempered distributions. The Fourier series argument is discussed in the next section.

The Fourier transform is

$$\text{Fourier transform of } f \;=\; \widehat{f}(\xi) \;=\; \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\, dx$$

The Poisson summation formula is applied to

$$f(x) \;=\; \varphi(\sqrt{y}\cdot x) \quad \text{with} \quad \varphi(x) \;=\; e^{-\pi x^2}$$

The Gaussian $\varphi(x) = e^{-\pi x^2}$ has the useful property that it is its own Fourier transform, easily proven by completing the square and a contour integration:

$$\widehat{\varphi}(\xi) \;=\; \int_{\mathbb{R}} e^{-\pi x^2}\, e^{-2\pi i x\xi}\, dx \;=\; \int_{\mathbb{R}} e^{-\pi(x+i\xi)^2 - \pi\xi^2}\, dx \;=\; e^{-\pi\xi^2} \int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\, dx$$

The latter integral is

$$\int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\, dx \;=\; \int_{\mathbb{R}+i\xi} e^{-\pi x^2}\, dx \;=\; \int_{\mathbb{R}} e^{-\pi x^2}\, dx$$

by moving the contour of integration, which is easily justified. Thus, the integral is a independent of ξ. In fact, the constant is 1. Again by a straightforward change of variables, Fourier transform behaves well with respect to *dilations*:

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{R}} \varphi(\sqrt{y}\,x)\, e^{-2\pi i x\xi}\, dx \;=\; \frac{1}{\sqrt{y}} \int_{\mathbb{R}} \varphi(x)\, e^{-2\pi i x\xi/\sqrt{y}}\, dx \;=\; \frac{1}{\sqrt{y}} \widehat{\varphi}(\xi/\sqrt{y}) \;=\; \frac{1}{\sqrt{y}} e^{-\pi\xi^2/y}$$

by replacing x by $x/\sqrt{y}$. Applying Poisson summation to $f(x) = e^{-\pi x^2 y}$,

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 y} \;=\; \frac{1}{\sqrt{y}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/y}$$

This gives

$$\theta(iy) \;=\; \frac{1}{\sqrt{y}} \theta(-1/iy)$$

which yields the asserted identity. ///

[2.2.7] Remark: For $z \in \mathfrak{H}$, also $-1/z \in \mathfrak{H}$, and the series for $\theta(z)$ and $\theta(-1/z)$ are nicely convergent. The identity just proven for θ is $\theta(-1/z) = \sqrt{-iz}\,\theta(z)$ on the imaginary axis. The *Identity Principle* from complex analysis implies that the same equality holds for all $z \in \mathfrak{H}$.

[2.2.8] Remark: The leading factor $\pi^{-s/2}\Gamma(\frac{s}{2})$ should *not* be construed as objectionable in any way, but, rather, as something that really does *belong* with $\zeta(s)$. The $\pi^{-s/2}\Gamma(\frac{s}{2})$ is called the **gamma factor** for $\zeta(s)$. In the context of the *Euler product* the modern viewpoint is that the gamma factor is a further Euler factor corresponding to the *prime*[7] ∞.

[7] An insight of modern times is that the completion $\mathbb{R}$ should whenever possible be put on an even footing with the other completions $\mathbb{Q}_p$ of $\mathbb{Q}$. Thus, although there is no actual prime ∞ in $\mathbb{Z}$ (or anywhere else), the objects that accompany genuine primes p and completions $\mathbb{Q}_p$ often have analogues for $\mathbb{R}$, so we *backform* to refer to the *prime* ∞. One attempt to be less bold in this regard is to speak of *places* rather than *primes*, but there's little point in fretting about this.

