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Harmonic analysis on spheres, I

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Harmonic analysis on the *circle* is the theory of Fourier series, which studies the expressibility of functions and *generalized* functions as sums of exponential functions. The exponential functions are *simple* functions, being *eigenfunctions* of the translation-invariant differential operator $\frac{d}{dx}$, and also being *group homomorphisms* to $\mathbb{C}^\times$.

Harmonic analysis on the *line* is the theory of Fourier *transforms*, more complicated than Fourier series, due to the line's non-compactness. On $\mathbb{R}$ the exponential functions, while still eigenfunctions for $\frac{d}{dx}$ and still giving group homomorphisms, are no longer in $L^2(\mathbb{R})$. Entangled with this point is the fact that Fourier inversion expresses functions as *integrals* of exponential functions, not as *sums*.

Both the circle and the real line are *groups*, and are *abelian*. These two features lend simplicity to their harmonic analysis. By contrast, unsurprisingly, harmonic analysis related to *non-abelian* groups is more complicated.

One immediate complication is that for non-abelian G not all subgroups are normal, so not all quotients G/H (with subgroup H) are *groups*. This already has consequences for quotients of *finite* groups. By contrast, the quotient $\mathbb{R}/\mathbb{Z}$ of the real line by the integers presents the circle as a *group*, not merely as a *quotient space* of a group.

Beyond *finite* groups and their quotients, a more typical situation is illuminated by spheres $S^{n-1} \subset \mathbb{R}^n$, which are quotients [1]

$$S^{n-1} \approx SO(n-1) \backslash SO(n)$$

of rotation groups $SO(n)$. Spheres themselves are rarely groups, but *are* acted-upon transitively by rotation groups. An essential simplicity *is* preserved, the *compactness*.

Fourier series of functions on spheres are sometimes called *Fourier-Laplace series*.

Later examples of harmonic analysis related to *non-compact* non-abelian groups are significantly more complicated than the compact (non-abelian) case.

Functions on spheres have surprising connections to the harmonic analysis of certain *non-compact* groups, such as $SL_2(\mathbb{R})$. [2]

[1] Rotation groups as *orthogonal groups* and *special orthogonal groups* are reviewed below.

[2] Relevant keywords for this unobvious connection are *oscillator representation* and *Segal-Shale-Weil representation*.

1. *Calculus on spheres*

To emphasize and exploit the rotational symmetry of spheres, we want *eigenfunctions* for rotation-invariant *differential operators* on spheres, and expect that these eigenfunctions will be the analogues of exponential functions on the circle or line. Thus, we must identify rotation-invariant differential operators on spheres. We will need a rotation-invariant *measure* or *integral* on spheres. Rather than writing formulas in coordinates, we describe these objects by their desired properties.

For a positive integer n let $S = S^{n-1}$ be the usual unit $(n-1)$-sphere

$$S \;=\; S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \;:\; x_1^2 + \ldots + x_n^2 = 1\}$$

Write Δ^S for the desired rotation-invariant second-order [3] differential operator on functions on S, and let $\int_S f$ denote the desired rotation-invariant (positive) integral. We call Δ^S the **Laplacian** on the sphere. All functions here are indefinitely differentiable. [4]

Two desired properties are

$$\begin{array}{lcll} \int_S (\Delta^S f) \cdot \varphi & = & \int_S f \cdot (\Delta^S \varphi) & \textbf{(self-adjointness)} \\ \int_S (\Delta^S f) \cdot \overline{f} & \leq & 0 & \textbf{(definiteness)} \end{array}$$

with equality only for f *constant*. We also assume that Δ^S has *real coefficients*, in the abstracted sense that

$$\overline{\Delta^S f} \;=\; \Delta^S \, \overline{f}$$

There is the natural complex hermitian inner product

$$\langle f, g \rangle \;=\; \int_S f \cdot \overline{g} \qquad\qquad \text{(for differentiable functions } f, g \text{ on } S)$$

A typical linear algebra conclusion, via a typical argument:

[1.0.1] Corollary: Granting existence of invariant Δ^S and invariant measure on S^{n-1}, with the self-adjointness and definiteness properties, eigenvectors f, g for Δ^S with *distinct* eigenvalues are orthogonal with respect to the inner product $\langle , \rangle$. Any eigenvalues are *non-positive real* numbers.

Proof: Let $\Delta^S f = \lambda \cdot f$ and $\Delta^S g = \mu \cdot g$. Assume $\lambda \neq 0$ (or else interchange the roles of λ and μ). Then

$$\langle f, f \rangle \;=\; \frac{1}{\lambda} \int_S (\Delta^S f) \cdot \overline{f} \;=\; \frac{1}{\lambda} \int_S f \; \overline{\Delta^S f} \;=\; \frac{\overline{\lambda}}{\lambda} \int_S f \; \overline{f}$$

Since $\lambda \neq 0$, f is not identically 0, so the integral of $f \cdot \overline{f}$ is not 0, and $\lambda = \overline{\lambda}$, so λ is *real*. The negative definiteness of Δ^S and positive-ness of the invariant measure on S give

$$\lambda \cdot \langle f, f \rangle \;=\; \int_S (\Delta^S f) \cdot \overline{f} \;<\; 0$$

[3] Unlike the circle and line, there are no rotation-invariant *first-order* differential operators on higher-dimensional spheres. By contrast, suitably symmetric *second-order* operators are ubiquitous.

[4] The notion of *differentiability* for functions on a sphere can be given in several ways, all equivalent. At one extreme, the most pedestrian is to declare a function f on S^{n-1} differentiable if the function $F(x) = f(x/|x|)$ on $\mathbb{R}^n - 0$ is differentiable on $\mathbb{R}^n - 0$. At the other extreme one gives S^{n-1} its usual structure of *smooth manifold*, which incorporates a notion of differentiable function. Happily, the choice of definition doesn't matter much, since we won't be attempting to directly *compute* derivatives, but only use *properties* of differentiation.

so $\lambda < 0$. Next,

$$\langle f, g\rangle \;=\; \frac{1}{\lambda}\int_S (\Delta^S f)\cdot \overline{g} \;=\; \frac{1}{\lambda}\int_S f\cdot \overline{\Delta^S g} \;=\; \frac{\overline{\mu}}{\lambda}\int_S f\cdot \overline{g}$$

The eigenvalues λ, μ are real, so for $\mu/\lambda \neq 1$ necessarily the integral is 0. ///

The standard **special orthogonal group** [5] is

$$SO(n) \;=\; \{g \in GL_n(\mathbb{R}) \,:\, g^\top g = 1_n \quad\text{and}\quad \det g = 1\}$$

and acts on S by *right* [6] matrix multiplication,

$$k \times x \longrightarrow xk \qquad\qquad (\text{for } x \in S^{n-1} \text{ and } k \in O(n))$$

considering elements of $\mathbb{R}^n$ as *row* vectors. Refer to the action of elements of $SO(n)$ on $S = S^{n-1}$ as **rotations**. [7] It is useful that for $g \in SO(n)$, inverting both sides of $g^\top g = 1$ gives $g^{-1}(g^\top)^{-1} = 1$, and then $1 = gg^\top$.

