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Asymptotics at regular singular points

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

Differential equations [1]

$$x^2u'' + bxu' + cu = 0 \qquad \text{(with constants } b, c\text{)}$$

have easy-to-understand solutions on $(0, +\infty)$: linear combinations of x^α, x^β for α, β solutions of the indicial equation

$$X(X-1) + bX + c = 0$$

Therefore, we imagine that a differential equation of the form

$$x^2u'' + xb(x)u' + c(x)u = 0$$

with b, c analytic near 0 has solutions *asymptotic*, as $x \to 0^+$, to solutions of the differential equation $x^2u'' + b(0)xu' + c(0)u = 0$ obtained by *freezing* the coefficients $b(x), c(x)$ of the original at $x = 0^+$. That is, solutions of the variable-coefficient equation should be asymptotic to x^α for solutions α to the indicial equation $X(X-1) + b(0)X + c(0) = 0$. An equation of that form, with b, c analytic near 0, is said to have a *regular singular point* at 0. Discussion below explains the behavior of solutions to such equations.

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1. *Examples*

We give two useful examples from the non-Euclidean geometry on the upper half-plane. Recall that the $SL_2(\mathbb{R})$-invariant [2] Laplacian on the upper half-plane $\mathfrak{H}$ is [3]

$$\Delta^{\mathfrak{H}} = y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)$$

[1] These differential equations are examples of *Euler-type* or *Cauchy-type* equations, which are well understood. See the appendix.

[2] As usual, $SL_2(\mathbb{R})$ acts by linear fractional transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$ on $\mathfrak{H}$. In particular, there are *real translations* $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}(z) = z + t$ for $t \in \mathbb{R}$, and *positive real dilations* $\begin{pmatrix} \sqrt{t} & 0 \\ 0 & 1/\sqrt{t} \end{pmatrix}(z) = tz$ for $t > 0$.

[3] It is not trivial to verify that this differential operator is $SL_2(\mathbb{R})$ invariant. Better, this operator is obtained by computing in coordinates the image of the Casimir operator for $SL_2(\mathbb{R})$. We accept the outcome for the present discussion.

[1.1] Translation-equivariant eigenfunctions

We ask for $\Delta^{\mathfrak{H}}$-eigenfunctions $f(z)$ of the special form

$$f(x+iy) \;=\; e^{2\pi i x} u(y)$$

That is, such an eigenfunction is *equivariant* under *translations*:

$$f(z+t) \;=\; e^{2\pi i(x+t)} u(y) \;=\; e^{2\pi i t} \cdot \Big(e^{2\pi i x} u(y)\Big) \;=\; e^{2\pi i t} \cdot f(z) \qquad \text{(with } t \in \mathbb{R} \text{ and } z \in \mathfrak{H}\text{)}$$

The eigenfunction condition is the partial differential equation

$$(\Delta^{\mathfrak{H}} - \lambda)\, e^{2\pi i x} u(y) \;=\; 0$$

Since the dependence on x is completely specified, this partial differential equation simplifies to the ordinary differential equation [4]

$$y^2 u'' - \Big(4\pi^2 y^2 + \lambda\Big) u \;=\; 0$$

The point $y = 0$ is *not* an *ordinary point* for this equation, because in the form

$$u'' - \Big(4\pi^2 + \frac{\lambda}{y^2}\Big) u \;=\; 0$$

the coefficient of u has a pole at 0. But $y = 0$ is a *regular singular point*, because that pole is of order at most 2. Thus, following the idea to freeze $y^2 u'' + y b(y) u' + c(y)$ to $y^2 u'' + y b(0) u' + c(0) u$, the indicial equation of the frozen equation is

$$X(X-1) - \lambda \;=\; 0$$

Expressing λ as $\lambda = s(s-1)$, the roots of the indicial equation are $s, 1-s$. The frozen equation has distinct solutions y^s and y^{1-s} for $s \neq \frac{1}{2}$. Thus, we could hope that solutions would have asymptotics as $y \to 0^+$ beginning

$$u(y) \;=\; A y^s (1 + O(y)) + B y^{1-s} (1 + O(y)) \qquad \text{(as } y \to 0^+\text{)}$$

Indeed, this is the case, as we see below. It seems more difficult to obtain the asymptotics at 0^+ from *integral representations* of solutions of the differential equation.

[1.1.1] Remark: As we discuss below, $y^2 u'' - (4\pi^2 y^2 + \lambda) u = 0$ has an *irregular* singular point at $+\infty$, so other methods are needed to obtain asymptotics for solutions as $y \to +\infty$.

[1.1.2] Remark: Up to choices of normalizations, the function u above, depending on the spectral parameter λ or s, is called a *Whittaker function* or *Bessel function*, and they enjoy an enormous literature. One point here is to have direct access to their properties, as examples of simple general phenomena.