3. *Poisson summation*

The simplest form of the Poisson summation formula is

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n) \qquad\qquad \text{(for suitable functions } f\text{, with Fourier transform } \widehat{f}\text{)}$$

with Fourier transform

$$\text{Fourier transform of } f \;=\; \widehat{f}(\xi) \;=\; \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\, dx$$

[3.1] The idea An easy way to understand why this should be true is the following heuristic. Given f a function on $\mathbb{R}$, form the *periodic* version of f

$$F(x) \;=\; \sum_{n\in\mathbb{Z}} f(x+n)$$

A periodic function should be represented by its *Fourier series*, so

$$F(x) \;=\; \sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell x} \int_0^1 F(x)\, e^{-2\pi i\ell x}\, dx$$

The Fourier *coefficients* of F expand to be seen as the Fourier *transform* of f:

$$\int_0^1 F(x)\, e^{-2\pi i\ell x}\, dx \;=\; \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)\, e^{-2\pi i\ell x}\, dx$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x)\, e^{-2\pi i\ell x}\, dx \;=\; \int_{\mathbb{R}} f(x)\, e^{-2\pi i\ell x}\, dx \;=\; \widehat{f}(\ell)$$

Evaluating at 0, we should have

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; F(0) \;=\; \sum_{\ell\in\mathbb{Z}} \widehat{f}(\ell)$$

[3.2] What would it take to legitimize this? Certainly f must be of sufficient decay so that the integral for its Fourier transform is convergent, and so that summing its translates by $\mathbb{Z}$ is convergent. We'd want f to be continuous, probably differentiable, so that we can talk about pointwise values of F, and to make plausible the hope that the Fourier series of F converges to F pointwise. For f and several derivatives rapidly decreasing, the Fourier transform $\widehat{f}$ will be of sufficient decay so that its sum over $\mathbb{Z}$ does converge.

A simple sufficient hypothesis for convergence is that f be in the *Schwartz space* of infinitely-differentiable functions all of whose derivatives are of *rapid decay*, that is,

$$\text{Schwartz space} \;=\; \{\text{smooth } f \;:\; \sup_x\, (1+x^2)^\ell |f^{(i)}(x)| < \infty \text{ for all } i, \ell\}$$

Representability of a periodic function by its Fourier series is a serious question, with several possible senses. The following section gives a result sufficient for the moment.

4. *Pointwise convergence of Fourier series*

A special, self-contained argument gives a good-enough result for immediate purposes. [8]

Consider ($\mathbb{Z}$-)periodic functions on $\mathbb{R}$, that is, complex-valued functions f on $\mathbb{R}$ such that $f(x+n) = f(x)$ for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$. For periodic f sufficiently nice so that integrals

$$\widehat{f}(n) \;=\; \int_0^1 f(x)\, e^{-2\pi i n x}\, dx \qquad\qquad (n^{th} \text{ Fourier coefficient of } f)$$

make sense, the **Fourier expansion** of f is

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{2\pi i n x} \qquad\qquad (\text{Fourier expansion of } f)$$

We want simple sufficient conditions on f and on points x_o so that

$$f(x_o) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{2\pi i n x_o} \qquad\qquad (\text{as convergent double sum of complex numbers})$$

Consider periodic *piecewise*-C^o [9] functions which are left-continuous and right-continuous [10] at any discontinuities.

[4.0.1] Theorem: For periodic f piecewise-C^o functions left-continuous and right-continuous at its discontinuities, for points x_o at which f is C^0 and *left-differentiable*[11] and *right-differentiable*, the Fourier series of f evaluated at x_o *converges* to $f(x)$:

$$f(x_o) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{2\pi i n x_o}$$

That is, for such functions, at such points, the Fourier series *represents* the function *pointwise*.

[4.0.2] Remark: The most notable missing conclusion in the theorem is *uniform* pointwise convergence. For more serious applications, pointwise convergence not known to be uniform is often useless.

