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Harmonic analysis on spheres, II

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The extrinsically defined $O(n)$-invariant Laplacian $\Delta^{\rm ext}$ on the sphere S^{n-1} is intended to be *obviously* $O(n)$-invariant, because the Laplacian Δ on the ambient $\mathbb{R}^n$ is $O(n)$-invariant. One way to express this extrinsic $O(n)$-invariant Laplacian is

$$(\Delta^{\rm ext} f)(x) \;=\; \Delta^{\mathbb{R}^n} f\Big(\frac{x}{|x|}\Big) \qquad\qquad (f \text{ defined on } S^{n-1}, \text{ for } x \in S^{n-1})$$

where

$$\Delta^{\mathbb{R}^n} \;=\; \Delta \;=\; \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

The trick of replacing the argument of f by $f(x/|x|)$ extends the definition of f to $\mathbb{R}^n - 0$ by making it constant on rays from 0, thus making sense of application of Δ.

Differentiability of functions on S^{n-1} should involve more than $\Delta^{\rm ext}$ and its iterates, just as in $\mathbb{R}^n$ there are the individual partial derivatives $\partial/\partial x_i$ as well as Δ.

Despite $O(n)$'s non-abelian-ness, it would be very convenient if $\Delta^{\rm ext}$ were nice enough to *commute* with suitable $O(n)$-analogues of the *constant-coefficient* operators on $\mathbb{R}^n$. One advantage would be that these alleged operators would all stabilize the *eigenspaces* $\mathcal{H}_d$ of $\Delta^{\rm ext}$ in functions on S^{n-1} (namely, homogeneous degree-d harmonic polynomials). In that scenario, the comparison of sup-norm and L^2-norm on $\mathcal{H}_d$ would immediately apply to these derivatives, as well.

In contrast, for operators that smear out $\mathcal{H}_d$ across many other subspaces $\mathcal{H}_{d'}$, comparisons are difficult, although perhaps interesting.

With a congenial collection of first-order derivatives, we can prove *Sobolev inequalities* on the sphere, giving essentially sharp comparisons of C^k norms and weighted L^2 norms.

The Sobolev inequalities give simple criteria for the legitimacy of differentiating Fourier-Laplace expansions of C^k functions on S^{n-1} term-by-term.

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1. *A doomed-but-plausible trial calculation*

The idea of this subsection is elementary and natural. If it could succeed, it would be worth knowing. However, it fails. Worse, it is not obvious from an elementary viewpoint that it *should* fail. Behaving innocently, we only discover its failure at the end of an attempted execution.

To describe first-order operators on S^{n-1}, we could try the same extension trick to use extrinsically-defined operators, and define

$$(D_i^{\rm ext} f)(x) \;=\; \frac{\partial}{\partial x_i} f\big(\frac{x}{|x|}\big) \qquad\qquad (f \text{ defined on } S^{n-1}, \text{ for } x \in S^{n-1})$$

To assess whether or not $D_i^{\rm ext} \circ \Delta^{\rm ext} = \Delta^{\rm ext} \circ D_i^{\rm ext}$, we must be careful: even after making a positive-homogeneous function $x \to f(x/|x|)$ of degree 0, application of $\Delta^{\rm ext}$ produces a positive-homogeneous function of degree -2, and $D_i^{\rm ext}$ produces a positive-homogeneous function of degree -1. Thus, correcting the homogeneity, what is really being asked is whether or not

$$|x| \frac{\partial}{\partial x_i}\big(|x|^2 \Delta f\Big(\frac{x}{|x|}\Big)\big) \;=\; |x|^2 \Delta\big(|x| \frac{\partial}{\partial x_i} f\Big(\frac{x}{|x|}\Big)\big) \qquad\qquad (???)$$

To check this, it suffices to take f positive-homogeneous of degree 0, since both sides of the hypothetical equation take $f(x/|x|)$ as input. Thus, we are asking whether or not

$$\frac{\partial}{\partial x_i}\big(|x|^2 \Delta f(x)\big) \;=\; |x| \Delta\big(|x| \frac{\partial}{\partial x_i} f(x)\big) \qquad\qquad (??? \text{ for } f \text{ positive-homogeneous of degree } 0)$$

Letting $r = |x|$, and abbreviating partial derivatives as $f_i = \partial f / \partial x_i$, the left-hand side is

$$\frac{\partial}{\partial x_i}\big(|x|^2 \Delta\big) \;=\; 2x_i \Delta f + r^2 \frac{\partial}{\partial x_i} \Delta f \;=\; 2x_i \Delta f + r^2 \Delta f_i$$

while the right-hand side is

$$|x| \Delta\big(|x| \frac{\partial}{\partial x_i} f\big) \;=\; r \sum_j \frac{\partial}{\partial x_j}\Big(\frac{x_j}{r} f_i + r f_{ij}\Big) \;=\; r \sum_j \Big(\frac{1}{r} f_i - \frac{x_j^2}{r^3} f_i + \frac{2x_j}{r} f_{ij} + r f_{ijj}\Big)$$

$$=\; n f_i - f_i - 2 f_i + r \Delta f_i \;=\; (n-3) f_i + r^2 \Delta f_i \qquad\qquad (\text{Euler's identity, } f_i \text{ homogeneous degree } -1)$$

Thus, we ask whether or not

$$2x_i \Delta f + r^2 \Delta f_i \;=\; (n-3) f_i + r^2 \Delta f_i \qquad\qquad (??? \; f \text{ homogeneous degree } 0)$$

