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The simplest Eisenstein series

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We explain some essential aspects of the simplest Eisenstein series for $SL_2(\mathbb{Z})$ on the upper half-plane $\mathfrak{H}$.

There are many different proofs of meromorphic continuation and functional equation of the simplest Eisenstein series for $\Gamma = SL_2(\mathbb{Z})$. We will follow [Godement 1966a] rewriting of a Poisson summation argument that appeared in [Rankin 1939], if not earlier. This argument is the most elementary and least messy of all the meromorphic continuation proofs I know, but is less informative than arguments that engage more seriously with the spectral theory itself. Nevertheless, it is best to obtain decisive information in this simple case. Arguments based on *Fourier expansions* do unnecessary work, and risk confusion over peripheral details.

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1. *Statements of results*

Let $G = SL_2(\mathbb{R})$ act on the upper half-plane $\mathfrak{H}$ by linear fractional transformations, as usual. Let P be upper-triangular matrices in G. Use coordinates $z = x + iy$ on $\mathfrak{H}$. The Eisenstein series E_s arises in spectral theory in the form

$$E_s(z) \;=\; \sum_{\gamma \in P\cap\Gamma\backslash\Gamma} \mathrm{Im}\,(\gamma z)^s$$

[1.1] Convergence: statement

For $\mathrm{Re}\,(s) > 1$ the series defining E_s converges absolutely, uniformly for z in compacts.

[1.2] Meromorphic continuation and functional equation: statement

The usual $\zeta(s)$ with its gamma factor is $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, with functional equation $\xi(1-s) = \xi(s)$.

The *meromorphic continuation* assertion for E_s is that $s(1-s)\xi(2s)\cdot E_s$ has an analytic continuation to an *entire* function [1] of s. The *functional equation* is

$$\xi(2s)\, E_s \;=\; \xi(2-2s)\, E_{1-s}$$

[1.3] Location of poles: statement

$E_s(z)$ has no pole in $\mathrm{Re}\,(s) > \frac{1}{2}$ other than at $s = 1$. The pole of E_s at $s = 1$ is simple with residue the constant function $3/\pi$.

In $0 < \mathrm{Re}\,(s) < \frac{1}{2}$, the Eisenstein series has poles at $\rho/2$ for all non-trivial zeros ρ of $\zeta(s)$.

[1.4] Constant term: statement

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[1] The Eisenstein series is a function-valued function of s. Nevertheless, we mostly consider the scalar-valued functions $s \to E_s(z)$ with fixed $z \in \mathfrak{H}$.

By definition, the *constant term* is a sort of 0^{th} Fourier coefficient:

$$c_P E_s(z) \;=\; \int_0^1 E_s(z+t)\,dt$$

The form of the constant term of E_s dictates the *functional equation* and other features of E_s. The constant term is

$$c_P E_s(x+iy) \;=\; y^s \;+\; \frac{\xi(2s-1)}{\xi(2s)}\, y^{1-s}$$

[1.5] Vertical growth in s: statement

Both in the convergent region and when analytically continued, the Eisenstein series is of *moderate growth*

$$|E_{\sigma+it}(z)| \;=\; O(|t|^N) \qquad (\text{for } \sigma \ge \tfrac{1}{2}, \text{ as } |t| \to +\infty,\ z \in \mathfrak{H} \text{ in a fixed compact})$$

This is necessary in moving contours in the integrals expressing pseudo-Eisenstein series as integrals of Eisenstein series.