Proof: First, treat the special case $x_o = 0$ and $f(0) = 0$. Representability of $f(0)$ by the Fourier series is the assertion that

$$0 \;=\; f(0) \;=\; \lim_{M,N\to+\infty} \sum_{-M\le n<N} \widehat{f}(n)\, e^{2\pi i n\cdot 0} \;=\; \lim_{M,N\to+\infty} \sum_{-M\le n<N} \widehat{f}(n)$$

[8] The virtue of the argument here is mainly its immediacy and lack of prerequisites. However, this approach is inadequate in most other situations. For example, for many purposes, we want not mere pointwise convergence, but *uniform* pointwise convergence.

[9] A function is *piecewise*-C^o when it is C^o *except* for a discrete set of points, at which it may fail to be continuous.

[10] As usual, a function has a *left-continuous* at x_o if the limit of $f(x)$ as x approaches x_o from the *left* exists. Similarly, f is *right-continuous* if the limit approaching from the *right* exists. Note that there is no purpose in asking whether these limits are the value $f(x_o)$, since if they had that common value, then the function would be continuous at x_o, and the notion of one-sided continuity would be irrelevant.

[11] As usual, a function f is *left-differentiable* at x_o if the limit of $[f(x) - f(x_o)]/[x - x_o]$ exists as x approaches x_o *from the left*. Right-differentiability at x_o is similar. Admittedly, this is a clumsy notion, but is relevant to treatment of functions that are not entirely smooth, but not too badly behaved.

Substituting the defining integral for the Fourier coefficients:

$$\sum_{-M\le n<N} \widehat{f}(n) \;=\; \sum_{-M\le n<N} \int_0^1 f(u)\, e^{-2\pi i n u}\, du$$

$$= \int_0^1 \sum_{-M\le n<N} f(u)\, e^{-2\pi i n u}\, du \;=\; \int_0^1 f(u)\cdot \frac{e^{2\pi i M u} - e^{-2\pi i N u}}{1-e^{-2\pi i u}}\, du$$

To prove the representability of $f(0)$ by the Fourier series, we will show that

$$\lim_{\ell\to\pm\infty} \int_0^1 \frac{f(u)\cdot e^{-2\pi i \ell u}}{1-e^{-2\pi i u}}\, du \;=\; 0$$

We claim that the function

$$g(x) \;=\; \frac{f(x)}{1-e^{-2\pi i x}}$$

is piecewise-C^o, and left-continuous and right-continuous at discontinuities. The only issue is at integers, and by the periodicity it suffices to prove continuity at 0. To prove continuity at 0, we can forget about periodicity for a moment, and write

$$\frac{f(x)}{1-e^{-2\pi i x}} \;=\; \frac{f(x)}{x}\cdot\frac{x}{1-e^{-2\pi i x}}$$

The two-sided limit

$$\lim_{x\to 0} \frac{x}{1-e^{-2\pi i x}} \;=\; \frac{d}{dx}\Big|_{x=0} \frac{x}{1-e^{-2\pi i x}} \;=$$

exists, by differentiability. Similarly, we have left and right limits

$$\lim_{x\to 0^-} \frac{f(x)}{x} \;=\; \text{left derivative at } 0$$

and

$$\lim_{x\to 0^+} \frac{f(x)}{x} \;=\; \text{left derivative at } 0$$

by the one-sided differentiability of f. Combining these two one-sided limits, both limits

$$\lim_{x\to 0^-} \frac{f(x)}{1-e^{-2\pi i x}} \qquad\qquad \lim_{x\to 0^+} \frac{f(x)}{1-e^{-2\pi i x}}$$

exist, proving the one-sided continuity of g at 0.

We want to prove an easy instance of a *Riemann-Lebesgue lemma*, namely, that that the Fourier coefficients of a periodic, piecewise-C^o function g, with left and right limits at discontinuities, go to 0.