[1.2] Dilation-equivariant eigenfunctions

For complex β, we can consider a dilation-equivariance condition

$$f(t \cdot z) \;=\; t^\beta \cdot f(z) \qquad \text{(for } t > 0 \text{ and } z \in \mathfrak{H}\text{)}$$

of $\Delta^{\mathfrak{H}}$-eigenfunctions f. Thus,

$$f(x+iy) \;=\; f(y \cdot \Big(\frac{x}{y} + i\Big)) \;=\; y^\beta \cdot f\Big(\frac{x}{y} + i\Big)$$

[4] This equation is a type of *Bessel* equation, with solutions which are K-type and I-type Bessel functions.

With $u(x) = f(x+i)$, the eigenfunction condition is

$$(\Delta^{\mathfrak{H}} - \lambda)\big(y^\beta \cdot u(\frac{x}{y})\big) \;=\; 0$$

To expand this, first dispatch one laborious computation:

$$\frac{\partial^2}{\partial y^2}\big(y^\beta \cdot u(\frac{x}{y})\big) \;=\; \frac{\partial}{\partial y}\Big(\beta y^{\beta-1} u(\frac{x}{y}) - y^\beta \frac{x}{y^2} u'(\frac{x}{y})\Big)$$

$$= \beta(\beta-1)y^{\beta-2}u(\frac{x}{y}) - 2\beta y^{\beta-1}\frac{x}{y^2}u'(\frac{x}{y}) + y^\beta \frac{2x}{y^3}u'(\frac{x}{y}) + y^\beta \frac{x^2}{y^4}u''(\frac{x}{y})$$

$$= y^{\beta-4}x^2u'' + 2y^{\beta-3}x(1-\beta)u' + \beta(\beta-1)y^{\beta-2}u$$

Thus, keeping in mind that the arguments of u, u', u'' are x/y,

$$(\Delta^{\mathfrak{H}} - \lambda)\big(y^\beta \cdot u(\frac{x}{y})\big) \;=\; y^\beta u'' + \Big(y^{\beta-2}x^2u'' + 2y^{\beta-1}x(1-\beta)u' + \beta(\beta-1)y^\beta u\Big) - \lambda u$$

$$= y^\beta \cdot \Big((1 + (\frac{x}{y})^2)u'' + 2(\frac{x}{y})(1-\beta)u' + \big(\beta(\beta-1) - \lambda\big)u\Big)$$

Thus, dividing through by y^β and setting $y = 1$, the eigenfunction condition $(\Delta^{\mathfrak{H}} - \lambda)y^\beta u(x/y) = 0$ becomes an ordinary differential equation in x: letting $\lambda_\beta = \beta(\beta-1)$,

$$(1+x^2)u'' + 2x(1-\beta)u' + (\lambda_\beta - \lambda)u \;=\; 0$$

For this equation, $x = 0$ is an ordinary point, so solutions admit convergent power series expansions there, and their behavior is clear. Behavior as $x \to +\infty$ can be explained by verifying that $+\infty$ is a regular singular point, by converting to coordinates $z = 1/x$ at infinity.

Let $u(x) = v(1/x)$. Then

$$u'(x) \;=\; \frac{-1}{x^2}v'(1/x) \qquad \text{and} \qquad u''(x) \;=\; \frac{1}{x^4}v''(1/x) + \frac{2}{x^3}v'(1/x)$$

Putting $z = 1/x$, this is

$$u' \;=\; -z^2v' \qquad \text{and} \qquad u'' \;=\; z^4v'' + 2z^3v' \qquad (\text{with } u = u(x),\ v = v(z),\ z = 1/x)$$

The equation becomes

$$(1 + \frac{1}{z^2})\Big(z^4v'' + 2z^3v'\Big) + \frac{2}{z}(1-\beta)\Big(-z^2v'\Big) + (\lambda_\beta - \lambda)v \;=\; 0$$

which is

$$(z^2+1)z^2v'' + 2z(z^2+\beta)v' + (\lambda_\beta - \lambda)v \;=\; 0$$

or

$$z^2v'' + z\frac{2(z^2+\beta)}{z^2+1}v' + \frac{\lambda_\beta - \lambda}{z^2+1}v \;=\; 0$$

The coefficients $b(z) = 2(z^2+\beta)/(z^2+1)$ and $c(z) = (\lambda_\beta - \lambda)/(z^2+1)$ are analytic near $z = 0$, so this equation has a *regular singular point* at $z = 0$. The indicial equation is

$$0 \;=\; X(X-1) + b(0)X + c(0) \;=\; X(X-1) + 2\beta X + \lambda_\beta - \lambda$$

Writing $\lambda = \lambda_s = s(s-1)$, the roots of the indicial equation are

$$\tfrac{1}{2} - \beta \pm \sqrt{(\tfrac{1}{2}-\beta)^2 + s(s-1) - \beta(\beta-1)} \;=\; (\tfrac{1}{2}-\beta) \pm (w - \tfrac{1}{2}) \;=\; \begin{cases} -\beta + w \\ \\ -\beta + 1 - w \end{cases}$$

Thus, solution to the differential equation should be asymptotic to $z^{-\beta+s}$ and $z^{-\beta+1-s}$ as $z \to 0$, that is, to $x^{\beta-s}$ and $x^{\beta-1+s}$ as $x \to +\infty$. We will see that this is correct.