The second terms agree, so we ask whether or not

$$2x_i \Delta f \;=\; (n-3) f_i \qquad\qquad (??? \; f \text{ homogeneous degree } 0)$$

Surely this does not hold in general. Being sick of computing by this point, we can finesse the issue. For example, if this equality held, multiply both sides by x_i and sum over i, to allegedly obtain (by Euler's identity)

$$2r^2 \Delta f \;=\; 0 \qquad\qquad (??? \; f \text{ homogeneous degree } 0)$$

However, for $f(x) = F(x/|x|)$ with F a harmonic homogeneous polynomial of degree d, our earlier *ad hoc* computation found

$$r^2 \Delta f \;=\; \Delta^{\rm ext} F \;=\; -d(d+n-2) \cdot F$$

This is rarely 0. The extrinsically defined first-order operators $D_i^{\rm ext}$ definitely do *not* commute with $\Delta^{\rm ext}$. Too bad. But there seems to be no elementary way to see this without doing the computation.

[1.0.1] Remark: It is important to be able, and occasionally *willing*, to carry out computations which fail, to learn of the failure. Still, it is obviously better to be able to see futility *in advance*.

2. *Matrix exponentiation*

It turns out to be not-such-a-good-idea to try to put coordinates on spheres $X = S^{n-1}$ for $n-1 > 1$. It's not a good idea to try to put coordinates on orthogonal groups, either. The coordinates on the ambient Euclidean spaces are useful, but only in a limited way.

Nevertheless, no one can deny the convenience and tractability of Euclidean spaces. The appeal of Euclidean spaces and the requirement of *intrinsic-ness* are manifest in the following.

The combination of two observations will allow an *intrinsic* description of special derivatives on S^{n-1}, and, also, a non-computational proof that all these special derivatives commute with the Laplacian on the sphere.

The first observation is that the *exponential map*

$$e^A \;=\; \exp(A) \;=\; \sum_{\ell=0}^{\infty} \frac{A^\ell}{\ell!}$$

gives coordinates on a *big enough* part of $G = O(n)$ to give a uniform description of derivatives on G itself. The second observation is that the transitive action of G on $X = S^{n-1}$ gives analogous derivatives on X.

For example, in 2-by-2 matrices,

$$\exp\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{1!}\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & -\theta^3 \\ \theta^3 & 0 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} \theta^4 & 0 \\ 0 & \theta^4 \end{pmatrix} + \ldots$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

A caution: for matrices x, y, usually $e^{x+y} \neq e^x\, e^y$. That is, the exponential map is *not* a group homomorphism from the additive group of n-by-n matrices to the multiplicative group $GL_n(\mathbb{R})$ of invertible n-by-n real matrices. This is disappointing, but is reasonable because the matrix multiplication is non-abelian, while matrix addition is abelian.

There is also a matrix logarithm

$$\log(1_n + A) \;=\; A - \frac{A^2}{2} + \frac{A^3}{3} - \ldots \qquad \text{(only for } A \text{ small)}$$

which gives an inverse near 1_n to the exponential map near 0_n. The *Lie algebra* $\mathfrak{g}$ of G can be defined to be

$$\mathfrak{g} \;=\; \{\text{real } n\text{-by-}n \text{ matrices } \gamma \;:\; \exp(t\gamma) \in G \text{ for all } t \in \mathbb{R}\}$$

Even though the exponential map does not generally respect addition, we *do have*

$$e^{(s+t)A} \;=\; e^{sA} \cdot e^{tA} \qquad \text{(for } s,t \in \mathbb{R})$$

3. *Rotation-invariant derivatives on* $\mathbb{R}^2$

Exponentiation $\theta \to e^{i\theta}$ gives the usual way to take derivatives of functions f on the circle $S^1 \subset \mathbb{C} \approx \mathbb{R}^2$, by

$$\lim_{\theta\to 0} \frac{f(x\cdot e^{i\theta}) - f(x)}{\theta} \;=\; \frac{\partial}{\partial\theta}\Big|_{\theta=0} f(x\cdot e^{i\theta}) \qquad \text{(for } x \in S^1)$$

Since we've written it so that the exponential map parametrizes the *group acting*, rather than the physical *space* on which the group act, the parametrized circle acts on the whole $\mathbb{R}^2$, not just on the circle itself. This action on the ambient space can be written without complex numbers as

$$\lim_{\theta\to 0} \frac{f((x,y)\cdot \exp\begin{pmatrix} 0 & \theta \\ -\theta & 0\end{pmatrix}) - f(x,y)}{\theta} \;=\; \frac{\partial}{\partial\theta}\Big|_{\theta=0} f((x,y)\cdot \exp\begin{pmatrix} 0 & \theta \\ -\theta & 0\end{pmatrix})$$

$$= \frac{\partial}{\partial\theta}\Big|_{\theta=0} f(x\cos\theta - y\sin\theta,\; x\sin\theta + y\cos\theta)$$

$$= \Big(f_1(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)\cdot(-x\sin\theta - y\cos\theta)$$

$$+\; f_2(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)\cdot(x\cos\theta - y\sin\theta)\Big)\Big|_{\theta=0}$$

$$= -yf_1(x,y) + xf_2(x,y) \qquad\qquad (\text{for } (x,y)\in S^1\subset\mathbb{R}^2)$$