Since the usual inner product $\langle , \rangle$ on $\mathbb{R}^n$ is

$$\langle x, y\rangle \;=\; xy^\top \qquad\qquad (\text{for } \textit{row} \text{ vectors } x, y)$$

it follows that $SO(n)$ preserves lengths and angles:

$$\langle xk, yk\rangle \;=\; (xk)(yk)^\top \;=\; x\,(kk^\top)\,y^\top \;=\; x\,y^\top \;=\; \langle x, y\rangle$$

For any k such that $k^\top k = 1$

$$(\det k)^2 \;=\; \det k^\top \cdot \det k \;=\; \det(k^\top k) \;=\; \det 1_n \;=\; 1$$

Thus, $\det k = \pm 1$.

[1.0.2] Claim: The action of $SO(n)$ on S^{n-1} is *transitive.* [8]

Proof: We show that, given $x \in S$ there is $k \in SO(n)$ such that $e_1 k = x$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. That is, we construct $k \in SO(n)$ such that the top row of k is x. Indeed, complete x to an $\mathbb{R}$-basis $x, x_2, x_3, \ldots, x_n$ for $\mathbb{R}^n$. Then apply the *Gram-Schmidt* process [9] to find an orthonormal (with

[5] The *full* orthogonal group is defined by $g^\top g = 1_n$ *without* the determinant condition. The determinant condition preserves *orientation*, however orientation is defined. The modifier *special* refers to the determinant condition.

[6] The choice of *right* matrix multiplication on *row* vectors is not essential, but is made with some forethought. Specifically, to discuss *functions on* the sphere, it right action on vectors has minor advantages that will become clear. Additionally, the contemporary conventions about groups acting on *functions on* sets strongly chooses the action on the *set* to be on the right, while the action on functions is on the *left*. As late as the mid-1960's the convention was ambiguous, but no longer.

[7] *Proof* that $SO(n)$ is exactly all the rotations would require a precise definition of *rotation*, which we avoid.

[8] Recall that *transitivity* means that for all $x, y \in S$ there is $g \in O(n)$ such that $gx = y$. If $SO(n)$ is all rotations of the sphere, then physical intuition makes the transitivity plausible. But this requires *two* assumptions: that all rotations are given by $SO(n)$, and that intuition about rotations is accurate in higher dimensions. We can do better.

[9] Recall that, given a basis $v_1, \ldots, v_n$ for a (real or complex) vector space with an inner product (real-symmetric or complex hermitian), the Gram-Schmidt process produces an *orthogonal* or *orthonormal* basis, as follows. Replace v_1 by $v_1/|v_1$ to give it length 1. Then replace v_2 first by $v_2 - \langle v_2, v_1\rangle v_1$ to make it orthogonal to v_1 and then by $v_2/|v_2|$ to give it length 1. Then replace v_3 first by $v_3 - \langle v_3, v_1\rangle v_1$ to make it orthogonal to v_1, then by $v_3 - \langle v_3, v_2\rangle v_2$ to make it orthogonal to v_2, and then by $v_3/|v_3|$ to give it length 1. And so on.

respect to the standard inner product) basis $x, v_2, \ldots, v_n$ for $\mathbb{R}^n$. The condition $kk^\top = 1$, when expanded, is the assertion that the rows of k form an orthonormal basis, so taking $x, v_2, \ldots, v_n$ as the rows of k, we have k such that $e_1 k = x$. As noted just above, the determinant of this k is ± 1. To ensure that it is 1, replace v_n by $-v_n$ if necessary. This still gives $e_1 k = x$, giving the transitivity. ///

The *isotropy group* $SO(n)_{e_n}$ of the last standard basis vector $e_n = (0, \ldots, 0, 1)$ is

$$\text{(isotropy group)} \;=\; SO(n)_{e_n} \;=\; \{\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} : A \in SO(n-1)\} \;\approx\; SO(n-1)$$

Thus, by transitivity, as $SO(n)$-spaces

$$S^{n-1} \;\approx\; SO(n-1)\backslash SO(n)$$

The action of $k \in SO(n)$ on *functions* f on the sphere $S = S^{n-1}$ (or on the ambient $\mathbb{R}^n$) is [10]

$$(k \cdot f)(x) \;=\; f(xk)$$

The rotation invariance conditions are

$$\begin{aligned} \textstyle\int_S k \cdot f &= \textstyle\int_S f & \text{(for } k \in SO(n)\text{)} \\ \Delta^S(k \cdot f) &= k \cdot (\Delta^S f) & \text{(for } k \in SO(n)\text{)} \end{aligned}$$

[1.0.3] Remark: To prove *existence* of an invariant integral and Laplacian, there are several approaches, of varying technical expense, and of varying quality. For immediate purposes, we choose a pedestrian *ad hoc* approach, despite its ugliness. We will give a more sensible and prettier argument later, requiring more technique.

[1.0.4] Remark: One difficulty is lack of adequate language or technical set-up to prove *uniqueness* of either the invariant integral or the invariant second-order operator. This leaves us in the awkward position of worrying that varying constructions may give varying results.

[10] Here is one benefit of letting the group act on the *right* on the *set*. If, instead, we try to let the group act by *left* multiplication, and then try to define $(k \cdot f)(x) = f(kx)$, we have a problem, as follows. For $g, h \in SO(n)$, *associativity* fails, since

$$((gh) \cdot f)(x) \;=\; f((gh)\,x) \;=\; f(g(hx)) \;=\; (g \cdot f)(hx) \;=\; (h \cdot (g \cdot f))(x)$$

instead of $g \cdot (h \cdot f)$. If we insist on a left action on the set, then we *can* succeed at some cost, by defining $(k \cdot f)(x) = f(k^{-1}x)$. One does see this in the literature, and, indeed, there's nothing wrong with this. In fact, often a set has groups acting both on left and right, and the inverse is inevitable. However, with just a one-sided action, avoiding the inverse is a pleasant (if minor) economy.