[1.3] An irregular singular point

Returning to the translation-equivariant eigenfunctions on $\mathfrak{H}$, we check that $y = +\infty$ is *not* an ordinary point nor a regular singular point. Given

$$u'' - \Big(4\pi^2 + \frac{\lambda}{y^2}\Big) u \;=\; 0$$

again let $u(x) = v(1/x)$ and put $z = 1/x$, obtaining

$$\Big(z^4 v'' + 2z^3 v'\Big) - (4\pi^2 + \lambda z^2) v \;=\; 0$$

or

$$z^2 v'' + 2zv' - \Big(\frac{4\pi^2}{z^2} + \lambda\Big) v \;=\; 0$$

Since the coefficient of v has a pole at $z = 0$, this equation falls outside the present discussion. Instead, a different *freezing* idea succeeds: letting $y \to +\infty$ freezes the original equation at $+\infty$, giving a *constant-coefficient* equation

$$u'' - 4\pi^2 u \;=\; 0$$

with easily-understood solutions $e^{\pm 2\pi y}$. Happily the solutions to the original equation *do* have asymptotics with main terms $e^{\pm 2\pi y}$. Further details and proofs will be given later, in a discussion of *irregular singular points*.

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2. *Regular singular points*

A homogeneous ordinary differential equation of the form

$$x^2 u'' + x b(x) u' + c(x) u \;=\; 0 \qquad \text{(with } b, c \text{ analytic near } 0\text{)}$$

is said to have a *regular singular point*[5] at 0. Similarly,

$$(x - x_o)^2 u'' + (x - x_o) b(x) u' + c(x) u \;=\; 0 \qquad \text{(with } b, c \text{ analytic near } x_o\text{)}$$

has a regular singular point at x_o. Obviously it suffices to treat $x_o = 0$, and is notationally convenient. The coefficients in an expansion of the form

$$u(x) \;=\; x^\alpha \cdot \sum_{n=0}^{\infty} a_n\, x^n \qquad \text{(with } a_0 \neq 0,\ \alpha \in \mathbb{C}\text{)}$$

[5] I must have learned about regular singular points first from [Ahlfors 1966]. While the latter mentions several attributions by name, it has no bibliography whatsoever. Few current complex analysis textbooks in English discuss regular singular points. [Whittaker-Watson 1926] has extensive bibliographic notes, and treats many useful examples.

are determined recursively, but we see below that this recursion succeeds only when α satisfies the *indicial equation*

$$\alpha(\alpha-1)+b(0)\alpha+c(0) \;=\; 0$$

Further, when the two roots α, α' of the indicial equation have a relation $n+\alpha-\alpha'=0$ for $0<n\in\mathbb{Z}$, the recursion for α may fail, although the recursion for α' will succeed. These conditions are easily discovered, as in the following discussion.

The convergence of the recursively defined series is important both because it produces a genuine function, and because it can be differentiated termwise, by Abel's theorem.

[2.1] The recursion

The equation is

$$x^{\alpha+2}\cdot\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n\,x^{n-2}+b(x)\,x^{\alpha+1}\sum_{n=0}^{\infty}(n+\alpha)a_n\,x^{n-1}+c(x)\,x^{\alpha}\sum_{n=0}^{\infty}a_n\,x^n \;=\; 0$$

Dividing through by x^α and grouping,

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n\,x^{n}+b(x)\sum_{n=0}^{\infty}(n+\alpha)a_n\,x^{n}+c(x)\sum_{n=0}^{\infty}a_n\,x^n \;=\; 0$$

The vanishing of the sum of coefficients of x^0, and $a_0\neq 0$, give the *indicial equation.* The coefficients a_n with $n>0$ are obtained recursively, from the expected

$$\big[(n+\alpha)(n+\alpha-1)+b(0)(n+\alpha)+c(0)\big]\cdot a_n \;=\; (\text{in terms of } a_0,a_1,\ldots,a_{n-1})$$

The coefficient of a_n simplifies by invoking the indicial equation and the fact that the sum of the two roots α,α' is $1-b(0)$:

$$(n+\alpha)(n+\alpha-1)+b(0)(n+\alpha)+c(0) \;=\; n(n+(2\alpha-1)+b(0)) \;=\; n(n+\alpha-\alpha')$$

That is,

$$n(n+\alpha-\alpha')\cdot a_n \;=\; (\text{in terms of } a_0,a_1,\ldots,a_{n-1}) \qquad\qquad (\text{for } n>0)$$