That is, in the ambient coordinates on $\mathbb{R}^2$, the natural rotation-invariant differential operator, expressed via the exponential map, is

$$X \;=\; -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Write $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$ for brevity. To avoid needing to use a dummy function f to describe operators, keep in mind that, *as an operator*, x is really multiplication-by-x. In that context,

$$\partial_x\, x \;=\; \partial_x \circ x \;=\; 1 + x\partial_x$$

since Leibniz' rule is

$$\partial_x\, xf \;=\; f + x\partial_x f$$

With these conventions,

$$X^2 \;=\; (-y\partial_x + x\partial_y)\circ(-y\partial_x + x\partial_y) \;=\; y^2\partial_x^2 - y\partial_y - yx\partial_x\partial_y - x\partial_x - xy\partial_y\partial_x + x^2\partial_y^2$$

$$= y^2\partial_x^2 + x^2\partial_y^2 - 2xy\partial_x\partial_y - x\partial_x - y\partial_y$$

In this situation, with a two-dimensional ambient space, we wish to compare a more-intrinsic version of a rotation-invariant second-order differential operator with our earlier completely-extrinsic description. Abbreviate

$$E \;=\; x\partial_x + y\partial_y \qquad\qquad (\text{operator in Euler's identity})$$

To see what second-order operators are expressible in terms of E, compute

$$E^2 \;=\; 2xy\partial_x\partial_y + E + (x^2\partial_x^2 + y^2\partial_y^2)$$

Use this to rearrange

$$X^2 \;=\; y^2\partial_x^2 + x^2\partial_y^2 - 2xy\partial_x\partial_y - x\partial_x - y\partial_y \;=\; \Big(r^2\Delta - (x^2\partial_x^2 + y^2\partial_y^2)\Big) - \Big(E^2 - E - (x^2\partial_x^2 + y^2\partial_y^2)\Big) - E \;=\; r^2\Delta - E^2$$

Now return to the comparison of this more-intrinsic rotation-invariant differential operator with the extrinsically defined $\Delta^{\text{ext}} f = \Delta f(x/|x|)$ with $\Delta = \Delta^{\mathbb{R}^n}$. It suffices to compare on positive-homogeneous functions of degree 0, restricting to the sphere after differentiation. Thus, we ask whether

$$\Delta f(\frac{x}{|x|})\Big|_{S^{n-1}} \;=\; (r^2\Delta f - E^2 f)\Big|_{S^{n-1}} \qquad\qquad (\text{??? for } f \text{ positive-homogeneous degree } 0)$$

Indeed, on positive-homogeneous degree s functions F we have $EF = sF$, and on the sphere $r = 1$, so we have equality of the two differential operators on functions on the sphere.

Alternatively, on homogeneous harmonic polynomials f of degree d on $\mathbb{R}^2$, with eigenvalue $-d(d+n-2) = -d^2$ for $\Delta^{\rm ext}$ from earlier computations, indeed

$$\Delta^{\rm ext} f \;=\; -d^2\cdot f \;=\; 0 - d^2\cdot f \;=\; r^2\Delta f - d^2\cdot f \;=\; X^2\, f \qquad\qquad (\text{for } f\in\mathcal{H}_d \text{ on } \mathbb{R}^2)$$

Finally, *obviously* X commutes with X^2, giving the *unobvious* formulaic relation

$$\big(-y\partial_x + x\partial_y\big)\circ\big(r^2\Delta - E^2\big) \;=\; X\circ X^2 \;=\; X^2\circ X \;=\; \big(r^2\Delta - E^2\big)\circ\big(-y\partial_x + x\partial_y\big)$$

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4. *Intrinsic derivatives*

Similarly, the matrix exponential gives a way to take derivatives on $G = O(n)$ and any reasonable physical space on which G acts. This succeeds despite the non-abelian-ness of G and despite the consequent failure of the matrix exponential to be a group homomorphism. The saving point is that *directional derivatives* only depend on action along one line at a time, not on the interactions between the directions.

First, identify the Lie algebra $\mathfrak{g}$ of G:

[4.0.1] Claim: The Lie algebra $\mathfrak{g}$ of $G = O(n)$ is *skew-symmetric* matrices:

$$\mathfrak{g} \;=\; \mathfrak{so}(n) \;=\; \{\text{real } n\text{-by-}n \text{ matrices } \gamma \;:\; \gamma^\top = -\gamma\}$$

Proof: Transpose interacts nicely with exponentiation:

$$(\exp A)^\top \;=\; \Big(\sum_{\ell=0}^\infty \frac{A^\ell}{\ell!}\Big)^\top \;=\; \sum_{\ell=0}^\infty \Big(\frac{A^\ell}{\ell!}\Big)^\top \;=\; \sum_{\ell=0}^\infty \frac{(A^\top)^\ell}{\ell!} \;=\; \exp(A^\top)$$

Using this, for γ skew-symmetric,

$$(e^{t\gamma})^\top\cdot e^{t\gamma} \;=\; (e^{t\gamma^\top})\cdot e^{t\gamma} \;=\; (e^{-t\gamma})\cdot e^{t\gamma} \;=\; (e^{(-t+t)\gamma}) \;=\; e^0 \;=\; 1_n$$