2. *Existence of the spherical Laplacian*

For one direct proof of *existence* of the invariant Laplacian on $S = S^{n-1}$ we *cheat* by using the imbedding of the sphere $S = S^{n-1}$ in $\mathbb{R}^n$. [11] We grant that the usual Euclidean Laplacian

$$\Delta \;=\; \left(\frac{\partial}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial}{\partial x_n}\right)^2 \qquad\qquad (\text{on } \mathbb{R}^n)$$

is $SO(n)$-invariant [12] on differentiable functions on $\mathbb{R}^n$. For a function f on S, create a function F on $\mathbb{R}^n - 0$ by $F(x) = f(x/|x|)$, and define

$$\Delta^S f \;=\; (\text{restriction to } S \text{ of }) \; \Delta F$$

The map $f \to F$ that creates from f on S the degree-zero[13] positive-homogeneous[14] function F on $\mathbb{R}^n - 0$ *commutes with* the action of $SO(n)$. [15] From the definition, it is clear that

$$\Delta^S \,\overline{f} \;=\; \overline{\Delta^S f}$$

The $SO(n)$-invariance of the spherical Laplacian follows from the $SO(n)$-invariance of the usual Laplacian: for $k \in SO(n)$

$$\Delta^S(k \cdot f) \;=\; (\Delta(k \cdot F))|_S \;=\; (k \cdot (\Delta F))|_S \;=\; k \cdot (\Delta F)|_S$$

since restriction to the sphere commutes with $SO(n)$, as does $f \to F$. Thus, Δ^S is $SO(n)$-invariant.

[11] The imbedding of S^{n-1} in $\mathbb{R}^n$ does have subtler ramifications, too, so using relatively *obvious* aspects is not so unconscionable.

[12] To verify that the usual Euclidean Laplacian is $SO(n)$-invariant is at worst a matter of direct computation. In fact, the Laplacian is invariant under the *full* orthogonal group $O(n)$. Let $k \in O(n)$ with ij^{th} entry k_{ij}. For F on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$,

$$\Delta(k \cdot F)(x) \;=\; \sum_\ell (\frac{\partial}{\partial x_\ell})^2 F(\ldots, \sum_i x_i k_{ij}, \ldots) \;=\; \sum_\ell \frac{\partial}{\partial x_\ell} \sum_s k_{\ell s} F_s(\ldots, \sum_j x_i k_{ij}, \ldots)$$

where F_s is the partial derivative of F with respect to its s^{th} argument. Taking the next derivative gives $\sum_\ell \sum_{s,t} k_{\ell s} k_{\ell t} F_{st}(\ldots, \sum_i x_i k_{ij}, \ldots)$. Interchange the order of the sums and observe that $\sum_\ell k_{\ell s} k_{\ell t}$ is the $(s,t)^{th}$ entry of $k^\top \cdot k$, which is the $(s,t)^{th}$ entry of 1_n. Thus, the whole is

$$\sum_s F_{ss}(\ldots, \sum_i x_i k_{ij}, \ldots) \;=\; (\Delta F)(xk) \;=\; (k \cdot (\Delta F))(x)$$

[13] We make $F(x) = f(x/|x|)$ be homogeneous of degree 0, rather than $|x|^s f(x/|x|)$ of degree s, so the *constant* functions on the sphere become constant functions on $\mathbb{R}^n - 0$, and are *annihilated* by any differential operator.

[14] A function F on $\mathbb{R}^m$ is *positive-homogeneous* of degree $s \in \mathbb{C}$ if, for all $t > 0$ and for all $x \in \mathbb{R}^m$, $F(tx) = t^s \cdot F(x)$.

[15] The map $f \to F$ commutes with the action $(k \cdot F)(x) = F(xk)$ of $SO(n)$ on functions because $F(xk) = f(xk/|xk|) = f((x/|x|) \cdot k) = (kf)(x/|x|)$.

3. *Polynomial eigenvectors for the spherical Laplacian*

Once one has the idea of positive-homogeneous functions of various degrees to extend functions from the sphere to the ambient Euclidean space, one might also run things in the opposite direction, as an *experiment* to see how the spherical Laplacian behaves. As usual, a function φ on $\mathbb{R}^n$ (or an open subset) is **harmonic** if it is annihilated by the Euclidean Laplacian.

Indeed, this experiment works so well that, if we use some basic results about tempered distributions, we can completely classify the spherical Laplacian's eigenfunctions, which we do in the next section.

[3.0.1] Claim: For f positive-homogeneous of degree s on $\mathbb{R}^n - 0$

$$\Delta(|x|^{-s}\, f) \;=\; -s(s+n-2)|x|^{-(s+2)}\, f + |x|^{-s}\, \Delta f$$

[3.0.2] Corollary: For f positive-homogeneous of degree s and *harmonic*, the restriction $f|_S$ of f to the sphere S^{n-1} is an *eigenfunction* for Δ^S,

$$\Delta^S(f|_S) \;=\; -s(s+n-2)\cdot(f|_S)$$

with eigenvalue $-s(s+n-2)$. ///

Proof: *(of claim)* Computing directly, letting $r = |x|$, and letting f_i be the partial derivative with respect to the i^{th} argument,

$$\Delta^S(f|_S) \;=\; \Delta f(x/|x|) \;=\; \Delta\left(|x|^{-s}\cdot f\right) \;=\; \sum_i \frac{\partial^2}{\partial x_i^2}\left((r^2)^{-\frac{s}{2}}\cdot f\right)$$

$$= \sum_i \frac{\partial}{\partial x_i}\Big(-\frac{s}{2}\,(2x_i)\,(r^2)^{-(\frac{s}{2}+1)}\, f + (r^2)^{-s/2}\, f_i\Big) \;=\; \sum_i \frac{\partial}{\partial x_i}\Big(-sx_i\,(r^2)^{-(\frac{s}{2}+1)}\, f + (r^2)^{-s/2}\, f_i\Big)$$

$$= \sum_i \Big(-s\,(r^2)^{-(\frac{s}{2}+1)}\, f + sx_i\,(\frac{s}{2}+1)(2x_i)\,(r^2)^{-(\frac{s}{2}+2)}\, f - sx_i(r^2)^{-(\frac{s}{2}+1)}\, f_i$$

$$-\frac{s}{2}\,(2x_i)(r^2)^{-(\frac{s}{2}+1)}\, f_i \;+\; (r^2)^{-s/2}\, f_{ii}\Big)$$

$$= -ns\,(r^2)^{-(\frac{s}{2}+1)}\, f + sr^2\,(s+2)\,(r^2)^{-(\frac{s}{2}+2)}\, f - s(r^2)^{-(\frac{s}{2}+1)}\, sf + (r^2)^{-s/2}\,\Delta f$$

by using Euler's identity [16] that for positive-homogeneous f of degree s,

$$\sum_i x_i\, f_i(x) \;=\; s\cdot f$$

as well as the obvious $\sum_i x_i^2 = r^2$. Simplifying,

$$\Delta(|x|^{-s}\cdot f) \;=\; -ns\, r^{-(s+2)}\, f + s(s+2)\, r^{-(s+2)}\, f - 2s\, r^{-(s+2)}\, sf + r^{-s}\,\Delta f$$

$$= -s(n-(s+2)+2s)\, r^{-(s+2)}\, f + r^{-s}\,\Delta f \;=\; -s(n+s-2)\, r^{-(s+2)}\, f \;+\; r^{-s}\,\Delta f$$

as asserted. ///

[16] Euler's identity is readily proven by considering the function $g(t) = G(tx)$ for $t > 0$, differentiating with respect to t, and evaluating at $t = 1$.