Since $n>0$, the recursion can fail only when

$$n+\alpha-\alpha' \;=\; 0 \qquad\qquad (\text{for some } 0<n\in\mathbb{Z})$$

[2.2] Convergence

To complete the proof of existence, we prove convergence. Let $A, M\geq 1$ be large enough so that

$$b(x) \;=\; \sum_{n\geq 0} b_n\,x^n \qquad\qquad (\text{with } |b_n|\leq A\cdot M^n)$$

$$c(x) \;=\; \sum_{n\geq 0} c_n\,x^n \qquad\qquad (\text{with } |c_n|\leq A\cdot M^n)$$

Inductively, suppose that $|a_\ell|\leq (CM)^\ell$, with a constant $C\geq 1$ to be determined in the following. Then

$$|n(n+\alpha-\alpha')\cdot a_n| \;\leq\; A\sum_{i=1}^{n}|n-i+\alpha|M^i\cdot(CM)^{n-i}+A\sum_{i=1}^{n}M^i\cdot(CM)^{n-i} \;\leq\; AM^nC^{n-1}\Big(\frac{n(n+1)}{2}+n|\alpha|+n\Big)$$

Dividing through by $n|n+\alpha-\alpha'|$, this is

$$|a_n| \;\le\; AM^n \cdot C^{n-1} \frac{(n+1)+2|\alpha|+2}{2|n+\alpha-\alpha'|}$$

This motivates the choice

$$C \;\ge\; \sup_{1\le n\in\mathbb{Z}} \frac{(n+1)+2|\alpha|+2}{2|n+\alpha-\alpha'|}$$

which gives $|a_n| \le A(CM)^n$, and a positive radius of convergence.

[2.2.1] Remark: In fact, in light of the Monodromy theorem, this estimate is far from best possible, but it is infeasible and pointless to try for sharper estimates.

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3. *Regular singular points at infinity*

Let $u(x) = v(1/x)$. Then

$$u'(x) \;=\; \frac{-1}{x^2} v'(1/x) \qquad \text{and} \qquad u''(x) \;=\; \frac{1}{x^4} v''(1/x) + \frac{2}{x^3} v'(1/x)$$

Putting $z = 1/x$, this is

$$u' \;=\; -z^2 v' \qquad \text{and} \qquad u'' \;=\; z^4 v'' + 2z^3 v' \qquad (\text{with } u = u(x),\ v = v(z),\ z = 1/x)$$

A differential equation $u'' + p(x)u' + q(x)u = 0$ becomes

$$\Big(z^4 v'' + 2z^3 v'\Big) + p(x)\big(-z^2 v'\big) + q(x)v \;=\; 0$$

or

$$z^2 v'' + z\Big(2 - \frac{p(1/z)}{z}\Big)v' + \frac{q(1/z)}{z^2} v \;=\; 0$$

The point $z = 0$ is a *regular singular point* when the coefficients

$$2 - \frac{p(1/z)}{z} \qquad\qquad \frac{q(1/z)}{z^2}$$

are analytic at 0. That is, $z = 0$ is a regular singular point when p, q have expansions of the forms

$$\begin{cases} p(\frac{1}{z}) &=& p_1 z + p_2 z^2 + \ldots \\ q(\frac{1}{z}) &=& q_2 z^2 + q_3 z^3 + \ldots \end{cases} \qquad \text{or, equivalently} \qquad \begin{cases} p(x) &=& \frac{p_1}{x} + \frac{p_2}{x^2} + \ldots \\ q(x) &=& \frac{q_2}{x^2} + \frac{q_3}{x^3} + \ldots \end{cases}$$

4. *Examples reprise*

We return to the two earlier examples from non-Euclidean geometry on the upper half-plane.

[4.1] Translation-equivariant eigenfunctions

We ask for eigenfunctions $f(z)$ of the special form

$$f(x+iy) \;=\; e^{2\pi i x} u(y)$$

which simplifies to the ordinary differential equation

$$y^2 u'' - \Big(4\pi^2 y^2 + \lambda\Big) u \;=\; 0$$

with regular singular point at $y = 0$. The indicial equation is

$$X(X-1) - \lambda \;=\; 0$$

With $\lambda = s(s-1)$, the roots of the indicial equation are $s, 1-s$. By now we know that, unless $s-(1-s)$ is an integer, the equation has solutions of the form

$$u_s(y) \;=\; y^s \cdot \sum_{\ell \ge 0} a_\ell \, y^\ell \qquad\qquad u_{1-s}(y) \;=\; y^{1-s} \cdot \sum_{\ell \ge 0} b_\ell \, y^\ell$$

with coefficients a_ℓ and b_ℓ determined by the natural recursions. We emphasize that these power series *have positive radius of convergence*, so certainly give asymptotics as $y \to 0^+$. Further, convergent series can be *differentiated* termwise, by Abel's theorem.

