Context: Finiteness of class number and Dirichlet’s units theorem are corollaries of Fujisaki’s lemma, that $J^1/k^\times$ is compact. ... a corollary of

**Measure-theory pigeon-hole principle:** for discrete subgroup $\Gamma$ of a unimodular topological group $G$, with $\Gamma \backslash G$ of finite measure, if a set $E \subset G$ has measure strictly greater than $\Gamma \backslash G$, then there are $x \neq y \in E$ such that $x^{-1}y \in \Gamma$.

As expected, measure on $\Gamma$ is counting measure, and

$$\int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(\gamma g) \, dg = \int_G f(g) \, dg \quad \text{(for } f \in C^o_c(G))$$

Subsumes Minkowski’s *Geometry of Numbers* proposition: for lattices $L$ in $\mathbb{R}^n$, a convex subset $C$ of $\mathbb{R}^n$, symmetric about 0, with measure strictly greater than $2^n$ times the measure of $\mathbb{R}^n/L$, contains a point of $L$ other than 0.
Inspection of the arguments shows that we want very few things from (right-invariant) integrals on groups $G$, which characterize the integrals:

\[
\begin{align*}
&f \to \int_G f(g) \, dg \text{ defined on } C_c^0(G) \quad (\text{functionals on } C_c^0(G)) \\
&\int_G f(gh) \, dg = \int_G f(g) \, dg \text{ for } h \in G \quad (\text{right invariance}) \\
&f \geq 0 \implies \int_G f(g) \, dg \geq 0 \quad (\text{positivity})
\end{align*}
\]

In fact, the positivity condition implies that $f \to \int_G f$ is a continuous linear functional on $C_c^0(G)$ in its natural topology, but the arguments here only use the positivity.
Recap of abstracted argument: with \( f \) the characteristic function of \( E \), if there were no such \( x, y \), then \( \sum_{\gamma \in \Gamma} f(\gamma \cdot x) \leq 1 \). But then

\[
\text{meas}(\Gamma \setminus G) \leq \int_{G} f(g) \, dg = \int_{\Gamma \setminus G} \left( \sum_{\gamma \in \Gamma} f(\gamma \cdot g) \right) \, dg \leq \text{meas}(\Gamma \setminus G)
\]

Impossible. So there is \( 1 \neq x^{-1}y \notin \Gamma \). ///

Existence of suitable measure on the quotient does not depend on discreteness of \( \Gamma \), but on the condition \( \Delta_G|_H = \Delta_H \), and then, as we proved, there exists a unique measure on \( H \setminus G \) such that

\[
\int_{G} f(g) \, dg = \int_{H \setminus G} \left( \int_{H} f(h \cdot \hat{g}) \, dh \right) \, d\hat{g}
\]
Another interruption! ... for context. Finite volume of $\mathbb{R}^n / \mathbb{Z}^n$ is familiar, but we have essentially no experience with discrete subgroups $\Gamma$ in non-abelian $G$. The following is a prototype both for the assertion and for the proof mechanisms.

**Claim:** The quotient $\Gamma \backslash G = SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})$ has finite invariant volume (where $SL_n(R) = n \times n$ matrices with entries in ring $R$). In fact, in a natural normalization,

$$\text{vol} (SL(n, \mathbb{Z}) \backslash SL(n, \mathbb{R})) = \zeta(2) \zeta(3) \zeta(4) \zeta(5) \ldots \zeta(n)$$

**Remark:** Mysterious $\zeta$(odd) values appear.

Minkowski knew the finiteness, and Siegel computed the value. We grant the finiteness, and compute the volume without a fundamental domain.
Proof: (modernization of Siegel’s argument) The point is

\[ \int_{\Gamma \setminus G} \sum_{\gamma \in \Gamma} f(\gamma g) \, dg = \int_{G} f(g) \, dg \]

Treat \( n = 2, \ G = SL(2, \mathbb{R}), \) and \( \Gamma = SL(2, \mathbb{Z}). \) We showed that a right \( G \)-invariant measure on \( \Gamma \setminus G \) is described by integrals of \( C_c^0(\Gamma \setminus G) \). Every \( F \in C_c^0(\Gamma \setminus G) \) is expressible as

\[ F(g) = \sum_{\gamma \in \Gamma} f(\gamma \cdot g) \quad \text{ (for some } f \in C_c^0(G)) \]

and the integral of \( F \) is sufficiently-defined and well-defined by

\[ \int_{\Gamma \setminus G} F(g) \, dg = \int_{\Gamma \setminus G} \sum_{\gamma \in \Gamma} f(\gamma \cdot g) \, dg = \int_{G} f(g) \, dg \]
Although we do not describe the geometry of $\Gamma \backslash G$, we do need details about the Haar measure on $G$, since a constant ambiguous by a constant is not interesting.