The essential property of g is that on $[0,1]$ it is approximable by *step functions*[12] in the sense[13] that, given $\varepsilon > 0$ there is a *step function* $s(x)$ such that

$$\int_0^1 |s(x) - g(x)|\, dx \;<\; \varepsilon$$

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[12] As usual, a *step function* φ is a function that assumes only finitely-many values, and is of the form

$$\varphi(x) \;=\; \begin{cases} y_1 & (\text{for } x_0 \le x < x_1) \\ y_2 & (\text{for } x_1 \le x < x_2) \\ \ldots & \\ y_{k-1} & (\text{for } x_{k-2} \le x < x_{k-1}) \\ y_k & (\text{for } x_{k-1} \le x < x_k) \end{cases}$$

for some collection of intervals $[x_0, x_1), [x_1, x_2), \ldots, [x_{k-1})$ and corresponding values $y_1, \ldots, y_k$.

[13] In standard language, this assertion of approximability is that continuous functions on $[0,1]$ can be approximated by step functions *in L^1-norm.* The L^1 norm $|f|_{L^1}$ of a function on $[0,1]$ is simply the integral of the absolute value: $\int_0^1 |f(x)|\, dx$.

With such s,

$$|\widehat{s}(n) - \widehat{g}(n)| \;\le\; \int_0^1 |s(u) - g(u)|\,du \;<\; \varepsilon \qquad\qquad \text{(for all } \varepsilon > 0\text{)}$$

Thus, it suffices to prove that Fourier coefficients of *step functions* go to 0, and, thus, that Fourier coefficients of *characteristic functions of intervals* go to 0. The latter is an easy computation:

$$\int_a^b e^{-2\pi i\ell x}\,dx \;=\; [\frac{e^{-2\pi i\ell x}}{-2\pi i\ell}]_a^b \;=\; \frac{e^{-2\pi i\ell b} - e^{-2\pi i\ell a}}{-2\pi i\ell} \;\longrightarrow\; 0 \qquad\qquad \text{(as } \ell \to \pm\infty\text{)}$$

This proves a RIemann-Lebesgue lemma for any function L^1-approximable by step functions. Thus, the Fourier coefficients of g go to 0, proving that the Fourier series of f converges to $f(0)$ when f is C^1 at 0.

For arbitrary $x_o \in [0,1]$, replacing f by $f - f(x_o)$ reduces to the case that $f(x_o) = 0$. Note that the continuity of f at x_o is necessary for this reduction. Replacing $f(x)$ by $\varphi(x) = f(x + x_o)$ reduces to the case x_o, noting that the effect on the Fourier expansion is to multiply the Fourier coefficients by constants:

$$\widehat{\varphi}(n) \;=\; \int_0^1 f(x + x_o)\,e^{-2\pi i n x}\,dx \;=\; \int_{x_o}^{1+x_o} f(x)\,e^{-2\pi i n(x - x_o)}\,dx \;=\; e^{2\pi i n x_o}\int_{x_o}^{1+x_o} f(x)\,e^{-2\pi i n x}\,dx$$

For any $\mathbb{Z}$-periodic function h, using the periodicity, such a shifted integral can be converted back to an integral over $[0,1]$:

$$\int_{x_o}^{1+x_o} h(x)\,dx \;=\; \int_{x_o}^1 h(x)\,dx \;+\; \int_1^{1+x_o} h(x)\,dx \;=\; \int_{x_o}^1 h(x)\,dx \;+\; \int_0^{x_o} h(x+1)\,dx$$

$$= \int_{x_o}^1 h(x)\,dx \;+\; \int_0^{x_o} h(x)\,dx \;=\; \int_0^1 h(x)\,dx$$

Thus,

$$\widehat{\varphi}(n) \;=\; e^{2\pi i n x_o}\int_{x_o}^{1+x_o} f(x)\,e^{-2\pi i n x}\,dx \;=\; e^{2\pi i n x_o}\int_0^1 f(x)\,e^{-2\pi i n x}\,dx \;=\; e^{2\pi i n x_o}\,\widehat{f}(n)$$

Thus, the result at $x_o = 0$ for $\varphi(x) = f(x + x_o)$ gives the general case:

$$f(x_o) \;=\; \varphi(0) \;=\; \sum_n \widehat{\varphi}(n) \;=\; \sum_n \widehat{f}(n)\,e^{2\pi i n x_o}$$