[3.0.3] Remark: The most tractable homogeneous functions are homogeneous *polynomials*, so we look for *harmonic* homogeneous polynomials before doing anything subtler. [17]

Let

$$\mathcal{H}_d = \{\text{homogeneous (total) degree } d \text{ harmonic polynomials in } \mathbb{C}[x_1, \ldots, x_n]\}$$

Let $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$ be the *homogeneous* polynomials of degree d. Introduce a temporary complex-hermitian form [18]

$$\langle , \rangle \; : \; \mathbb{C}[x_1, \ldots, x_n] \times \mathbb{C}[x_1, \ldots, x_n] \longrightarrow \mathbb{C}$$

by

$$\langle P, Q\rangle = \overline{Q}(\partial)\,(P(x))\,|_{x=0}$$

where $Q(\partial)$ means to replace x_i by $\partial/\partial x_i$ in a polynomial, and $R|_{x=0}$ means to evaluate R at $x = 0$. The main point of this contruction is the immediate identity

$$\langle \Delta f, g\rangle = \langle f, r^2 g\rangle$$

where $r^2 = x_1^2 + \ldots + x_n^2$. That is, multiplication by r^2 is *adjoint* to application of Δ with respect to $\langle , \rangle$. This will be useful only after we prove that the form $\langle , \rangle$ is *non-degenerate*.

[3.0.4] Claim: The form $\langle , \rangle$ is positive-definite hermitian.

Proof: Looking at *monomials*, a basis for the $\mathbb{C}$-vectorspace $\mathbb{C}[x_1, \ldots, x_n]$, we see that $\langle P, Q\rangle = 0$ for *homogeneous* polynomials P, Q *unless* P, Q are of the same degree. When restricted to the *homogeneous* polynomials $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$ of degree d, the form $\langle , \rangle$ has an *orthogonal basis* of *distinct monomials*, since

$$(\partial x_1{}^{m_1} \ldots \partial x_n{}^{m_n})\,(x_1^{e_1} \ldots x_n^{e_n})|_{x=0} = \begin{cases} 0 & (\text{if } any\ m_i \neq e_i) \\ m_1! \ldots m_n! & (\text{if } every\ m_i = e_i) \end{cases}$$

Looking at the orthogonal basis of monomials, we see that $\langle , \rangle$ is *hermitian* and *positive definite* on $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$. ///

[3.0.5] Claim: The map

$$\Delta \; : \; \mathbb{C}[x_1, \ldots, x_n]^{(d)} \longrightarrow \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$$

is *surjective*. Harmonic polynomials f in $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$ are *orthogonal* to polynomials $r^2 h$ (with $h \in \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$) with respect to $\langle , \rangle$.

Proof: For $h \in \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$, if $\langle \Delta f, h\rangle = 0$ for all $f \in \mathbb{C}[x_1, \ldots, x_n]^{(d)}$, then

$$0 = \langle \Delta f, h\rangle = \langle f, r^2 h\rangle \qquad (\text{for all } f)$$

so $r^2 h = 0$, so $h = 0$, by the positive-definiteness of $\langle , \rangle$. This also proves the second assertion. ///

[17] And, gratifyingly, a slightly more sophisticated argument in the next section proves that there are no *other* eigenfunctions of the spherical Laplacian.

[18] This hermitian form is not as *ad hoc* as it may look. In effect, it pairs polynomials against *Fourier transforms* of polynomials, which are derivatives of Dirac delta at 0, which are *compactly-supported* distributions, so can be evaluated on polynomials, which are smooth. One need not think in these terms to use the properties here, which are much more elementary to prove. A more sophisticated interpretation *is* resurrected in the next section to give a definitive classification of eigenvectors.

Iterating the claim just proven,

$$\mathbb{C}[x_1,\ldots,x_n]^{(d)} \;=\; \mathcal{H}_d \oplus r^2\mathcal{H}^{d-2} \oplus r^4\mathcal{H}^{d-4} + \ldots$$

[3.0.6] Claim: Polynomials restricted to the n-sphere are equal to linear combinations of *harmonic* polynomials.

Proof: Use the observation

$$\mathbb{C}[x_1,\ldots,x_n]^{(d)} \;=\; \mathcal{H}_d \oplus r^2\mathcal{H}^{d-2} \oplus r^4\mathcal{H}^{d-4} + \ldots$$

to write a homogeneous polynomial as

$$f \;=\; f_0 + r^2 f_2 + r^4 f_4 + \ldots$$

with each f_i harmonic. Restricting to the sphere,

$$f|_S \;=\; (f_0 + r^2 f_2 + r^4 f_4 + \ldots)|_S \;=\; (f_0 + f_2 + f_4 + \ldots)|_S$$

since $r^2 = 1$ on the sphere. ///

[3.0.7] Remark: Again, from computations above,

$$\Delta^S f \;=\; -d(d+n-2)\cdot f \qquad\qquad (\text{for } f \in \mathcal{H}_d)$$

Since the degree d is non-negative,

$$\lambda_d \;=\; -d(d+n-2) = -(d+\frac{n-2}{2})^2 + (\frac{n-2}{2})^2$$

the eigenvalues $\lambda_d = -d(d+n-2)$ are non-positive, and 0 only for degree $d=0$. As $d \to +\infty$, the eigenvalues go to $-\infty$. Indeed, $\lambda_{d'} < \lambda_d \le 0$ for $d' > d$, so the spaces $\mathcal{H}_d$ are *distinguished* by their eigenvalues for the spherical Laplacian.