We execute a few steps of the recursion for the coefficients for y^s. The equation

$$\sum_{\ell \ge 0} (\ell+s)(\ell+s-1) a_\ell \, y^\ell - (4\pi^2 y^2 + \lambda) \sum_{\ell \ge 0} a_\ell \, y^\ell \;=\; 0$$

simplifies to

$$\ell(\ell+2s-1)\, a_\ell \;=\; 4\pi^2 a_{\ell-2} \qquad\qquad (\text{for } \ell \ge 1)$$

with $a_{-1} = 0$ by convention, and $a_0 = 1$. Thus, the odd-degree terms are all 0, and

$$u_s(y) \;=\; y^s \cdot \Big(1 + \frac{4\pi^2\, y^2}{2(1+2s)} + \frac{(4\pi^2)^2\, y^4}{2(1+2s)\cdot 4(3+2s)} + \ldots\Big)$$

Similarly, replacing s by $1-s$,

$$u_{1-s}(y) \;=\; y^{1-s} \cdot \Big(1 + \frac{4\pi^2\, y^2}{2(3-2s)} + \frac{(4\pi^2)^2\, y^4}{2(3-2s)\cdot 4(5-2s)} + \ldots\Big)$$

For $\mathrm{Re}(s) \ne \frac{1}{2}$, one of these solutions is obviously asymptotically larger than the other. For $\mathrm{Re}(s) = \frac{1}{2}$, they are the same size, so some cancellation can occur. Write $s = \frac{1}{2} + i\nu$, so $1 - s = \frac{1}{2} - i\nu$, and rewrite the expansions in those coordinates:

$$\begin{cases} u_{\frac{1}{2}+i\nu}(y) \;=\; y^{\frac{1}{2}+i\nu} \cdot \Big(1 + \dfrac{\pi^2\, y^2}{(1+i\nu)} + \dfrac{\pi^4\, y^4}{(1+i\nu)\cdot 2(2+i\nu)} + \ldots\Big) \\ u_{\frac{1}{2}-i\nu}(y) \;=\; y^{\frac{1}{2}-i\nu} \cdot \Big(1 + \dfrac{\pi^2\, y^2}{(1-i\nu)} + \dfrac{\pi^4\, y^4}{(1-i\nu)\cdot 2(2-i\nu)} + \ldots\Big) \end{cases}$$

Then, for example, visibly

$$\begin{cases} u_{\frac{1}{2}+i\nu} + u_{\frac{1}{2}-i\nu} &=& 2y^{\frac{1}{2}}\cos(\log y) + O(y^{\frac{3}{2}}) \\ u_{\frac{1}{2}+i\nu} - u_{\frac{1}{2}-i\nu} &=& 2y^{\frac{1}{2}}\sin(\log y) + O(y^{\frac{3}{2}}) \end{cases}$$

Further, behavior of the higher terms as functions of ν is clear.

[4.2] Dilation-equivariant eigenfunctions

Returning to dilation-equivariant eigenfunctions $f(x+iy) = y^\beta u(x/y)$, we have the differential equation

$$(1+x^2)u'' + 2x(1-\beta)u' + (\lambda_\beta - \lambda)u \;=\; 0 \qquad\qquad (\text{with } \lambda_\beta = \beta(\beta-1))$$

It was verified earlier that $+\infty$ is a regular singular point, by converting to coordinates $z = 1/x$ at infinity: with $u(x) = v(1/x)$ and $z = 1/x$, the equation becomes

$$z^2 v'' + z\frac{2(z^2+\beta)}{z^2+1}v' + \frac{\lambda_\beta - \lambda}{z^2+1}v \;=\; 0$$

The indicial equation is

$$0 \;=\; X(X-1) + b(0)X + c(0) \;=\; X(X-1) + 2\beta X + \lambda_\beta - \lambda$$

With $\lambda = s(s-1)$ the roots of the indicial equation are $-\beta+s, -\beta+1-s$. For $s-(1-s)$ not an integer, there are solutions asymptotic to $z^{-\beta+s}$ and $z^{-\beta+1-s}$ as $z \to 0^+$. That is, solutions to

$$(1+x^2)u'' + 2x(1-\beta)u' - (\lambda - \lambda_\beta)u \;=\; 0$$

are asymptotic to $x^{\beta-s}$ and $x^{\beta-1+s}$ as $x \to +\infty$. We execute a few steps of the recursion for $\beta = 0$. Let

$$v(z) \;=\; z^s \cdot \sum_{\ell\ge 0} a_\ell\, z^\ell$$

Using

$$\frac{1}{z^2+1} \;=\; 1 - z^2 + z^4 - z^6 + \ldots \qquad\qquad (\text{for } z \text{ near } 0)$$

the equation

$$\sum_{\ell\ge 0}(\ell+s)(\ell+s-1)a_\ell\, z^\ell - \frac{2z^2}{z^2+1}\sum_{\ell\ge 0}(\ell+s)a_\ell\, z^\ell - \frac{\lambda}{z^2+1}\sum_{\ell\ge 0} a_\ell\, z^\ell \;=\; 0$$

becomes

$$\ell(\ell+2s-1)a_\ell \;=\; (2-\lambda)\cdot\Big((\ell-2+s)a_{\ell-2} + (\ell-4+s)a_{\ell-4} + \ldots\Big)$$