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2. *Proofs*

[2.1] Convergence for $\mathrm{Re}\,(s) > 1$

Since $\mathbb{Z}$ is a principal ideal domain, there is a bijection

$$(P\cap\Gamma)\backslash\Gamma \;\longleftrightarrow\; \mathbb{Z}^\times \backslash \big\{(c\ d) : c, d \text{ coprime integers}\big\} \qquad \text{by} \qquad \begin{pmatrix} * & * \\ c & d \end{pmatrix} \longleftrightarrow (c\ d)$$

Recall

$$\mathrm{Im}\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) \;=\; \frac{\mathrm{Im}\,(z)}{|cz+d|^2} \qquad \left(\text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})\right)$$

Since $\mathbb{Z}^\times = \{\pm 1\}$ has cardinality 2, the Eisenstein series is

$$E_s(z) \;=\; \tfrac{1}{2} \sum_{c,d \text{ coprime}} \frac{y^s}{|cz+d|^{2s}} \;=\; \tfrac{1}{2} y^s \sum_{c,d \text{ coprime}} \frac{1}{\big[(cx+d)^2 + (cy)^2\big]^s}$$

Since E_s is Γ-invariant, it suffices to consider z in a fixed compact C inside the usual fundamental domain

$$\{z = x+iy \in \mathfrak{H} \;:\; |z| \ge 1,\ -\tfrac{1}{2} \le x \le \tfrac{1}{2}\}$$

For such z,

$$(cx+d)^2 + (cy)^2 \;=\; (x^2+y^2)c^2 + 2x\cdot cd + d^2 \;\ge\; c^2 - |cd| + d^2 \;\ge\; \tfrac{1}{2}(c^2+d^2)$$

Also, the sum over coprime (c,d) is certainly dominated by the sum over *all* (c,d), not both 0. Thus, in fact, the Eisenstein series is uniformly dominated by

$$\sum_{c,d \text{ not both } 0} \frac{1}{(c^2+d^2)^{\mathrm{Re}\,(s)}}$$

An adaptation of an integral test proves that this converges for $\mathrm{Re}\,(s) > 1$.

[2.2] Analytic continuation and functional equation

For $(c\ d) = v \in \mathbb{R}^2$, consider the Gaussian

$$\varphi(v) \;=\; e^{-\pi|v|^2} \;=\; e^{-\pi(c^2+d^2)}$$

where $v \to |v|$ is the usual length function on $\mathbb{R}^2$. For $g \in GL_2(\mathbb{R})$, define

$$\Theta(g) \;=\; \sum_{v\in\mathbb{Z}^2} \varphi(v\cdot g) \;=\; \sum_{(c,d)\in\mathbb{Z}^2} e^{-\pi|(c,d)g|^2}$$

where $v \in \mathbb{R}^2$ is a row vector. Consider the integral (a Mellin transform)

$$\int_0^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t}$$

where the t in the argument of Θ simply acts by scalar multiplication on $g \in GL_2(\mathbb{R})$. On one hand, integrating term-by-term gives

$$\int_0^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} \;=\; \sum_{v\neq(0,0)} \int_0^\infty t^{2s}\,e^{-\pi|tvg|^2}\,\frac{dt}{t}$$

Since

$$\pi|tvg|^2 \;=\; (t\cdot\sqrt{\pi}|vg|)^2$$

we can change variables by replacing t by $t/(\sqrt{\pi}|vg|)$ to obtain

$$\sum_{v\neq(0,0)} (\sqrt{\pi}|vg|)^{-2s} \int_0^\infty t^{2s}\,e^{-t^2}\,\frac{dt}{t} \;=\; \frac{1}{2}\pi^{-s} \sum_{v\neq(0,0)} |vg|^{-2s} \int_0^\infty t^s\,e^t\,\frac{dt}{t}$$

$$=\; \frac{1}{2}\pi^{-s}\,\Gamma(s) \sum_{v\neq(0,0)} |vg|^{-2s}$$

Now we want $g \in SL(2,\mathbb{R})$ of a simple sort chosen to map $i \to x+iy$. One reasonable choice is

$$g \;=\; \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}$$

Using this choice of G and writing out $v = (c,d)$ gives

$$vg \;=\; (c,d)g \;=\; (c\quad d) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \;=\; (c\sqrt{y},\ (cx+d)/\sqrt{y})$$

and thus

$$\sum_v |vg|^{-2s} \;=\; \sum_v |(c\sqrt{y},(cx+d)/\sqrt{y})|^{-2s} \;=\; \sum_v (c^2y+(cx+d)^2/y)^{-s}$$

$$=\; \sum_v \frac{y^s}{(c^2y^2+(cx+d)^2)^s} \;=\; \sum_v \frac{y^s}{|ciy+cx+d|^{2s}} \;=\; \sum_v \frac{y^s}{|cz+d|^{2s}}$$