[3.0.8] Remark: For the circle S^1, the 0-eigenspace is 1-dimensional and for $d > 0$ the $(-d^2)$-eigenspace is 2-dimensional, with basis $(x \pm iy)^d$. By contrast, for $n > 1$ the dimensions of eigenspaces are *unbounded* as the degree d goes to $+\infty$. Specifically,

[3.0.9] Claim: The dimension of $\mathcal{H}_d$ is

$$\dim_{\mathbb{C}} \mathcal{H}_d \;=\; \dim \mathbb{C}[x_1,\ldots,x_n]^{(d)} - \dim \mathbb{C}[x_1,\ldots,x_n]^{(d-2)} \;=\; \binom{n+d-1}{n-1} - \binom{n+d-3}{n-1}$$

Proof: From above,

$$\Delta \;:\; \mathbb{C}[x_1,\ldots,x_n]^{(d)} \;\longrightarrow\; \mathbb{C}[x_1,\ldots,x_n]^{(d-2)}$$

is surjective, so the dimension of $\mathcal{H}_d$ is the indicated difference of dimensions.

The dimension of the space of degree d polynomials in n variables can be counted, by counting the number of monomials $x_1^{e_1}\ldots x_n^{e_n}$ with $\sum_i e_i = d$, as follows. Imagine each exponent as consisting of the corresponding number of marks, lined up, with $n-1$ additional marks to separate the marks corresponding to the n distinct variables x_i, for a total of $n+d-1$. The choice of location of the separating marks is the binomial coefficient. ///

[3.0.10] Corollary: The dimension of $\mathcal{H}_d$ grows like d^{n-2} as $d \to +\infty$. ///

[3.0.11] Corollary: The dimensions of the polynomial λ-eigenspaces for the spherical Laplacian Δ grow like $|\lambda|^{\frac{n}{2}-1}$. ///

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4. *Complete determination of eigenvectors*

The reversibility of the computation of the spherical Laplacian on restrictions of positive-homogeneous functions suggests a converse:

[4.0.1] Claim: Let f be an eigenvector for Δ^S on $S = S^{n-1}$, with eigenvalue λ. For any complex s such that $-s(s+n-2) = \lambda$, the function

$$F(x) \;=\; |x|^s \cdot f(x/|x|)$$

on $\mathbb{R}^n - 0$ is *harmonic.*

Proof: By construction, the function $x \to f(x/|x|)$ is positive-homogeneous of degree 0. The Euclidean Laplacian reduces degree of homogeneity[19] by 2, so $\Delta f(x/|x|)$ is positive-homogeneous of degree -2. The hypothesis implies that $(\Delta f(x/|x|))|_S = \lambda \cdot f(x/|x|)|_S$. By homogeneity, for x not necessarily on the sphere, the hypothesis implies that

$$|x|^2 \cdot \Delta f(x/|x|) \;=\; -s(s+n-2) \cdot f(x/|x|)$$

By the computation of the previous section, since F is positive homogeneous of degree s,

$$|x|^2 \cdot \Delta(|x|^{-s}\, F(x)) \;=\; -s(s+n-2)\, |x|^{-s} F(x) + |x|^{-s+2} \Delta F(x)$$

Putting these together, since f and F agree on the sphere,

$$-s(s+n-2)\, |x|^{-s} F(x) \;=\; -s(s+n-2)\, f(x/|x|) \;=\; |x|^2 \cdot \Delta f(x/|x|)$$

$$=\; |x|^2 \cdot \Delta |x|^{-s}\, F(x) \;=\; -s(s+n-2)\, |x|^{-s} F(x) + |x|^{-s+2} \Delta F(x)$$

Cancelling,

$$0 \;=\; |x|^{-s+2} \Delta F(x)$$

so F is harmonic. ///

Recall the earlier linear-algebra observation (from self-adjointness and definiteness) that eigenvalues λ of Δ^S are *negative* real numbers (apart from $\lambda = 0$ for constants). Thus, given $\lambda \le 0$, solving for $s \in \mathbb{C}$ such that $-s(s+n-2) = \lambda$ gives

$$-\lambda \;=\; |\lambda| \;=\; s(s+n-2) \;=\; s^2 + (n-2)s \;=\; (s + \tfrac{n-2}{2})^2 - (\tfrac{n-2}{2})^2$$

Thus,

$$(s + \tfrac{n-2}{2})^2 \;=\; (\tfrac{n-2}{2})^2 - \lambda$$

and

$$s \;=\; -\tfrac{n-2}{2} \pm \sqrt{(\tfrac{n-2}{2})^2 + |\lambda|}$$

Thus, there are choices $s > 0$ and $s < 0$. The $s > 0$ choice makes $F(x) = |x|^s\, f(x/|x|)$ at least *continuous* at $x = 0$.

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[19] For $\Phi(x)$ positive-homogeneous of degree s on $\mathbb{R}^n$, any $\partial\Phi/\partial x_j$ is positive-homogeneous of degree $t-1$, seen as follows. Differentiate $\Phi(tx) = t^s \Phi(x)$ to obtain $t \cdot \partial\Phi(tx)/\partial x_j = t^s \partial\Phi(x)/\partial x_j$.

[4.0.2] Lemma: A function $F(x) = |x|^s f(x/|x|)$ with $s > 0$ and f continuous on S gives a *tempered distribution* (by *integrating against* it).

Proof: Since f is continuous on the compact S, it is *bounded*, by some constant C. Then $|F(x)| \le C \cdot |x|^s$ is of polynomial growth on $\mathbb{R}^n$. The same inequality shows that as $|x| \to 0$ the function $F(x)$ goes to 0. Thus, F is *continuous* on $\mathbb{R}^n$, so *locally integrable.* [20] Local integrability is sufficient for (integration against) F to be a *distribution*. The moderate growth then assures that (integration against) F is a *tempered* distribution. ///

Use notation $(t \cdot \varphi)(x) = \varphi(xt)$ for $t \in \mathbb{R}^\times$ and φ continuous on $\mathbb{R}^n$. Say that a distribution u on $\mathbb{R}^n$ is *positive-homogeneous* of degree s when

$$u(t^{-1} \cdot \varphi) \;=\; t^{n+s}\, u(\varphi) \qquad\qquad (\text{for } \varphi \in \mathscr{S}(\mathbb{R}^n) \text{ and } t > 0)$$

This is compatible with homogeneity of *functions*, since for $u = u_f$ being *integration against* a positive-homogeneous function f of degree s,

$$u_f(t^{-1} \cdot \varphi) \;=\; \int_{\mathbb{R}^n} f(x)\, \varphi(t^{-1}x)\, dx \;=\; t^n \int_{\mathbb{R}^n} f(tx)\, \varphi(x)\, dx \;=\; t^{n+s} \int_{\mathbb{R}^n} f(x)\, \varphi(x)\, dx \;=\; t^{n+s}\, u_f(\varphi)$$

The extra exponent n arises from the change of measure.

One may recall that something like the following is true, but a reprise of the proof helps us be sure that we correctly recall how the exponent behaves.