Thus, we have proven that piecewise-C^1 functions with left and right limits at discontinuities are pointwise represented by their Fourier series at points where they're differentiable. ///

[4.0.3] **Remark:** In fact, the argument above shows that for a function f and point x_o such that

$$\frac{f(x) - f(x_o)}{e^{2\pi i x} - e^{2\pi i x_o}}$$

is in $L^1[0,1]$, the Fourier series at x_o converges to $f(x_o)$. This holds, for example, when f satisfies a *Lipschitz condition*

$$|f(x) - f(x_o)| \;\le\; |x - x_o|^\alpha \qquad\qquad \text{(as } x \to x_o\text{, with some } \alpha > 0\text{)}$$

and is in $L^1[0,1]$.

5. *Appendix: Perron identities*

These contour-integral identities were used classically to extract elementary information from spectral identities and function-theoretic identities.

However, *one* spectral identity is merely transformed into *another*, by a Fourier transform. Often implicitly, choices are made to heighten an *asymmetry*, wherein one side is seemingly elementary and uncomplicated, and the other is whatever it must be.

[5.1] Heuristic The best-known identity starts from the *idea* that for $\sigma > 0$

$$\int_{\sigma-i\infty}^{\sigma-i\infty} \frac{X^s}{s}\, ds \;=\; \begin{cases} 1 & (\text{for } X > 1) \\ \\ 0 & (\text{for } 0 < X < 1) \end{cases} \qquad\qquad (\text{convergence?})$$

The *idea* of the proof of this identity is that, for $X > 1$, the contour of integration slides indefinitely to the left, eventually vanishing, picking up the residue at $s = 0$, while for $0 < X < 1$, the countour slides indefinitely to the right, eventually vanishing, picking up *no* residues.

The *idea* of the application is that this identity can extract *counting* information from a meromorphic continuation of a Dirichlet series: for example, from

$$\sum_n \frac{a_n}{n^s} \;=\; f(s) \qquad\qquad (\text{left-hand side convergent for } \mathrm{Re}\, s > 1)$$

we would have

$$\sum_{n<X} a_n \;=\; \text{sum of residues of } X^s\, f(s)/s$$

That is, the *counting* function $\sum_{n<X} a_n$ is *extracted* from the analytic object $\sum_\lambda a_n/n^s$ by the contour integration. With f a logarithmic derivative, such as $f(s) = \zeta'(s)/\zeta(s)$, the poles of f are mostly the zeros of ζ.

However, the tails of these integrals are fragile.

[5.2] Simple precise assertion The elegant simplicity of the idea about moving lines of integration must be elaborated for correctness: for fixed $\sigma > 0$, for $T > 0$, we claim that

$$\int_{\sigma-iT}^{\sigma-iT} \frac{X^s}{s}\, ds \;=\; \begin{cases} 1 + O_\sigma\big(\frac{X^\sigma}{T\cdot\log X}\big) & (\text{for } X > 1) \\ \\ O_\sigma\big(\frac{X^\sigma}{T\cdot|\log X|}\big) & (\text{for } 0 < X < 1) \end{cases}$$

The proof is a precise form of the idea of sliding vertical contours. That is, for $X > 1$, consider the contour integral around the rectangle with *right* edge $\sigma \pm iT$, namely, with vertices $\sigma - iT$, $\sigma + it$, $-B + iT$, $-B - iT$, with $B \to +\infty$. For $0 < X < 1$ consider the contour integral around the rectangle with *left* edge $\sigma \pm iT$, namely, with vertices $\sigma - iT$, $\sigma + it$, $B + iT$, $B - iT$, with $B \to +\infty$.