Letting $1 \le \delta = \gcd(c,d)$, this is

$$\sum_v \frac{y^s}{|cz+d|^{2s}} \;=\; \sum_\delta \frac{1}{\delta^{2s}} \sum_{\text{coprime } c,d} \frac{y^s}{|cz+d|^{2s}} \;=\; 2\,\zeta(2s)\cdot E_s(z)$$

The expression

$$2\,\zeta(2s)\,E_s(z) \;=\; \sum_{(c,d)\neq(0,0)} \frac{y^s}{|cz+d|^{2s}} \qquad\qquad \text{(summing } (c,d) \text{ over } \textit{all} \text{ non-zero vectors in } \mathbb{Z}^2\text{)}$$

is convenient, being a sum over a lattice with 0 removed.

Thus, we see that the integral representation yields the Eisenstein series with a leading power of π, a gamma function, and a factor of $\zeta(2s)$:

$$\int_0^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} \;=\; 2\,\pi^{-s}\,\Gamma(s)\,\zeta(2s)\,E_s(g)$$

On the other hand, to prove the meromorphic continuation, use the integral representation as in Riemann's corresponding argument for $\zeta(s)$, first breaking the integral into two parts, one from 0 to 1, and the other from 1 to $+\infty$. Keep $g \in SL(2,\mathbb{R})$ in a compact subset of $SL(2,\mathbb{R})$. Then

$$\int_1^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} \;=\; \text{entire in } s$$

since elementary estimates show that the integral is uniformly and absolutely convergent. Apply Poisson summation to the kernel: first note that the Gaussian $\varphi(v) = e^{-\pi|v|^2}$ is its own Fourier transform, and that

$$\text{Fourier transform of } (v \to \varphi(tvg)) \;=\; (v \to t^{-2}\det(g)^{-1}\cdot\varphi(t^{-1}v\,{}^\top g^{-1}))$$

where ${}^\top g$ is g-transpose. Then Poisson summation asserts

$$\Theta(tg) \;=\; t^{-2}\,\det(g)^{-1}\cdot\Theta(t^{-1}\,{}^\top g^{-1})$$

The modification for the kernel gives

$$\Theta(tg)-1 \;=\; t^{-2}\,\det(g)^{-1}\cdot[\Theta(t^{-1}\,{}^\top g^{-1})-1] + t^{-2}\,\det(g)^{-1} - 1$$

Then tranform the integral from 0 to 1: at first only for $\mathrm{Re}\,(s) > 1$,

$$\int_0^1 t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} \;=\; \int_0^1 t^{2s}\,\big(t^{-2}\,\det(g)^{-1}\cdot[\Theta(t^{-1}\,{}^\top g^{-1})-1] + t^{-2}\,\det(g)^{-1} - 1\big)\,\frac{dt}{t}$$

Replacing t by $1/t$ turns this into

$$\int_1^\infty t^{-2s}\,\big(t^2\,\det(g)^{-1}\cdot[\Theta(t\,{}^\top g^{-1})-1] + t^2\,\det(g)^{-1} - 1\big)\,\frac{dt}{t}$$

Explicitly evaluating the last two elementary integrals of powers of t from 1 to ∞, using $\mathrm{Re}\,(s) > 1$, this is

$$\det(g)^{-1}\int_1^\infty t^{2-2s}\,(\Theta(t\,{}^\top g^{-1})-1)\,\frac{dt}{t} + \frac{\det(g)^{-1}}{2s-2} - \frac{1}{2s}$$

That g has determinant 1 to simplifies this to

$$\int_1^\infty t^{2-2s}\,(\Theta(t\,{}^\top g^{-1})-1)\,\frac{dt}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