[4.0.3] Claim: For a tempered distribution u, positive-homogeneous of degree s, $\widehat{u}$ is positive homogeneous of degree $-(s+n)$.

Proof: Using the definition,

$$\widehat{u}(t^{-1} \cdot \varphi) \;=\; u(\widehat{t^{-1}\varphi}) \;=\; t^n\, u(t\,\widehat{\varphi}) \;=\; t^n\, t^{-(s+n)}\, u(\widehat{\varphi}) \;=\; t^{n-(s+n)}\, \widehat{u}(\varphi)$$

With the normalization above to match integration against homogeneous functions, this is as desired. ///

A degree s positive-homogeneous F with $\Delta F = 0$ has a Fourier transform $\widehat{F}$ positive homogeneous of degree $-(s+n)$ and satisfying

$$(-2\pi i)^2 \cdot |x|^2 \cdot \widehat{F} \;=\; 0$$

since the Fourier transform converts the differential operator ∂x_j into multiplication by $-2\pi i x_j$.

This shows that $\widehat{F}$ *has support* $\{0\}$. Essentially by the definitions and existence of *Taylor-Maclaurin expansions* distributions supported at 0 are exactly derivatives of the Dirac delta at 0. Granting this, we can sort them by degree of homogeneity:

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[20] As usual, *local integrability* of a function F is the condition that the integral of $|F|$ on an arbitrary compact K is finite.

[4.0.4] Claim: The distributions supported at 0 are

$$\bigoplus_{d=0,-1,-2,\ldots} \Big(\bigoplus_{-|\alpha|=d} \mathbb{C}\cdot\delta^{(\alpha)} \Big)$$

over $d = 0, -1, -2, -3, \ldots$ of positive-homogeneous distributions of degree d

$$\text{positive-homogeneous degree } d \;=\; \bigoplus_{|\alpha|=d} \mathbb{C}\cdot\delta^{(\alpha)}$$

where α is summed over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| = \sum_i \alpha_i$.

Proof: Granting that distributions at 0 are finite linear combinations of derivatives of δ, the homogeneity is easy to gauge:

$$\delta^{(\alpha)}(t^{-1}\cdot\varphi) \;=\; (-1)^{|\alpha|}\,(t^{-1}\cdot\varphi)^{(\alpha)}(0) \;=\; (-1)^{|\alpha|}\,t^{-|\alpha|}\,\varphi^{(\alpha)}(0) \;=\; t^{-|\alpha|}\,\delta^{(\alpha)}(\varphi)$$

Thus, the positive-homogeneous degree d distributions make up the direct sum over α with $-|\alpha| = d$, as asserted. ///

[4.0.5] Corollary: The only degrees of positive-homogeneity of (tempered or not) distributions supported at 0 are $0, -1, -2, -3, \ldots$. ///

[4.0.6] Corollary: The only degrees $s > 0$ for positive-homogeneous harmonic F are $0 < s \in \mathbb{Z}$, and, further, F is a harmonic *polynomial*.

Proof: Fourier transforms of derivatives of Dirac delta are polynomials. ///

[4.0.7] Corollary: The eigenfunctions for Δ^S are exactly the restrictions to S^{n-1} of homogeneous harmonic polynomials of degree d, with eigenvalue $-d(d+n-2)$. ///

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5. *Existence of invariant integrals on spheres*

We have used the assumed *existence* of an $SO(n)$-invariant integral on S^{n-1} to be sure that eigenvalues for the spherical Laplacian Δ^S are non-positive, in determining all eigenvectors.

To *prove* existence of an invariant integral we can write a *formula* [21] as follows, using the fact that the measure on $\mathbb{R}^n$ is $SO(n)$-invariant, since the absolute value of the determinant of an element of $SO(n)$ is 1. For a continuous function f on S, define

$$\int_S f \;=\; \int_{R^n} \gamma(|x|^2)\; f(x/|x|)\; dx$$

where γ is a fixed smooth non-negative function on $[0, \infty)$ with

$$\int_{\mathbb{R}^n} \gamma(|x|^2)\, dx \;=\; 1$$

[21] If we were already happy with *Haar measure* on $SO(n)$ and invariant measures on quotient spaces such as S^{n-1}, then we would not need an explicit construction. We rarely need more than the *existence* (and essential uniqueness) of such an integral.

For convenience, we may at some moments suppose that γ has compact support and vanishes identically on a neighborhood of 0. [22] For $k \in SO(n)$ we have the $SO(n)$-invariance of the integral:

$$\int_S k \cdot f \;=\; \int_{R^n} \gamma(|x|^2)\, f(\frac{xk}{|xk|})\, dx \;=\; \int_{R^n} \gamma(|xk^{-1}|^2)\, f(\frac{x}{|x|})\, dx \;=\; \int_{R^n} \gamma(|x|^2)\, f(\frac{x}{|x|})\, dx \;=\; \int_S f$$

by changing variables to replace x by xk^{-1}, and using $|xk^{-1}| = |x|$. Less trivial is proof of the desired integration-by-parts-twice result from this clunky viewpoint:

[5.0.1] Proposition: For differentiable functions f, φ on S^n,

$$\int_S (\Delta^S f) \cdot \varphi \;=\; \int_S f \cdot \Delta^S \varphi$$

Further, Δ^S is *negative-definite* in the sense that

$$\int_S (\Delta^S f) \cdot \overline{f} \le 0$$

with equality only for f *constant.*

Proof: Let $F(x) = f(x/r)$ and $\Phi(x) = \varphi(x/r)$, where $r = |x|$. By definition,

$$\int_S (\Delta^S f) \cdot \varphi \;=\; \int_{\mathbb{R}^n} \gamma(r^2)\, r^2 \cdot (\Delta F)(x)\, \Phi(x)\, dx$$

where the r^2 is inserted so that $r^2 \Delta F$ is positive-homogeneous of degree 0 as required by the integration formula. [23] Integrating by parts, this becomes

$$-\int_{\mathbb{R}^n} \sum_i \frac{\partial F}{\partial x_i} \frac{\partial}{\partial x_i} \left(r^2 \cdot \gamma(r^2)\, \Phi(x) \right)\, dx$$

Let $\delta(r^2) = r^2 \gamma(r^2)$. Then

$$\frac{\partial}{\partial x_i} \left[r^2 \cdot \gamma(r^2)\, \Phi(x) \right] \;=\; \frac{\partial}{\partial x_i} \left[\delta(r^2)\, \Phi(x) \right] \;=\; 2x_i \delta'(r^2) \Phi(x) + \delta(r^2) \frac{\partial \Phi}{\partial x_i}$$