For both $X > 1$ and $0 < X < 1$, the $\pm(B \pm iT)$ edge of the rectangle is dominated by

$$\int_{-T}^{T} \frac{e^{-B|\log X|}}{|B \pm it|}\, dt \;\ll\; T \cdot \frac{e^{-B|\log X|}}{B} \;\to\; 0 \qquad\qquad (\text{as } B \to +\infty)$$

in both cases, the top and bottom edges of the rectangle are dominated by

$$X^\sigma \cdot \int_0^\infty \frac{e^{-u|\log X|}}{|(\sigma \pm u) + iT|}\, du \;\ll\; X^\sigma \cdot \int_0^\infty \frac{e^{-u|\log X|}}{T}\, du \;\ll\; \frac{X^\sigma}{T \cdot |\log X|}$$

This proves the claim. Replacing X by e^X in the estimate gives the equivalent

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma-iT}\frac{e^{sX}}{s}\,ds \;=\; \begin{cases} 1+O_\sigma(\frac{e^{\sigma X}}{T\cdot X}) & (\text{for } X>0) \\ O_\sigma(\frac{e^{\sigma X}}{T\cdot|X|}) & (\text{for } X<0) \end{cases}$$

[5.3] Hazards When the quantity X above is summed, especially if the summation is over a set whose precise specifications are difficult, the denominators of the big-O error terms may blow up. In situations such as

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\Big(\sum_j a_j\,e^{-sX_j}\Big)\frac{e^{sX}}{s}\,ds \;=\; \sum_{j\,:\,X_j<X} a_j \;+\; \sum_j a_j\cdot O_\sigma\Big(\frac{e^{\sigma(X-X_j)}}{T\cdot|X-X_j|}\Big)$$

the distribution of the values X_j has an obvious effect on the convergence of the error term.

[5.4] The other side of the equation A desired and plausible conclusion such as

$$\lim_T \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma-iT} f(s)\,\frac{e^{sX}}{s}\,ds \;=\; \big(\text{sum of } \mathrm{Res}_{s=\rho}\, f(s)\cdot\tfrac{e^{\rho X}}{\rho}\big)$$

summed over poles ρ of f in the left half-plane $\mathrm{Re}\, s<\sigma$, requires that the contour integrals over the other three sides of the rectangle with side $\sigma\pm iT$ go to 0, and that the tails of the vertical integral go to 0. The integral over the large rectangle will be evaluated with X large positive, so the decay condition applies to f to the *left*. The left side of the rectangle will go to 0 for large enough positive X when $f(s)$ has at worst exponential growth to the left, that is, when $f(s)\ll e^{-C\cdot|\mathrm{Re}\, s|}$ for *some* large-enough C and $\mathrm{Re}\, s\to-\infty$. The top and bottom are more fragile, since e^{sX}/s does not have strong decay vertically.

Not unexpectedly, the *poles* of f near $\sigma+iT$ may *bunch up* as T grows, so that a countour integral must be **threaded** between them, and the corresponding integral will be somewhat larger simply because of proximity to these poles. This contribution to vertical growth of f is significant in examples, and motivates alternatives, as below.

[5.5] Variant identities When e^{sX}/s is altered to help convergence of the integral against the *counting* aspect is inevitably altered. The proofs of variants follow the same straightforward line as above for the simplest case. Rather than replacing e^{sX}/s with e^{sX}/s^2, a better effect is achieved with $e^{sX}/s(s+1)$. In fact, for $\theta>0$ and $1\le\ell\in\mathbb{Z}$

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma-iT}\frac{e^{sX}}{s(s+\theta)(s+2\theta)\ldots(s+\ell\theta)}\,ds \;=\; \begin{cases} \frac{1}{\ell!\theta^\ell}(1-e^{-\theta X})^\ell+O_\sigma(\frac{e^{\sigma X}}{T^2\cdot X}) & (\text{for } X>0) \\ O_\sigma(\frac{e^{\sigma X}}{T^2\cdot|X|}) & (\text{for } X<0) \end{cases}$$