Conversely, assume

$$(e^{t\gamma})^\top\cdot e^{t\gamma} \;=\; 1_n$$

Apply d/dt to both sides: the familiar scalar relation $(e^{ct})' = ce^{ct}$ holds here. Leibniz' rule applies to matrix-valued functions A, B, just keeping track of the order of multiplication:

$$(A\cdot B)' \;=\; A'\cdot B + A\cdot B'$$

Then

$$\gamma^\top e^{t\gamma^\top}\cdot e^{t\gamma} \;+\; e^{t\gamma^\top}\cdot\gamma e^{t\gamma} \;=\; \frac{d}{dt}1_n \;=\; 0_n$$

Setting $t=0$ gives

$$\gamma^\top + \gamma \;=\; 0_n$$

as claimed. ///

For $\gamma\in\mathfrak{g}$ the associated (right) differentiation R_γ on (differentiable) functions f on G is

$$(R_\gamma f)(g) \;=\; \lim_{t\to 0}\frac{f(g\cdot e^{t\gamma}) - f(g)}{t} \;=\; \frac{\partial}{\partial t}\Big|_{t=0} f(g\,e^{t\gamma})$$

The same expression gives a differential operator X_γ on functions on any nice space on which G acts: writing the action on the physical space on the *right*, the differential operator is written on the *left*, and

$$(X_\gamma f)(x) \;=\; \lim_{t\to 0} \frac{f(x\cdot e^{t\gamma}) - f(x)}{t} \;=\; \frac{\partial}{\partial t}\Big|_{t=0} f(x\, e^{t\gamma}) \qquad\qquad (\text{for } x \in X)$$

The Lie algebra $\mathfrak{g} = \mathfrak{so}(n)$ of $G = O(n)$ has a basis $\{\theta_{ij} : 1 \le i < j \le n\}$ of elements

$$\theta_{ij} \;=\; \begin{pmatrix} \ddots & & & & \\ & 0 & \ldots & 1 & \\ & \vdots & & \vdots & \\ & -1 & \ldots & 0 & \\ & & & & \ddots \end{pmatrix} \qquad\qquad (\text{all 0s except the 1 and } -1)$$

behaving like $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the Lie algebra $\mathfrak{so}(2)$ of $SO(2)$ under exponentiation:

$$\exp\, t\theta_{ij} \;=\; \begin{pmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & \cos t & & \ldots & & \sin t & & \\ & & & & 1 & & & & & \\ & & & \vdots & & \ddots & & \vdots & & \\ & & & & & & 1 & & & \\ & & & -\sin t & & \ldots & & \cos t & & \\ & & & & & & & & 1 & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{pmatrix}$$

where the diagonal is 1s except the $\cos t$'s, and off-diagonal entries are 0s except the $\pm \sin t$'s. Right multiplication of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ by $\exp t\theta_{ij}$ has no effect on any coordinates other than the i^{th} and j^{th}, and the latter are rotated just as in the $SO(2)$ situation:

$$x \cdot \exp t\theta_{ij} \;=\; (x_1, \ldots, x_{i-1}, x_i \cos t - x_j \sin t, x_{i+1}, \ldots, x_{j-1}, x_j \cos t + x_i \sin t, x_{j+1}, \ldots, x_n)$$

Let X_{ij} be the corresponding differential operator on functions on $\mathbb{R}^n$: by the chain rule, letting $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial_j = \frac{\partial}{\partial x_j}$, we obtain an expression completely parallel to the $SO(2)$ situation,

$$X_{ij} f(x) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} f(x \cdot \exp t\theta_{ij}) \;=\; -x_j \partial_i f + x_i \partial_j f$$

That is, as an operator,

$$X_{ij} \;=\; -x_j \partial_i + x_i \partial_j$$

Anticipating quasi-clairvoyantly that this is the right thing to do, define a somewhat-more-intrinsic S^{n-1} Laplacian candidate by a differential operator on the ambient $\mathbb{R}^n$:

$$\Delta^{\text{smi}} \;=\; \sum_{i<j} X_{ij}^2$$

First, we will compare this to the extrinsic S^{n-1} Laplacian from earlier. Second we will see why $\sum_{i<j} X_{ij}^2$ is even better than it appears: it has a completely intrinsic description, given later.

5. *Aside: invariance of the Euler operator by* $GL_n(\mathbb{R})$

The Euler operator in $\mathbb{R}^n$ is

$$E \;=\; x_1\partial_1 + \ldots + x_n\partial_n$$

In addition to Euler's identity [1] $Ef = s \cdot f$ for positive-homogeneous degree s functions f, E is invariant under the group $GL_n(\mathbb{R})$ of invertible n-by-n real matrices, as the following straightforward computation shows. In particular, E is $O(n)$-invariant. The usual definition of matrix multiplication:

$$(xg)_j \;=\; \sum_\ell x_\ell g_{\ell j} \qquad\qquad (\text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n \text{ and } g \in GL_n(\mathbb{R}))$$

and the chain rule give

$$E(g\cdot f)(x) \;=\; \sum_i x_i\,\partial_i f(xg) \;=\; \sum_i x_i \sum_j f_j(xg)\,\partial_i(xg)_j$$

$$= \sum_i x_i \sum_j f_j(xg)\,\partial_i(\sum_\ell x_\ell g_{\ell j}) \;=\; \sum_i x_i \sum_j f_j(xg)\, g_{ij}$$

$$= \sum_j \Big(\sum_i x_i\, g_{ij}\Big) f_j(xg) \;=\; \sum_j (xg)_j f_j(xg) \;=\; (Ef)(xg) \;=\; \big(g\cdot(Ef)\big)(x)$$

This is the *commutativity* $g \circ E = E \circ g$. Equivalently, this relation can be expressed as the *invariance* $g \circ E \circ g^{-1} = E$.