The whole is

$$-\int_{\mathbb{R}^n} \sum_i \frac{\partial F}{\partial x_i} \left[2x_i \delta'(r^2) \Phi(x) + \delta(r^2) \frac{\partial \Phi}{\partial x_i} \right] dx \;=\; -\int_{\mathbb{R}^n} \sum_i \frac{\partial F}{\partial x_i}\, \delta(r^2) \frac{\partial \Phi}{\partial x_i}\, dx$$

since Euler's identity asserts that

$$\sum_i x_i \frac{\partial F}{\partial x_i} \;=\; 0$$

because F is positive-homogeneous of degree 0. That last expression for the integral is symmetric in F and Φ, giving the desired integration-by-parts result. Finally, with $\Phi = \overline{F}$, the last expression for the integral is visibly non-positive, and is 0 only if $\partial F/\partial x_i = 0$ for all i, only if F is constant, only if f is constant. ///

[5.0.2] Remark: This argument is unsatisfying, since it does not extend to more general situations. We will give a more universal existence argument later.

[22] Compact support and the vanishing condition would imply that there would be no boundary terms in integrating by parts on $\mathbb{R}^n$.

[23] Since $F(x) = f(x/r)$ is positive-homogeneous of degree 0, ΔF is positive-homogeneous of degree -2, so we need to adjust it by r^2.

6. L^2 spectral decompositions on spheres

The idea of **spectral decomposition** on the sphere is that functions on the sphere should be *sums of eigenfunctions* for the spherical Laplacian. For L^2 functions the convergence should be in L^2. For *smooth* functions, the sum should converge well in a *uniform pointwise* sense. As usual, L^2 convergence does not imply pointwise convergence.

[6.0.1] Theorem: The collection of finite linear combinations of homogeneous harmonic polynomials restricted to S^{n-1} is *dense* in $L^2(S^{n-1})$.

Proof: By Weierstraß' Approximation Theorem, polynomials are dense in $C^o(\mathbb{R}^n)$, meaning in the topology given by sups on compacts. Even though it has no interior, the sphere is compact, so restrictions of polynomials to S^{n-1} approximate continuous functions on the sphere in sup norm. Urysohn's lemma shows that $C^o(S^{n-1})$ is dense in $L^2(S^{n-1})$.

We showed that every homogeneous degree d polynomial f is expressible as

$$f \;=\; f_d + r^2 \cdot f_{d-2} + r^4 \cdot f_{d-4} + \ldots \qquad \text{(where } f_j \in \mathcal{H}_j\text{)}$$

Restricting to the sphere, $r = 1$, so

$$f \;=\; f_d + f_{d-2} + f_{d-4} + \ldots \qquad \text{(restricted to the sphere)}$$

Thus, harmonic polynomials are dense in L^2. ///

Thus, every L^2 function f on S^n has at least an L^2 Fourier-Laplace expansion

$$f \;=\; \sum_{d=0}^{\infty} f_d \qquad \text{(in } L^2 \text{ sense))}$$

where f_d is the orthogonal (in L^2) projection of f to the space $\mathcal{H}_d$ of homogeneous degree d harmonic polynomials (restricted to the sphere).

7. Sup-norms of spherical harmonics on S^{n-1}

To eventually assess the *pointwise* convergence of Fourier series on the sphere, we must first be aware that, unlike the functions $e^{in\theta}$ on circle S^1, for $f \in \mathcal{H}_d$ on S^n with $n > 1$ there is *no* instantaneous comparison [24] of the two norms

$$|f|_{C^o} = \sup_{x \in S^{n-1}} |f(x)| \qquad |f|_{L^2} = \left(\int_S |f(x)|^2 \, dx \right)^{1/2}$$

Nevertheless, a little work *does* give a useful comparison: [25]

[24] Of course, since the spaces $\mathcal{H}_d$ are finite-dimensional, each one has a unique topology compatible with the vector space structure. (The latter fact is not trivial to prove, but is not too hard.) Thus, the fact that the sup-norm and L^2-norm are comparable on these finite-dimensioal spaces is clear *a priori*. But this qualitative fact is irrelevant to the issue at hand. What is *not* clear is how the comparison constants grow in the parameter d, and this greater precision is needed for Sobolev-type estimates.

[25] In fact, the proof of the following proposition uses few of the specifics of this situation, and, indeed, the same argument proves that for a compact group K acting transitively on a set X, with a finite-dimensional K-stable space V of functions on X, for $f \in V$ we have the same comparison of sup-norm and L^2-norm, namely $|f|_{C^o} \le \sqrt{(\dim V)/\mathrm{vol}\,(X)} \cdot |f|_{L^2}$. This inequality is interesting even for *finite* groups.

[7.0.1] Proposition: Let $f \in \mathcal{H}_d$. Then

$$|f|_{C^o} \;\le\; \sqrt{\frac{\dim \mathcal{H}_d}{\mathrm{vol}\,(S^{n-1})}} \;\cdot\; |f|_{L^2}$$

And the estimate is *sharp*, in the sense that there is a function in $\mathcal{H}_d$ for which equality holds.

[7.0.2] Remark: The occurrence of the square root of the total measure of S^{n-1} compensates for the fact that the square root of the total measure enters in the L^2-norm, and does *not* enter in the sup norm.

[7.0.3] Remark: Using the dimension computation from above, as $d \to +\infty$

$$\sqrt{\dim \mathcal{H}_d} \;\sim\; \sqrt{\binom{n+d-1}{n-1} - \binom{n+d-3}{n-1}} \;\sim\; d^{\frac{n}{2}-1}$$

Proof: For $x \in S$, from the Riesz-Fischer theorem, the functional $f \to f(x)$ is necessarily given by

$$f(x) \;=\; \langle f, F_x \rangle \qquad\qquad (\text{for some } F_x \in \mathcal{H}_d)$$

The Cauchy-Schwarz-Bunyakowsky inequality gives

$$|f(x)| \;=\; |\langle f, F_x \rangle| \;\le\; |f|_{L^2} \cdot |F_x|_{L^2}$$

which bounds the value $f(x)$ in terms of its L^2-norm with constant being the L^2-norm of F_x.