Thus, with $a_{-\ell} = 0$ for $-\ell < 0$, and with $a_0 = 1$, all the odd-degree coefficients are 0, and the expansions are something like

$$v \;=\; z^s \cdot \Big(1 + \frac{(2-\lambda)s}{2(1+2s)}z^2 + \frac{(2-\lambda)}{4(3+2s)}\Big(2\frac{(2-\lambda)s}{2(1+2s)} + s\Big)z^4 + \ldots\Big) \qquad\qquad (\text{as } z \to 0^+)$$

$$u \;=\; x^{-s} \cdot \Big(1 + \frac{(2-\lambda)s}{2(1+2s)}\frac{1}{x^2} + \frac{(2-\lambda)}{4(3+2s)}\Big(2\frac{(2-\lambda)s}{2(1+2s)} + s\Big)\frac{1}{x^4} + \ldots\Big) \qquad\qquad (\text{as } x \to +\infty)$$

still with $\lambda = s(s-1)$. These series have positive radius of convergence.

5. *Appendix: ordinary points*

The following discussion is well-known, although the convergence discussion is often omitted. This is the simpler case extended by the discussion of the regular singular points.

[5.1] Ordinary points A homogeneous ordinary differential equation of the form

$$u'' + b(x)u' + c(x)u \;=\; 0 \qquad\qquad \text{(with } b, c \text{ analytic near 0)}$$

is said to have an *ordinary point* at 0. The coefficients in a proposed expansion of the form

$$u(x) \;=\; \sum_{n=0}^{\infty} a_n\, x^n \qquad\qquad \text{(with } a_0 \neq 0\text{)}$$

are determined recursively from a_0 and a_1, as follows. The equation is

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{n-2} + b(x)\sum_{n=0}^{\infty} na_n\, x^{n-1} + c(x)\sum_{n=0}^{\infty} a_n\, x^n \;=\; 0$$

or

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{n-2} + b(x)\sum_{n=0}^{\infty}(n-1)a_{n-1}\, x^{n-2} + c(x)\sum_{n=0}^{\infty} a_{n-2}\, x^{n-2} \;=\; 0$$

The coefficients a_n with $n \geq 2$ are obtained recursively, from the expected

$$n(n-1)\cdot a_n \;=\; \text{(in terms of } a_0, a_1, \ldots, a_{n-1}\text{)}$$

To complete the proof of existence, we prove *convergence*. Take $A, M \geq 1$ large enough so that

$$\begin{cases} b(x) & = & \sum_{n\geq 0} b_n\, x^n & \text{(with } |b_n| \leq A\cdot M^n\text{)} \\ c(x) & = & \sum_{n\geq 0} c_n\, x^n & \text{(with } |c_n| \leq A\cdot M^n\text{)} \end{cases}$$

Inductively, suppose that $|a_\ell| \leq (CM)^\ell$, with a constant $C \geq 1$ to be determined in the following. Then

$$n(n-1)\cdot|a_n| \;\leq\; A\sum_{i=1}^{n}(n-i)M^{i-1}\cdot(CM)^{n-i} + A\sum_{i=2}^{n} M^{i-2}\cdot(CM)^{n-i} \;\leq\; AM^{n-1}\cdot C^{n-1}\Big(\frac{n(n+1)}{2} + n - 1\Big)$$

Dividing through by $n(n-1)$, this is

$$|a_n| \;\leq\; AM^{n-1}C^{n-1}\frac{n^2+3n-2}{n(n-1)}$$

This motivates taking

$$C \;\geq\; A \sup_{2\leq n\in\mathbb{Z}} \frac{n^2+3n-2}{n(n-1)}$$

which gives $|a_n| \leq (CM)^n$. In particular, for arbitrary a_0 and a_1 the resulting power series has a positive radius of convergence. In particular, these series can be differentiated termwise, by Abel's theorem.

[5.2] Ordinary points at infinity

Let $u(x) = v(1/x)$ and $z = 1/x$. Then

$$u'(x) \;=\; \frac{-1}{x^2} v'(1/x) \qquad \text{and} \qquad u''(x) \;=\; \frac{1}{x^4} v''(1/x) + \frac{2}{x^3} v'(1/x)$$

or

$$u' \;=\; -z^2 v' \qquad \text{and} \qquad u'' \;=\; z^4 v'' + 2z^3 v' \qquad (\text{with } u = u(x),\ v = v(z),\ z = 1/x)$$

A differential equation $u'' + b(x)u' + c(x)u = 0$ becomes

$$\Big(z^4 v'' + 2z^3 v'\Big) + p(x)\big(-z^2 v'\big) + q(x)v \;=\; 0$$

or

$$v'' + \frac{2z - p\big(\frac{1}{z}\big)}{z^2} v' + \frac{q\big(\frac{1}{z}\big)}{z^4} v \;=\; 0$$