6. $\sum_{i<j} X_{ij}^2$ *in coordinates on* $\mathbb{R}^n$

The expression for Δ^{smi} in coordinates $x_1, \ldots, x_n$ on $\mathbb{R}^n$ simplifies usefully. Note that

$$E^2 \;=\; \sum_{i<j} 2x_ix_j\partial_i\partial_j \;+\; \sum_i x_i^2\partial_i^2 \;+\; E$$

which gives

$$\sum_{i<j} 2x_ix_j\partial_i\partial_j \;=\; E^2 - E - \sum_i x_i^2\partial_i^2$$

Thus,

$$\Delta^{\text{smi}} \;=\; \sum_{i<j}(-x_j\partial_i + x_i\partial_j)^2 \;=\; \sum_{i<j}\Big(-2x_ix_j\partial_i\partial_j + x_j^2\partial_i^2 + x_i^2\partial_j^2 - x_j\partial_j - x_i\partial_i\Big)$$

$$= \sum_{i<j} -2x_ix_j\partial_i\partial_j + \sum_i (r^2 - x_i^2)\partial_i^2 - (n-1)\sum_i x_i\partial_i$$

$$= \; -E^2 + E + \sum_i x_i^2\partial_i^2 + \sum_i (r^2 - x_i^2)\partial_i^2 - (n-1)E \;=\; -E^2 - (n-2)E + r^2\Delta$$

with $\Delta = \Delta^{\mathbb{R}^n}$. Thus,

$$\Delta^{\text{smi}} \;=\; \sum_{i<j} X_{ij}^2 \;=\; \sum_{i<j}(-x_j\partial_i + x_i\partial_j)^2 \;=\; -E(E+n-2) + r^2\Delta$$

[1] Recall that Euler's identity $Ef = s\cdot f$ for positive-homogeneous degree s functions f is proven by considering $F(t) = f(tx) = t^s f(x)$ for $t > 0$. Apply $\partial/\partial t$ and set $t = 1$ to obtain Euler's identity.

On positive-homogeneous degree-0 functions, $\Delta^{\text{smi}} = r^2\Delta$, matching Δ^{ext}. Alternatively, for $f \in \mathcal{H}_d$ on $\mathbb{R}^n$, because $Ef = d \cdot f$,

$$\Delta^{\text{smi}} f \;=\; -d(d+n-2)f + r^2 \cdot 0 \;=\; -d(d+n-2)f$$

as desired. That is, the somewhat-more-intrinsic Δ^{smi} is equal to the extrinsic Δ^{ext}.

This expression verifies that $\Delta^{\text{smi}} = \sum_{i<j} X_{ij}^2$ is $O(n)$-invariant, since the Euler operator E is $O(n)$-invariant.

[6.0.1] **Remark:** However, the above gives no explanation of choice of the expression $\sum_{i<j} X_{ij}^2$, or how to know *in advance* that it would be invariant.

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7. *X_ℓ commutes with $\sum_{i<j} X_{ij}^2$*

The description of the first-order differential operators X_ℓ in terms of matrix exponentiation immediately proves that these operators commute with any $O(n)$-invariant differential operator, for simple reasons. To prove this, let $R(k)$ be the (right) translation operator by $k \in G$, meaning

$$\big(R(k)f\big)(x) \;=\; f(xk)$$

For any $\gamma \in \mathfrak{g} = \mathfrak{so}(n)$, and for any $O(n)$-invariant differential operator L,

$$X_\gamma \circ L \;=\; \frac{\partial}{\partial t}\Big|_{t=0} R(e^{t\gamma}) \circ L \;=\; \frac{\partial}{\partial t}\Big|_{t=0} L \circ R(e^{t\gamma}) \;=\; L \circ \frac{\partial}{\partial t}\Big|_{t=0} R(e^{t\gamma}) \;=\; L \circ X_\gamma$$

Thus, since the $\mathbb{R}^n$-coordinate expression for $\Delta^{\text{smi}} = \sum_{i<j} X_{ij}^2$ commutes with $O(n)$, it commutes with each X_γ.

[7.0.1] **Remark:** Below, we use the *universal enveloping algebra* below to see that the expression $\sum_{i<j} X_{ij}^2$ itself is merely an expression of an *intrinsic* object in coordinates. The $O(n)$-invariance will likewise be clear *a priori*.

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8. *Stabilization of $\mathcal{H}_d$ by X_γ*

Since all the operators X_γ for $\gamma \in \mathfrak{g} = \mathfrak{so}(n)$ commute with the Laplacian Δ^S on the sphere S^{n-1}, all operators X_γ stabilize the eigenspaces of Δ^S: for $f \in \mathcal{H}_d$,

$$\Delta^S(X_\gamma f) \;=\; X_\gamma(\Delta^S f) \;=\; X_\gamma(\lambda_d \cdot f) \;=\; \lambda_d \cdot X_\gamma f \qquad \text{(where } \lambda_d = -d(d+n-2),\ f \in \mathcal{H}_d)$$

Conveniently, the eigenvalues

$$\lambda_d \;=\; -d(d+n-2)$$

of the Laplacian Δ^S on S^{n-1} on $\mathcal{H}_d$ on $\mathbb{R}^n$ are *different* for different degrees d, since completing the square

$$d(d+n-2) \;=\; d^2 + (n-2)d \;=\; \big(d + \frac{n-1}{2}\big)^2 - \frac{(n-1)^2}{4}$$

shows that as d moves farther from $(n-1)/2$ the eigenvalue strictly increases. Therefore, all the first-order differential operators X_γ stabilize $\mathcal{H}_d$.