Further, for g in $SL(2)$,

$${}^\top g^{-1} \;=\; wgw^{-1}$$

where w is the *long Weyl element*

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since $\mathbb{Z}^2 - (0,0)$ is stable under w, and since the length function $v \to |v|^2$ is invariant under w,

$$\Theta(g) = \Theta(wg) = \Theta(gw^{-1})$$

so

$$\Theta({}^\top g^{-1}) = \Theta(g)$$

Thus, the original integral from 0 to 1 becomes

$$\int_1^\infty t^{2-2s}\,(\Theta(tg) - 1)\,\frac{dt}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

and the *whole* equality, with g of the special form above, is

$$\frac{1}{2}\pi^{-s}\Gamma(s)\,\zeta(2s)\,E_s(z) = \int_1^\infty t^{2s}\,(\Theta(tg) - 1)\,\frac{dt}{t} + \int_1^\infty t^{2-2s}\,(\Theta(tg) - 1)\,\frac{dt}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

or (multiplying through by 2)

$$\pi^{-s}\Gamma(s)\,\zeta(2s)\,E_s(z) = 2\int_1^\infty t^{2s}\,(\Theta(tg) - 1)\,\frac{dt}{t} + 2\int_1^\infty t^{2-2s}\,(\Theta(tg) - 1)\,\frac{dt}{t} - \frac{1}{1-s} - \frac{1}{s}$$

The integral from 1 to ∞ is nicely convergent for all $s \in \mathbb{C}$, uniformly in g in compacts. The elementary rational expressions of s have meromorphic continuations. Thus, the right-hand side gives a meromorphic continuation of the Eisenstein series, and is visibly invariant under $s \to 1 - s$.

It is also visible that the only poles are at $s = 1, 0$, that the residue at $s = 1$ is the constant function 1, and at $s = 0$ the residue is the constant function 0. At $s = 1$ the factor $\pi^{-s}\Gamma(s)$ is holomorphic and has value $1/\pi$, so

$$\mathrm{Res}_{s=1}\,\zeta(2s)\,E_s = \pi$$

At $s = 0$ the factor $\pi^{-s}\Gamma(s)$ has a simple pole with residue 1, so $\zeta(2s)\,E_s$ itself is holomorphic at $s = 0$, and is the constant function 1.

Now we recover the assertions for E_s itself. The convergence of the infinite product

$$\zeta(2s) = \sum_n \frac{1}{n^{2s}} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-2s}}$$

for $\mathrm{Re}(s) > 1/2$ assures that $\zeta(2s)$ is not zero for $\mathrm{Re}(s) > 1/2$. And $\zeta(2) = \pi^2/6$. These standard facts and the previous discussion give the full result. ///

[2.3] Constant term

In the region of convergence, $\mathrm{Re}(s) > 1$, the *constant term* of E_s is

$$c_P E_s(x+iy) = \int_{\mathbb{R}/\mathbb{Z}} E_s(x+iy+t)\,dt = \tfrac{1}{2}\sum_{\text{coprime } c,d} y^s \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{[(c(x+t)+d)^2 + (cy)^2]^s}\,dt$$

The constant term does not depend on x, and, since $E_s(x+iy)$ is $\mathbb{Z}$-periodic in x, we can change variables by replacing t by $t - x$, and effectively take $x = 0$:

$$c_P E_s(x+iy) = \tfrac{1}{2} y^s \sum_{\text{coprime } c,d} \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{[(ct+d)^2 + (cy)^2]^s}\,dt$$

