Due Mon, 05 Mar 2012, preferably as PDF emailed to me.

[number theory 09.1] For $y \in \mathbb{Z}_p$, what are the eigenvalues and eigenvectors of the unitary operator $T : L^2(\mathbb{Z}_p) \to L^2(\mathbb{Z}_p)$ defined by $Tf(x) = f(x+y)$?

[number theory 09.2] With $G = \mathbb{Z}/n$, let $T : L^2(G) \to L^2(G)$ be the first difference operator

$$Tf(x) = f(x+1) - f(x)$$

Find the eigenvalues and eigenvectors of $T$.

[number theory 09.3] With $G = \mathbb{Z}/n$, let $T : L^2(G) \to L^2(G)$ be the second difference operator

$$Tf(x) = (f(x+1) - f(x)) - (f(x) - f(x-1)) = f(x+1) - 2f(x) + f(x-1)$$

Find the eigenvalues and eigenvectors of $T$.

[number theory 09.4] With $G = \mathbb{Z}/n \oplus \mathbb{Z}/n$, let $T : L^2(G) \to L^2(G)$ be the discrete Laplacian

$$Tf(x) = f(x+(1,0)) - 2f(x) + f(x+(1,0)) + f(x+(0,1)) - 2f(x) + f(x+(0,1))$$

Find the eigenvalues and eigenvectors of $T$.

[number theory 09.5] What are the eigenvalues and eigenvectors of the compact self-adjoint operator $T : L^2(\mathbb{Z}_p) \to L^2(\mathbb{Z}_p)$ defined by

$$Tf(x) = \int_{\mathbb{Z}_p} |x-y| \cdot f(y) \ dy$$

[number theory 09.6] * [Starred problems are optional] Define a Hilbert-Schmidt operator $T : L^2[0,1] \to L^2[0,1]$ by the continuous kernel

$$K(x, y) = \begin{cases} (x-1) \cdot y & \text{(for } 0 \leq y < x) \\ x \cdot (y-1) & \text{(for } x < y \leq 1) \end{cases}$$

Find mild hypotheses on $f$ so that the function $u = Tf$ solves the differential equation $u'' = f$. That is, the differential operator $d^2/dx^2$ has a (one-sided) inverse which is a compact operator.

[number theory 09.7] * [Starred problems are optional] Show that Volterra operator $V : L^2[0,1] \to L^2[0,1]$ given by

$$Vf(x) = \int_0^x f(t) \ dt$$

(for $f \in L^2[0,1]$)

is compact, although not self-adjoint. Show that $\lim_{n \to \infty} |V^n| = 0$, and that $V$ has no eigenvectors.