9. *Integration by parts for* X_γ

The nature of the differential operators X_γ gives an integration-by-parts principle, using only the $O(n)$-invariance of the integral on $S = S^{n-1}$, as follows. Letting $R(g)$ be right translation by $g \in O(n)$, first

$$\int_S X_\gamma f \cdot g \;=\; \int_S \frac{\partial}{\partial t}\Big|_{t=0} R(e^{t\gamma}) f \cdot g \;=\; \frac{\partial}{\partial t}\Big|_{t=0} \int_S R(e^{t\gamma}) f \cdot g$$

assuming we can justify moving the differentiation through the integral. [2] And then by changing variables, replacing x by $xe^{-t\gamma}$,

$$\int_S \big(R(e^{t\gamma})f\big)(x)\; g(x)\, dx \;=\; \int_S f(xe^{t\gamma})\; g(x)\, dx \;=\; \int_S f(x)\; g(xe^{-t\gamma})\, dx \;=\; \int_S f\; R(e^{-t\gamma})g$$

Moving the differentiation back inside the integral yields the integration-by-parts relation

$$\langle X_\gamma f, g\rangle \;=\; \int_S X_\gamma f \cdot g \;\;=\;\; \int_S f \cdot X_{-\gamma} g \;=\; -\int_S f \cdot X_\gamma g \;=\; \langle f, X_\gamma g\rangle$$

Combining this with the expression

$$\Delta^S \;=\; \sum_{i<j} X_{ij}^2$$

gives a much-more-intrinsic proof of the integration-by-parts property for Δ^S:

$$\langle \Delta^S f, g\rangle \;=\; \int_S \Delta^S f\; g \;=\; \int_S f\; \Delta^S g \;=\; \langle f, \Delta^S\rangle$$

Further, we have the *non-positivity*

$$\langle X_\gamma^2 f, f\rangle \;=\; -\langle X_\gamma f, X_\gamma f\rangle \;\le\; 0$$

Similarly,

$$\langle \Delta^S f, f\rangle \;\le\; 0$$

10. *Sup norms of derivatives of harmonic polynomials*

Recall that

$$\sup_{f\in\mathcal{H}_d} \frac{|f|_{C^o}}{|f|_{L^2}} \;\ll\; \sqrt{\dim \mathcal{H}_d} \qquad\qquad \text{(implied constant depending on measure on } S^{n-1})$$

where the C^o norm is the sup-norm on S^{n-1}. In the C^o topology, limits of C^o functions are again C^o. The C^1 norm should have the property that C^1-topology limits of C^1 functions are again C^1. Roughly, the C^1 topology looks at sup-norms of *first derivatives*. But *which* first derivatives?

[2] Perhaps this is not the moment to worry about interchange of integration and differentiation in a parameter. Nevertheless, in the present example, since the integral is over a compact space, standard ideas about *Gelfand-Pettis integrals*, also called *weak integrals*, would easily dispatch the issue: the integrand is a continuous $C^\infty(\mathbb{R})$-valued function (in t) of $x \in S^{n-1}$, and differentiation is a continuous linear map of $C^\infty(\mathbb{R})$ to itself. Since this aspect of the present example is not at all delicate, we will give the other aspects primary attention.

The question *which derivates?* has an easy adequate answer on $\mathbb{R}^n$: the partial derivatives of the partials $\partial f(x)/\partial x_i$ with respect to the standard coordinates x_i.

On the sphere, however, we saw earlier that naive forms of the ambient derivatives do not interact well with the invariant Laplacian Δ^S. Instead, the differential operators X_γ defined via exponentiation of the Lie algebra are the right things, since they *commute* with Δ^S. In particular, we may as well take the finite list X_{ij} with $i < j$, since these span $\mathfrak{g} = \mathfrak{so}(n)$. Thus, thinking that functions on the sphere S^{n-1} are defined on some larger open set in $\mathbb{R}^n$ containing S^{n-1}, define

$$|f|_{C^1(S^{n-1})} \;=\; |f|_{C^o} + \sup_{i<j} \big|X_{ij} f\big|_{C^o(S^{n-1})} \;=\; |f|_{C^o} + \sup_{i<j} \big|(-x_j\partial_i + x_i\partial_j)f\big|_{C^o(S^{n-1})}$$

It might seem that far more derivatives are included than necessary, but the expression of the Laplacian Δ^S using *all* the X_{ij} suggests that this more-symmetrical conception is good. Since X_{ij} commutes with Δ^S, and, therefore, stabilizes $\mathcal{H}_d$, the comparison to L^2 norms gives

$$\sup_{i<j} \big|X_{ij} f\big|_{C^o} \;\ll\; \sqrt{\dim \mathcal{H}_d} \;\cdot\; \big|X_{ij} f\big|_{L^2} \qquad \text{(constant depending only on measure of } S^{n-1})$$