The subsum $c = 0$ consists of two terms, divided by 2, and gives y^s. Let $\varphi(|c|)$ be Euler's totient function that counts integers d relatively prime to $|c|$ in the range $1 \le d \le |c|$. In the subsum $c \neq 0$, the c^{th} subsum is

$$\sum_{n\in\mathbb{Z}} \sum_{d \bmod c,\, \text{prime to } c} \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{[(ct+nc+d)^2+(cy)^2]^s}\,dt$$

$$= \frac{1}{|c|^{2s}} \sum_{n\in\mathbb{Z}} \sum_{d \bmod c,\, \text{prime to } c} \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{[(t+n+\frac{d}{c})^2+y^2]^s}\,dt$$

$$= \frac{\varphi(|c|)}{|c|^{2s}} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{[(t+n)^2+y^2]^s}\,dt \;=\; \frac{\varphi(|c|)}{|c|^{2s}} \int_{\mathbb{R}} \frac{1}{[t^2+y^2]^s}\,dt \;=\;=\; \frac{\varphi(|c|)}{|c|^{2s}} y^{1-2s} \int_{\mathbb{R}} \frac{1}{[t^2+1]^s}\,dt$$

by unwinding the integral and replacing t by ty. The sum over c is

$$\tfrac{1}{2}\sum_{c\neq 0} \frac{\varphi(|c|)}{|c|^{2s}} \;=\; \sum_{1\le c\in\mathbb{Z}} \frac{\varphi(c)}{c^{2s}} \;=\; \prod_{p \text{ prime}} \Big(\frac{1}{1^{2s}} + \frac{p-1}{p^{2s}} + \frac{p^2-p}{p^{4s}} + \ldots\Big) \;=\; \prod_{p \text{ prime}} \Big(1 + \frac{(p-1)p^{-2s}}{1-p^{1-2s}}\Big)$$

$$= \prod_{p \text{ prime}} \Big(\frac{1-p^{1-2s}+p^{1-2s}-p^{-2s}}{1-p^{1-2s}}\Big) \;=\; \prod_{p \text{ prime}} \Big(\frac{1-p^{-2s}}{1-p^{1-2s}}\Big) \;=\; \frac{\zeta(2s-1)}{\zeta(2s)}$$

The integral is computed via the usual trick involving $\Gamma(s)$:

$$\int_{\mathbb{R}} \frac{dt}{(t^2+1)^s} \;=\; \frac{1}{\Gamma(s)} \int_{\mathbb{R}} \int_0^\infty u^s\, e^{-u(t^2+1)}\, \frac{du}{u}\, dt \;=\; \frac{2}{\Gamma(s)} \int_0^\infty \int_0^\infty u^s\, e^{-u(t^2+1)}\, \frac{du}{u}\, dt$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty t^{\frac{1}{2}}\, u^s\, e^{-u(t+1)}\, \frac{du}{u}\, \frac{dt}{t} \;=\; \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty t^{\frac{1}{2}}\, u^{s-\frac{1}{2}}\, e^{-(t+u)}\, \frac{du}{u}\, \frac{dt}{t}$$

$$= \frac{\Gamma(\frac{1}{2})\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} \;=\; \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} \;=\; \frac{\pi^{-(s-\frac{1}{2})}\,\Gamma(s-\frac{1}{2})}{\pi^{-s}\,\Gamma(s)}$$

Assembling all this, with $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, the constant term is

$$c_P E_s(x+iy) \;=\; y^s \;+\; \frac{\xi(2s-1)}{\xi(2s)}\, y^{1-s}$$

[2.4] Vertical growth in s

As should be expected, estimates on vertical growth are applications of Phragmén-Lindelöf to the entire function

$$\widetilde{E}_s(z) \;=\; s(1-s)\cdot \pi^{-s}\,\Gamma(s)\,\zeta(2s)\cdot E_s(z)$$

for z in a fixed compact subset of $\mathfrak{H}$.

We delay this discussion to a context where it can be treated more gracefully.

Bibliography

[Godement 1966a] R. Godement, *Decomposition of $L^2(\Gamma\backslash G)$ for $\Gamma = SL(2,Z)$*, in Proc. Symp. Pure Math. **9** (1966), AMS, 211-24.

[Rankin 1939] R. Rankin, *Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetic functions, I*, Proc. Cam. Phil. Soc. **35** (1939), 351-372.