Indeed, the residues at the poles 0, $-\theta$, -2θ, $\ldots$, $-\ell\theta$ sum to

$$\frac{e^{0\cdot X}}{(0+\theta)(0+2\theta)\cdots(0+(\ell-1)\theta)(0+\ell\theta)}+\frac{e^{-\theta\cdot X}}{(-\theta+0)(-2\theta+\theta)\cdots(-\theta+(\ell-1)\theta)(-\theta+\ell\theta)}$$

$$+\;\frac{e^{-2\theta\cdot X}}{(-2\theta+0)(-2\theta+\theta)\cdots(-2\theta+\ell\theta)}+\ldots+\frac{e^{-\ell\theta\cdot X}}{(-\ell\theta+0)(-\ell\theta+\theta)\cdots(-\ell\theta+(\ell-1)\theta)}$$

$$=\;\frac{1}{\ell!\,\theta^\ell}-\frac{e^{-\theta X}}{1!\,(\ell-1)!\,\theta^\ell}+\frac{e^{-2\theta X}}{2!\,(\ell-2)!\,\theta^\ell}+\ldots\pm\frac{e^{-\ell\theta X}}{\ell!\,0!\,\theta^\ell}\;=\;\frac{(1-e^{-\theta X})^\ell}{\ell!\,\theta^\ell}$$

6. *Appendix:* $\Gamma(s)\cdot\Gamma(1-s) = \pi/\sin\pi s$

This useful identity is proven by a residue integration trick that has other applications, as well.

Take $0 < \mathrm{Re}(s) < 1$ for convergence of both integrals, and compute

$$\Gamma(s)\cdot\Gamma(1-s) \;=\; \int_0^\infty\int_0^\infty u^s\,e^{-u}\cdot v^{1-s}\,e^{-v}\,\frac{du}{u}\,\frac{dv}{v} \;=\; \int_0^\infty\int_0^\infty u\,e^{-u(1+v)}\,v^{1-s}\,\frac{du}{u}\,\frac{dv}{v}$$

by replacing v by uv. Replacing u by $u/(1+v)$ (another instance of the basic *gamma identity*) and noting that $\Gamma(1) = 1$ gives

$$\int_0^\infty \frac{v^{-s}}{1+v}\,dv$$

Replace the path from 0 to ∞ by the *Hankel contour* H_ε described as follows. Far to the right on the real line, start with the branch of v^{-s} given by $(e^{2\pi i}v)^{-s} = e^{-2\pi is}v^{-s}$, integrate from $+\infty$ to $\varepsilon > 0$ along the real axis, clockwise around a circle of radius ε at 0, then back out to $+\infty$, now with the standard branch of v^{-s}. For $\mathrm{Re}(-s) > -1$ the integral around the little circle goes to 0 as $\varepsilon \to 0$. Thus,

$$\int_0^\infty \frac{v^{-s}}{1+v}\,dv \;=\; \lim_{\varepsilon\to 0}\frac{1}{1-e^{-2\pi is}}\int_{H_\varepsilon}\frac{v^{-s}}{1+v}\,dv$$

The integral of this integrand over a large circle goes to 0 as the radius goes to $+\infty$, for $\mathrm{Re}(-s) < 0$. Thus, this integral is equal to the limit as $R \to +\infty$ and $\varepsilon \to 0$ of the integral

from R to ε
from ε clockwise back to ε
from ε to R
from R counterclockwise to R

This integral is $2\pi i$ times the sum of the residues inside it, namely, that at $v = -1 = e^{\pi i}$. Thus,

$$\Gamma(s)\cdot\Gamma(1-s) \;=\; \int_0^\infty \frac{v^{-s}}{1+v}\,dv \;=\; \frac{2\pi i}{1-e^{-2\pi is}}\cdot(e^{\pi i})^{-s} \;=\; \frac{2\pi i}{e^{\pi is}-e^{-\pi is}} \;=\; \frac{\pi}{\sin\pi s}$$

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