Integrating by parts,

$$\big|X_{ij} f\big|^2_{L^2} \;=\; \langle -X_{ij}^2 f, f\rangle \;\le\; \sum_{i<j} \langle -X_{ij}^2 f, f\rangle \;=\; \langle -\Delta^S f, f\rangle \;=\; -\lambda_d \cdot |f|^2_{L^2}$$

where Δ^S acts on $\mathcal{H}_d$ by $\lambda_d = -d(d+n-2)$. Taking square roots,

$$\big|X_{ij} f\big|_{L^2} \;\le\; \sqrt{|\lambda_d|} \cdot |f|_{L^2} \qquad \text{(for } f \in \mathcal{H}_d)$$

Combining this with the comparison of C^o and L^2 norms on $\mathcal{H}_d$ gives

$$\big|X_{ij} f\big|_{C^o} \;\le\; \sqrt{\dim \mathcal{H}_d} \cdot \sqrt{|\lambda_d|} \cdot |f|_{L^2} \qquad \text{(for } f \in \mathcal{H}_d)$$

Thus,

$$|f|_{C^1} \;\le\; \sqrt{\mathcal{H}_d} \;\cdot\; (1 + \sqrt{|\lambda_d|}) \;\cdot\; |f|_{L^2} \qquad \text{(for } f \in \mathcal{H}_d)$$

Similarly, the C^k norm is such that C^k-limits of C^k functions are C^k. It can be defined as[3]

$$|f|_{C^k} \;=\; |f|_{C^o} + \sup_{i<j} |X_{ij} f|_{C^{k-1}}$$

The same argument proves

$$|f|_{C^k} \;\le\; \sqrt{\mathcal{H}_d} \;\cdot\; \big(1 + \sqrt{|\lambda_d|}\big)^k \;\cdot\; |f|_{L^2} \qquad \text{(for } f \in \mathcal{H}_d)$$

The expression for $\dim \mathcal{H}_d$ in binomial coefficients gives $\dim \mathcal{H}_d \ll d^{n-2}$. Similarly, $\lambda_d \ll d^2$ for large d. Thus,

$$|f|_{C^k} \;\ll\; d^{\frac{n}{2}-1+k} \;\cdot\; |f|_{L^2} \qquad \text{(for } f \in \mathcal{H}_d \text{ on } \mathbb{R}^n\text{, large } d)$$

To avoid trouble at $d = 0$, we could write

$$|f|_{C^k} \;\ll\; (1+d)^{\frac{n}{2}-1+k} \;\cdot\; |f|_{L^2} \qquad \text{(for } f \in \mathcal{H}_d \text{ on } \mathbb{R}^n)$$

Expression in terms of λ_d is also useful, using $d \ll \sqrt{|\lambda_d|}$:

$$|f|_{C^k} \;\ll\; (1 + \sqrt{|\lambda_d|})^{\frac{n}{2}-1+k} \;\cdot\; |f|_{L^2} \qquad \text{(for } f \in \mathcal{H}_d \text{ on } \mathbb{R}^n)$$

[3] There is some work to be done to prove that the C^k norm has the intended property!

These equivalent inequalities express a sort of *pre-Sobolev inequality*.

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11. *Termwise differentiation of Fourier-Laplace series*

A C^k function f on S^{n-1} has a Fourier-Laplace expansion

$$f = \sum_d f_d \qquad \text{(in an } L^2 \text{ sense, } f_d \in \mathcal{H}_d)$$

This is interesting and useful because the spectral components f_d are Δ^S-eigenfunctions. That is, this expansion *diagonalizes* the action of Δ^S. That is, *if this made sense*, obviously

$$\Delta^S f = \sum_d \lambda_d \cdot f_d \qquad (???)$$

The issue, as with Fourier series on the circle, is the sense in which the implied limit on the right-hand side exists at all, and is the left-hand side. For example, for $f \in C^2$, if the right-hand side *converges in* C^2, then Δ^S is a continuous operator $C^2 \to C^o$, and the termwise differentiation is justified.

However, even when f is C^k, its Fourier-Laplace series need not converge to it in the C^k topology, but only in a weaker $C^{k'}$-topology for $k' < k$. This seeming paradox already occurs with Fourier series on the circle.

The essential estimate uses the sup-norm estimates on the components $f_d \in \mathcal{H}_d$ from above. The C^o case already illustrates the point: for any $s \in \mathbb{R}$,

$$\sum_d |f_d|_{C^o} \ll \sum_d (1+d)^{\frac{n}{2}-1}|f_d|_{L^2} = \sum_d (1+d)^{\frac{n}{2}-1+s}|f_d|_{L^2} \cdot \frac{1}{(1+d)^s}$$

$$\le \Big(\sum_d (1+d)^{n-2+2s}|f_d|^2_{L^2}\Big)^{1/2} \cdot \Big(\sum_d \frac{1}{(1+d)^{2s}}\Big)^{1/2}$$

by Cauchy-Schwarz-Bunyakowsky. The elementary sum over d is *finite* exactly for $s > \frac{1}{2}$. The condition $s > \frac{1}{2}$ is equivalent to the condition that the exponent of $1+d$ in the sum involving the components f_d satisfies

$$n - 2 + 2s > n - 1 = \dim S^{n-1}$$

That is, replacing $\frac{n-2}{2} + s$ by s,

$$\sum_d |f_d|_{C^o} \ll_{n,s} \Big(\sum_d (1+d)^{2s}|f_d|^2_{L^2}\Big)^{1/2} \qquad (\text{for } s > \tfrac{1}{2}\dim S^{n-1})$$

The weighted L^2-norm on the right-hand side of the latter is the s^{th} *Sobolev norm* $|f|_s^2$:

$$s^{th} \text{ Sobolev norm-squared} = |f|_s^2 = \sum_d (1+d)^{2s} \cdot |f_d|^2_{L^2}$$

An exactly analogous computation gives a similar result for the C^k norm:

$$\sum_d |f_d|_{C^k} \ll_{n,s} \Big(\sum_d (1+d)^{2s}|f_d|^2_{L^2}\Big)^{1/2} \qquad (\text{for } s > k + \tfrac{1}{2}\dim S^{n-1})$$

That is, the Sobolev norm $|\cdot|_s$ *dominates* the C^k norm for $s > k + \frac{1}{2}\dim S^{n-1}$.