$G$ is unimodular, since $G = [G, G]$. (!) To describe the measure on $G$ usefully, we do need coordinates on $G$, but not the naive $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $K$ be the usual special orthogonal group

$$K = SO(2) = \{ g \in G : g^\top g = 1_2 \} = \{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \}$$

and

$$P^+ = \{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0, \ b \in \mathbb{R} \}$$

Compact $K$ is unimodular, while $P^+$ is not. The Iwasawa decomposition (not too hard an exercise, in this example!) is

$$G = P^+ \cdot K \approx P^+ \times K$$
Lemma: Haar measure on $G$ is $d(pk) = dp \cdot dk$, where $dp$ is left Haar measure on $P^+$, and $dk$ is right Haar on $K$. That is, 

$$\int_G \varphi(g) \, dg = \int_{P^+} \int_K \varphi(pk) \, dk \, dp$$

(for $\varphi \in C_c^0(G)$)

Proof: Let the group $P^+ \times K$ act on $G$ by $(p \times k)(g) = p^{-1}gk$. (The inverse is for associativity!) The isotropy subgroup in $P^+ \times K$ of $1 \in G$ is \{ $p \times k : p^{-1} \cdot 1 \cdot k = 1$ \} = $P^+ \cap K = \{1\}$. Thus, there is a unique $P^+ \times K$-invariant measure on $G$, and it fits into $\int_G = \int_{P^+} \int_K$. The Haar measure on $G$ gives such a thing, as does a Haar measure on $G$. \///
Now completely specify the Haar measure on $G$. Normalize the Haar measure on the circle (!) $K$ to have total measure $2\pi$. Normalize the left Haar measure $dp$ on $P^+$ to (!)

$$d\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \frac{1}{t^2} dx \frac{dt}{t} \quad (x \in \mathbb{R} \text{ and } t > 0)$$

Corresponding to a nice (Schwartz?) function $f$ on $\mathbb{R}^2$, let $F$ on $G$ be

$$F(g) = \sum_{v \in \mathbb{Z}^2} f(vg)$$

By design, this function $F$ is left $\Gamma$-invariant. Eevaluating

$$\int_{\Gamma \backslash G} F(g) \, dg$$

in two different ways will determine the volume of $\Gamma \backslash G$. 
Lemma: Given coprime $c, d \in \mathbb{Z}$, there exists \( \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \).

Proof: The ideal $\mathbb{Z}c + \mathbb{Z}d$ is $\mathbb{Z}$, so there are $a, b \in \mathbb{Z}$ such that $ad + bc = 1$. Then \( \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in \Gamma \). 

Thus, for a fixed positive integer $\ell$, the set $\{(c, d) : \gcd(c, d) = \ell\}$ is an orbit of $\Gamma$ in $\mathbb{Z}^2$. Take $(0, 1)$ as convenient base point and observe that

$$\mathbb{Z}^2 - \{0\} = \{\ell \cdot (0, 1) \cdot \gamma : \text{for } \gamma \in \Gamma, \ 0 < \ell \in \mathbb{Z}\}$$

Let

$$N = \{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in G \} \quad N_{\mathbb{Z}} = N \cap \Gamma$$
The stabilizer of \((0, 1)\) in \(\Gamma\) is \(N_{\mathbb{Z}}\), and there is a bijection
\[
\mathbb{Z}^2 - \{0\} \longleftrightarrow \{\ell > 0\} \times N_{\mathbb{Z}} \backslash \Gamma
\]
by
\[
\ell(0, 1)\gamma \leftarrow \ell \times N_{\mathbb{Z}}\gamma
\]
Then
\[
\int_{\Gamma \backslash G} F(g) \, dg = \int_{\Gamma \backslash G} f(0) \, dg + \int_{\Gamma \backslash G} \sum_{x \neq 0} f(xg) \, dg
\]
\[
= \int_{\Gamma \backslash G} f(0) \, dg + \sum_{\ell > 0} \int_{N_{\mathbb{Z}} \backslash G} f(\ell \cdot (0, 1)g) \, dg
\]
Writing the integral on $G$ as an iterated integral on $P^+$ and $K$, $\int_{\Gamma \setminus G} F$ is

$$\int_{\Gamma \setminus G} f(0) \, dg + \sum_{\ell > 0} \int_{N_{\mathbb{Z}} \setminus P} \int_{K} f(\ell \cdot (0,1)pk) \, dg$$