The point $z = 0$ is an *ordinary point* when the coefficient of v' is analytic and vanishes to first order at 0, and the coefficient of v is analytic. That is, $z = 0$ is an ordinary point when p, q have expansions at infinity of the form

$$\begin{cases} p\big(\frac{1}{z}\big) &=& 2z + p_2 z^2 + p_3 z^3 \ldots \\ q\big(\frac{1}{z}\big) &=& q_4 z^4 + q_5 z^5 + \ldots \end{cases}$$

[5.3] Not-quite-ordinary points

Consider a differential equation with coefficients having poles of at most first order at 0:

$$u'' + \frac{b(x)}{x} u' + \frac{c(x)}{x} u \;=\; 0$$

with b, c analytic at 0. The coefficients in a proposed expansion of the form

$$u(x) \;=\; \sum_{n=0}^{\infty} a_n\, x^n \qquad (\text{with } a_0 \neq 0)$$

are determined recursively as follows. The equation is

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{n-2} + b(x) \sum_{n=0}^{\infty} n a_n\, x^{n-2} + c(x) \sum_{n=0}^{\infty} a_n\, x^{n-1} \;=\; 0$$

or

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{n-2} + b(x) \sum_{n=0}^{\infty} n a_n\, x^{n-2} + c(x) \sum_{n=0}^{\infty} a_{n-1}\, x^{n-2} \;=\; 0$$

We expect to determine the coefficients a_n with $n \geq 2$ recursively, from

$$\big(n(n-1) + b(0)n\big) \cdot a_n \;=\; (\text{in terms of } a_0, a_1, \ldots, a_{n-1}) \qquad (\text{for } n \geq 1)$$

For $b(0)$ not a non-positive integer, the recursion succeeds, and a_0 determines all the other coefficients a_n.

For $b(0) = 0$, so that the coefficient of v' has no pole, the relation from the coefficient of x^{-1},

$$b(0)a_1 + c(0)\, a_0 \;=\; 0$$

implies that *either* $c(0) = 0$ and the coefficient of v has no pole, returning us to the ordinary-point case, *or* $a_0 = 0$, and there is no non-zero solution of this form.

For $b(0)$ a negative integer $-\ell$, the recursion for a_ℓ gives a_ℓ the coefficient 0, and imposes a non-trivial relation on the prior coefficients a_n.

To complete the proof of existence, we prove *convergence*, assuming $b(0)$ is not a non-positive integer. Dividing through by a constant if necessary, we can take $M \geq 1$ large enough so that

$$\begin{cases} b(x) &=& \sum_{n\geq 0} b_n\, x^n \quad (\text{with } |b_n| \leq M^n) \\ c(x) &=& \sum_{n\geq 0} c_n\, x^n \quad (\text{with } |c_n| \leq M^n) \end{cases}$$

Inductively, suppose that $|a_\ell| \leq (CM)^\ell$, with a constant $C \geq 1$ to be determined in the following. Then

$$\big(n(n-1) + b(0)n\big) \cdot |a_n| \;=\; \Big|\sum_{i=1}^{n} (n-i) M^{i-1} (CM)^{n-i} + \sum_{i=1}^{n} M^{i-1}(CM)^{n-i}\Big| \;\leq\; M^{n-1}C^{n-1}\Big(\frac{n(n+1)}{2} + n\Big)$$

Dividing through by $n(n-1) + b(0)n$, this is

$$|a_n| \;\leq\; M^{n-1}C^{n-1}\frac{n^2+3n}{n(n-1)+b(0)n}$$

This motivates taking

$$C \;\geq\; \sup_{2\leq n\in\mathbb{Z}} \frac{n^2+3n}{n(n-1)+b(0)n}$$

which gives $|a_n| \leq (CM)^n$. In particular, for arbitrary a_0 the resulting power series has a positive radius of convergence. For example, the series can be differentiated termwise, by Abel's theorem.

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6. *Appendix: Euler-Cauchy equations*

The differential operator $x\frac{d}{dx}$ has readily-understood eigenfunctions on $(0, +\infty)$: from $xu' = \lambda u$ we have $u'/u = \lambda/x$, then $\log u = \lambda \log x + C$, and

$$u \;=\; \text{const} \cdot x^\lambda \qquad\qquad (\text{for } x > 0)$$

Differential operators

$$x^2\frac{d^2}{dx^2} + bx\frac{d}{dx} + c \qquad\qquad (\text{with constants, } b, c)$$

or

$$x^k\frac{d^k}{dx^k} + c_{k-1}x^{k-1}\frac{d^{k-1}}{dx^{k-1}} + \ldots + c_1 x\frac{d}{dx} + c_0$$

where the power of x matches the order of differentiation can be understood as composites of operators of the form $x\frac{d}{dx} - \alpha$. These differential operators are of *Euler type*, or *Cauchy type*, or *Euler-Cauchy type.* In the order-two case,

$$\Big(x\frac{d}{dx} - \alpha\Big)\Big(x\frac{d}{dx} - \beta\Big) \;=\; x^2\frac{d^2}{dx^2} + (1-\alpha-\beta)x\frac{d}{dx} + \alpha\beta$$

That is, given coefficients $b, c \in \mathbb{C}$, the parameters α, β are solutions of the *indicial equation*

$$X(X-1) + bX + c \;=\; 0$$

Then the differential equation

$$x^2u'' + bxu' + cu \;=\; 0$$

has solutions x^α and x^β. When the roots α, β *coincide*, a second solution for $x > 0$ is $x^\alpha \log x$. This can be verified by computation, or we can use a more general principle, as follows.