In particular, this dominance gives corresponding estimates on *tails* of infinite sums:

$$\sum_{k\le d\le \ell} |f_d|_{C^k} \ll_{n,s} \Big(\sum_{k\le d\le \ell} (1+d)^{2s}|f_d|^2_{L^2}\Big)^{1/2} \qquad (\text{for } s > k + \tfrac{1}{2}\dim S^{n-1})$$

Thus, if the partial sums $\sum_{d\le\ell} f_d$ form a Cauchy sequence in the $|\cdot|_s$-topology for $s > k + \frac{n-1}{2}$, then they form a Cauchy sequence in the C^k-topology.

Finally, take $f \in C^{2\ell}(S^{n-1})$. Then f, $\Delta^S f$, $(\Delta^S)^2 f$, ..., $(\Delta^S)^\ell f$ are in $C^o(S^{n-1})$, so are certainly in $L^2(S^{n-1})$, since S^{n-1} is compact. Let $\mathrm{pr}_d : L^2(S^{n-1}) \to \mathcal{H}_d$ be the orthogonal projection. We claim that

$$\mathrm{pr}_d(\Delta^S f) = \lambda_d \cdot \mathrm{pr}_d(f)$$

Indeed, for any $g \in \mathcal{H}_d$,

$$\langle \Delta^S f, g\rangle = \langle f, \Delta^S g\rangle = \lambda_d \langle f, g\rangle$$

The projection can be expressed in terms of an orthonormal basis $\{g_i\}$ for $\mathcal{H}_d$:

$$\mathrm{pr}_d(f) = \sum_i \langle f, g_i\rangle \cdot g_i$$

Thus,

$$(\Delta^S)^i f = \sum_d \mathrm{pr}_d(\Delta^S f) = \sum_d \lambda_d^i \cdot \mathrm{pr}_d(f) \qquad (\text{in an } L^2 \text{ sense})$$

By Plancherel,

$$\sum_d \big|\lambda_d^i \cdot \mathrm{pr}_d(f)\big|^2 = |(\Delta^S)^i f|^2_{L^2} < \infty$$

Thus, taking a linear combination of such inequalities for $i = 0, 1, \ldots, \ell$, we have finiteness of the $2\ell^{th}$ Sobolev norm

$$\sum_d (1+|\lambda_d|)^\ell \cdot \mathrm{pr}_d(f)\big|^2 < \infty$$

Equivalently, since $d^2 \ll \lambda_d \ll d^2$,

$$|f|^2_{2k} = \sum_d (1+d)^{2\ell} \cdot \mathrm{pr}_d(f)\big|^2 < \infty$$

From above, the $2\ell^{th}$ Sobolev dominates the k^{th} for $2\ell > k + \frac{n-1}{2}$. Thus, the Fourier-Laplace series for $f \in C^{2\ell}(S^{n-1}$ converges to f in C^k for $k < 2\ell - \frac{n-1}{2}$. That is, termwise differentiation is justified in that range.

[11.0.1] Remark: The index shift by $\frac{1}{2}\dim S^{n-1}$ is an instance of a very general phenomenon.

[11.0.2] Remark: We might worry that this last result is too weak. For example, we might imagine that termwise differentiation of Fourier-Laplace series of $C^{2\ell}$ functions is justified up through $2\ell^{th}$ derivatives, rather than merely $2\ell - \frac{n-1}{2} - \epsilon$. However, a Baire category argument shows that, typically, the apparent discrepancy indicated above is genuine.

[11.0.3] Remark: The Sobolev space $H^s(S^{n-1})$ is the completion of $C^\infty(S^{n-1})$ with respect to the s^{th} Sobolev norm. The inequalities above prove the *Sobolev imbedding theorem*

$$H^s(S^{n-1}) \subset C^k(S^{n-1}) \qquad (\text{for } s > k + \tfrac{1}{2}\dim S^{n-1})$$

The nested intersection (projective limit) is therefore

$$H^\infty(S^{n-1}) = \bigcap_{s>0} H^s(S^{n-1}) = \bigcap_k C^k(S^{n-1}) = C^\infty(S^{n-1})$$

[11.0.4] Remark: Since $H^\infty(S^{n-1}) = C^\infty(S^{n-1})$, the space of *distributions* $C^\infty(S^{n-1})^*$ is also a *colimit*

$$C^\infty(S^{n-1})^* \;=\; \mathrm{colim}_{s>0}\, H^s(S^{n-1})^*$$

Apart from possible choices of topology on them, the duals of the Hilbert spaces $H^s(S^{n-1})$ are easy to understand: by Riesz-Fischer the duals are (conjugate-) isomorphic to themselves.