With $f$ rotation invariant, so $f(\ell(0,1)p k) = f(\ell(0,1)p)$, the integral is

$$\int_{\Gamma \setminus G} f(0) \, dg + 2\pi \cdot \sum_{\ell > 0} \int_{N_{\mathbb{Z}} \setminus P} f(\ell(0,1)p) \, dp$$

since the total measure of $K$ is $2\pi$. Expressing the Haar measure on $P^+$ in coordinates as above, the integral is
\[ \int_{\Gamma \backslash G} f(0) \, dg + 2\pi \sum_{\ell} \int_{0}^{\infty} \int_{\mathbb{Z} \backslash \mathbb{R}} f(\ell(0,1) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) \, dx \, \frac{dt}{t^2} \]

Note that \( N \) fixes \((0,1)\), so the integral over \( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \) is \( \int_{\mathbb{Z} \backslash \mathbb{R}} 1 \, dx = 1 \), and the whole integral is

\[ \int_{\Gamma \backslash G} F(g) \, dg = \int_{\Gamma \backslash G} f(0) \, dg + 2\pi \sum_{\ell} \int_{M} f(\ell(0,1) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) \frac{1}{t^2} \frac{dt}{t} \]

\[ = \int_{\Gamma \backslash G} f(0) \, dg + 2\pi \sum_{\ell} \int_{0}^{\infty} f(\ell(0,t^{-1})) \frac{1}{t^2} \frac{dt}{t} \]

\[ = f(0) \cdot \text{vol}(\Gamma \backslash G) + 2\pi \sum_{\ell} \int_{0}^{\infty} f(0,\ell t) t^2 \frac{dt}{t} \]

replacing \( t \) by \( t^{-1} \).
Replacing $t$ by $t/\ell$ gives

\[
\int_{\Gamma \backslash G} F(g) \, dg = f(0) \cdot \text{vol}(\Gamma \backslash G) + 2\pi \cdot \sum_{\ell} \ell^{-2} \int_{0}^{\infty} f(0, t) \, t^{2} \frac{dt}{t}
\]

\[
= f(0) \cdot \text{vol}(\Gamma \backslash G) + 2\pi \zeta(2) \cdot \int_{0}^{\infty} f(0, t) \, t^{2} \frac{dt}{t}
\]

Using the rotation invariance of $f$,

\[
\int_{0}^{\infty} f(0, t) \, t^{2} \frac{dt}{t} = \int_{0}^{\infty} f(0, t) \, t \, dt = \frac{1}{2\pi} \int_{\mathbb{R}^{2}} f(x) \, dx = \frac{1}{2\pi} \hat{f}(0)
\]

The $2\pi$’s cancel, and

\[
\int_{\Gamma \backslash G} F(g) \, dg = \int_{\Gamma \backslash G} \sum_{x \in \mathbb{Z}^{2}} f(xg) \, dg = f(0) \cdot \text{vol}(\Gamma \backslash G) + \zeta(2) \hat{f}(0)
\]
On the other hand, by Poisson summation,

\[
\sum_{x \in \mathbb{Z}^2} f(xg) = \frac{1}{|\det g|} \sum_{x \in \mathbb{Z}^2} \hat{f}(x^\top g^{-1}) = \sum_{x \in \mathbb{Z}^2} \hat{f}(x^\top g^{-1})
\]

(since \(\det g = 1\)). \(\Gamma\) is stable under transpose-inverse, allowing an analogous computation with the roles of \(f\) and \(\hat{f}\) reversed, obtaining

\[
f(0) \cdot \text{vol} (\Gamma\backslash G) + \zeta(2) \hat{f}(0) = \int_{\Gamma \backslash G} F(g) \, dg
\]

\[
= \hat{f}(0) \cdot \text{vol} (\Gamma\backslash G) + \zeta(2) f(0)
\]

from which

\[
(f(0) - \hat{f}(0)) \cdot \text{vol} (\Gamma\backslash G) = (f(0) - \hat{f}(0)) \cdot \zeta(2)
\]

With \(f(0) \neq \hat{f}(0)\), \(\text{vol} (\Gamma\backslash G) = \zeta(2)\).///
Next: More about Haar measure...

**Change-of-measure and Haar measure on \( \mathbb{A} \) and \( k_v \):**

Another thing used in the proof of Fujisaki’s lemma was that, for *idele* \( \alpha \), the change-of-measure on \( \mathbb{A} \) is

\[
\frac{\text{meas}(\alpha E)}{\text{meas}(E)} = |\alpha| \quad \text{(for measurable } E \subset \mathbb{A})
\]

Naturally, this should be examined...