We show that $F_x(x)$ is independent of $x \in S$. Let $k \in SO(n)$, with action $(k \cdot f)(x) = f(xk)$. By design, the action of $k \in SO(n)$ on functions is *unitary* in the sense that

$$\langle k \cdot f, k \cdot F \rangle \;=\; \int_X f(xk)\,\overline{F(xk)}\,dx \;=\; \int_X f(x)\,\overline{F(x)}\,dx \;=\; \langle f, F \rangle$$

by replacing x by xk^{-1} in the integral. Then the functions F_x have natural relations among themselves, namely,

$$\langle f, F_{xk} \rangle \;=\; f(xk) \;=\; (k \cdot f)(x) \;=\; \langle k \cdot f, F_x \rangle \;=\; \langle f, k^{-1} \cdot F_x \rangle$$

Thus,

$$F_{xk} \;=\; k^{-1} \cdot F_x$$

In particular, the L^2-norm of the function F_x is the same for every $x \in S$. In particular,

$$F_x(x) \;=\; \langle F_x, F_x \rangle \;=\; \langle k^{-1} \cdot F_x, k^{-1} F_x \rangle \;=\; F_{xk}(xk)$$

Since K is transitive on the sphere, $F_x(x)$ is independent of x.

Further,

$$|F_x(y)| \;=\; |\langle F_x, F_y \rangle| \;\le\; |F_x|_{L^2} \cdot |F_y|_{L^2} \;=\; |F_x|^2_{L^2}$$

because all the L^2 norms are the same. This determines the sup-norm

$$|F_x|_{C^o} \;=\; F_x(x) = |F_x|^2_{L^2}$$

The L^2 norm is evaluated as follows. Express F_x in terms of an orthonormal basis $\{f_i\}$ for $\mathcal{H}_d$, as usual, by

$$F_x \;=\; \sum_i \langle F_x, f_i \rangle \cdot f_i$$

Evaluating both sides at x gives

$$F_x(x) \;=\; \sum_i \langle F_x, f_i\rangle \cdot f_i(x) \;=\; \sum_i \overline{f_i(x)} \cdot f_i(x) \;=\; \sum_i |f_i(x)|^2$$

Again, the value $F_x(x) = |F_x|^2_{L^2}$ is independent of x, since for $k \in SO(n)$ and $x \in S$

$$F_{xk}(xk) = \langle F_{xk}, F_{xk}\rangle = \langle k\cdot F_x, k\cdot F_x\rangle = \langle F_x, F_x\rangle = F_x(x)$$

Integrating $F_x(x) = \sum_i |f_i(x)|^2$ over S, using the fact that $F_x(x)$ is independent of $x \in S$,

$$\mathrm{vol}\,(S)\cdot F_x(x) \;=\; \dim_{\mathbb{C}}\, \mathcal{H}_d$$

Then

$$F_x(x) \;=\; |F_x|_{C^o} \;=\; |F_x|^2_{L^2}$$

gives

$$|F_x|_{L^2} \;=\; \sqrt{F_x(x)} \;=\; \sqrt{\frac{\dim \mathcal{H}_d}{\mathrm{vol}\,(S)}}$$

Combining this with $|f(x)| \;\le |F_x|_{L^2}\cdot|f|_{L^2}$ from above,

$$|f|_{C^o} \;\le\; |F_x|_{L^2}\cdot|f|_{L^2}\sqrt{\frac{\dim \mathcal{H}_d}{\mathrm{vol}\,(S)}}\cdot|f|_{L^2}$$

as claimed by the proposition. Further, we saw that

$$|F_x|_{C^o} \;=\; |F_x|_{L^2}\cdot|F_x|_{L^2} \;=\; \sqrt{\frac{\dim \mathcal{H}_d}{\mathrm{vol}\,(S)}}\cdot|F_x|_{L^2}$$

so the estimate is sharp. ///

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8. *Pointwise convergence of Fourier-Laplace series*

The comparison of sup-norms and L^2 norms on $\mathcal{H}_d$ yields useful criteria for pointwise convergence.

[8.0.1] Corollary: In an L^2-expansion $f = \sum_d f_d$ with $f_d \in \mathcal{H}_d$, the convergence is *uniformly pointwise*, so the partial sums converge to a continuous function, when

$$\sum_d (1+d)^{\frac{n-2}{2}}\cdot|f_d|_{L^2} \;<\; \infty$$

Proof: From the sup-norm estimate proposition, and from the formula for the dimension of spaces of harmonic polynomials,

$$|f_d|_{C^o} \;\ll_n\; (1+d)^{\frac{n-2}{2}}\cdot|f_d|_{L^2} \qquad \text{(implied constant depending only on } n\text{)}$$

The convergence assumption assures that the sequence $\sum_{d\le N} f_d$ of continuous functions on S is Cauchy in sup norm, so converges uniformly pointwise. Uniform limits of continuous functions are continuous. ///

[8.0.2] Remark: In the sum in the corollary the L^2 norms of the Fourier components f_d are *not* squared. Thus, this is not quite the L^2 condition we want. Still, it is a *natural* condition for uniform pointwise convergence. The case of S^1 is misleading because the exponentials $e^{\pm in\theta}$ are *uniformly bounded*, unlike the higher-dimensional case.

Giving away something, a more usable criterion for pointwise convergence is

[8.0.3] Corollary: The convergence of an L^2-expansion $f = \sum_d f_d$ with $f_d \in \mathcal{H}_d$ is *uniformly pointwise* when

$$\sum_d (1+d)^{n-2} \cdot |f_d|^2_{L^2} \;<\; \infty \qquad \text{(for any } s > \tfrac{n-2}{2}\text{)}$$

Proof: Starting from the previous corollary, by Cauchy-Schwarz-Bunyakowsky, for any $s \in \mathbb{R}$,

$$\sum_d (1+d)^{\frac{n-2}{2}} \cdot |f_d|_{L^2} \;=\; \sum_d (1+d)^{\frac{n-2}{2}+s} \cdot |f_d|_{L^2} \cdot \frac{1}{(1+d)^s}$$

$$= \; \Big(\sum_d (1+d)^{n-2+2s} \cdot |f_d|^2_{L^2}\Big)^{1/2} \cdot \Big(\sum_d \frac{1}{(1+d)^{2s}}\Big)^{1/2}$$

For $s > \frac{1}{2}$, the latter sum is finite. Replacing $\frac{n-2}{2} + s$ by s, the condition is $s > \frac{n-1}{2}$. Thus, when

$$\Big|\sum_d (1+d)^{\frac{n-2}{2}} \cdot |f_d|_{L^2}\Big| \;\ll_s\; \Big(\sum_d (1+d)^{2s} \cdot |f_d|^2_{L^2}\Big)^{1/2} \qquad \text{(for } s > \tfrac{1}{2}\dim S^{n-1}\text{)}$$

the Fourier-Laplace series $\sum_d f_d$ converges uniformly pointwise. ///

[8.0.4] Remark: The latter invocation of Cauchy-Schwarz-Bunyakowsky, thereby obtaining a Hilbert-space criterion for uniform pointwise convergence, obtains the simplest *Sobolev inequality* on the sphere.