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Number theory exercises 08

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Due Mon, 27 Feb 2012, preferably as PDF emailed to me.

[number theory 08.1] (Sample application of basic ideas from Minkowski) Let ω be an abstract square root of D for $0 < D \in \mathbb{Z}$, and map $\mathbb{Z}[\omega]$ to a lattice L in $\mathbb{R}^2$ by $a + b\omega \to (a + b\sqrt{D}, a - b\sqrt{D})$, where $\sqrt{D}$ is the positive real square root of D. Show that the rectangle $R_{T,\delta} = \{(x,y) : |x| \le T,\ |y| \le \delta\}$ contains a point of L other than $(0,0)$ when $T \cdot \delta > 8\sqrt{D}$.

Thus, obtain a corollary about *Diophantine approximation:* given $\varepsilon > 0$, there are integers a, b with $|b| < \frac{4\sqrt{D}}{\sqrt{2}\varepsilon} + \frac{\varepsilon}{2\sqrt{D}}$ such that $|a - b\sqrt{D}| < \varepsilon$. (Of course, rational numbers approximate real irrationals. The point is about how large the denominators must be.)

[number theory 08.2] * *[Starred problems are optional]* Contrive an interesting application of Minkowski's *geometry of numbers* similar to the previous example about $\sqrt{D}$, but for *cube roots*.

[number theory 08.3] Use Poisson summation to show that

$$\sum_{n \in \mathbb{Z}} \frac{1}{t^2 + n^2} = \sum_{n \in \mathbb{Z}} \frac{\pi}{t} e^{-2\pi |n| t} \qquad \text{(for } t \notin i\mathbb{R}\text{)}$$

Rearrange and let $t \to 0$ to show $\zeta(2) = \pi^2/6$.

[number theory 08.4] Show that $d^+x/|x|_p$ is a $\mathbb{Q}_p^\times$-invariant measure on $\mathbb{Q}_p^\times$, where d^+x refers to the additive Haar measure. Determine the measure of the local units $\mathbb{Z}_p^\times$. Observe that the infinite product over all primes p of the measure of the local units *diverges*.

[number theory 08.5] * *[Starred problems are optional]* Construct a $\mathbb{Q}_p^\times$-invariant integral on $C_c^o(\mathbb{Q}_p^\times)$ directly.

Hint: (a) Normalize the measure of $\mathbb{Z}_p^\times$ to be 1.

(b) Show that the compact-and-open $U_k = 1 + p^k\mathbb{Z}_p$ is of index $(p-1)p^{k-1}$ in $\mathbb{Z}_p^\times$ for $k > 0$, so the measure of U_k is $1/(p-1)p^{k-1}$.

(c) Show that the special simple functions consisting of linear combinations of characteristic functions of sets $y \cdot (1 + p^k\mathbb{Z}_p)$ with $0 < k \in \mathbb{Z}$ are *dense* in $C_c^o(\mathbb{Q}_p^\times)$, in the topology which gives $C_c^o(K)$ the sup-norm topology, where K is compact in $\mathbb{Q}_p^\times$.

(d) Define the integral of special simple functions directly, and extend by continuity.

[number theory 08.6] * *[Starred problems are optional]* Show that $\mathbb{Z}_p^\times$ is the *unique* maximal compact subgroup of $\mathbb{Q}_p^\times$.

In contrast, $\mathbb{Q}_p$ (with addition) has no maximal compact subgroup at all. In fact, the non-compact is the ascending union of compact subgroups $p^{-k}\mathbb{Z}_p$! Very different from the multiplicative group!