For brevity, let $D = x\frac{d}{dx}$. Suppose $(D - \alpha)u = 0$. Viewing u as a function of the spectral parameter α as well as the physical variable x, differentiating with respect to α gives

$$0 \;=\; \frac{\partial}{\partial\alpha}\Big((D-\alpha)u\Big) \;=\; -u + (D-\alpha)\frac{\partial u}{\partial \alpha}$$

That is,

$$(D-\alpha)\frac{\partial u}{\partial\alpha} \;=\; u \;\neq\; 0$$

Then

$$(D-\alpha)^2\frac{\partial u}{\partial\alpha} \;=\; (D-\alpha)u \;=\; 0$$

That is, $\partial u/\partial\alpha$ is a solution of $(D-\alpha)^2 v = 0$ and *not* a solution of $(D-\alpha)v = 0$.

In particular,

$$\frac{\partial u}{\partial\alpha}x^\alpha \;=\; \log x \cdot x^\alpha$$

The same discussion shows that

$$\Big(x\frac{\partial}{\partial x} - \alpha\Big)^{k+1}(\log x)^k \cdot x^\alpha \;=\; 0$$

while

$$\Big(x\frac{\partial}{\partial x} - \alpha\Big)^{k}(\log x)^k \cdot x^\alpha \;\neq\; 0$$

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7. *Appendix: Abel's theorem on power series*

[7.0.1] Theorem: (Abel) Let $f(z) = \sum_{n\geq 0} c_n\,(z - z_o)^n$ be a power series in one (real or complex) variable z. Suppose that the series is absolutely convergent for $|z - z_o| < r$. Then the function given by $f(z)$ is *differentiable* for $|z - z| < r$, and the derivative is given by the (absolutely convergent) series

$$\sum_{n\geq 0} nc_n\, z^{n-1}$$

[7.0.2] Corollary: By repeated differentiation,

$$f^{(k)}(z) = \sum_{n\geq 0} n(n-1)\ldots(n-k+1)\, c_n\, z^{n-k}$$

In particular, $f^{(k)}(z_o) = k(k-1)\ldots(k-k+1)\,c_k = k!\,c_k$, so the power series coefficients of $f(z)$ are *uniquely determined.* ///

Proof: Without loss of generality, $z_o = 0$. Fix $0 < \rho < r$, and $|\zeta| < \rho$, $|z| < r$. The obvious candidate for the derivative is

$$g(z) \;=\; \sum_{n\geq 0} nc_n\, z^{n-1}$$

Then

$$\frac{f(z) - f(\zeta)}{z - \zeta} - g(\zeta) \;=\; \sum_{n\geq 1} c_n \left(\frac{z^n - \zeta^n}{z - \zeta} - n\zeta^{n-1}\right)$$

For $n = 1$, the expression in the parentheses is 1. For $n > 1$, it is

$$z^{n-1} + z^{n-2}\zeta + z^{n-3}\zeta^2 + \ldots + z\zeta^{n-2} + \zeta^{n-1} - n\zeta^{n-1}$$

$$= (z^{n-1}-\zeta^{n-1})+(z^{n-2}\zeta-\zeta^{n-1})+(z^{n-3}\zeta^2-\zeta^{n-1})+\ldots+(z^2\zeta^{n-3}-\zeta^{n-1})+(z\zeta^{n-2}-\zeta^{n-1})+(\zeta^{n-1}-\zeta^{n-1})$$

$$= (z-\zeta)\left[(z^{n-2}+\ldots+\zeta^{n-2}) + \zeta(z^{n-3}+\ldots+\zeta^{n-3}) + \ldots + \zeta^{n-3}(z+\zeta) + \zeta^{n-2} + 0\right]$$

$$= (z-\zeta)\sum_{k=0}^{n-2} (k+1)\, z^{n-2-k}\, \zeta^k$$

For $|z|$ and $|\zeta|$ both smaller than ρ, the latter sum is dominated by

$$|z-\zeta|\,\rho^{n-2}\,\frac{n(n-1)}{2} < n^2\,|z-\zeta|\,\rho^{n-2}$$

Thus,

$$\left|\frac{f(z)-f(\zeta)}{z-\zeta} - g(\zeta)\right| \;\le\; |z-\zeta| \sum_{n\ge 2} |c_n|\, n^2\, \rho^{n-2}$$

Since $\rho < r$ the latter series converges absolutely, so the left-hand side goes to 0 as $z \to \zeta$. ///

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