Now, where were we...?

Context: Finiteness of class number and *Dirichlet's units theorem* are corollaries of *Fujisaki's lemma*, that $\mathbb{J}^1/k^\times$ is *compact*. ... a corollary of

Measure-theory pigeon-hole principle: for *discrete* subgroup Γ of a *unimodular* topological group G, with $\Gamma\backslash G$ of finite measure, if a set $E \subset G$ has measure strictly greater than $\Gamma\backslash G$, then there are $x \neq y \in E$ such that $x^{-1}y \in \Gamma$.

As expected, measure on Γ is *counting* measure, and

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(\gamma g)\, dg \;=\; \int_G f(g)\, dg \qquad\qquad (\text{for } f \in C_c^o(G))$$

As we saw, this subsumes Minkowski's *Geometry of Numbers* basic proposition: for *lattices* L in $\mathbb{R}^n$, for *convex* $C \subset \mathbb{R}^n$, symmetric about 0, with measure strictly greater than 2^n times the measure of $\mathbb{R}^n/L$, a point of L other than 0 lies in C.

Inspection of the arguments shows that we want very few things from (right-invariant) integrals on groups G, *characterizing* rather than *constructing* the integrals:

$$\begin{cases} f \to \int_G f(g)\,dg \text{ defined on } C^o_c(G) & \text{(functionals on } C^o_c(G)) \\ \int_G f(gh)\,dg = \int_G f(g)\,dg \text{ for } h \in G & \text{(right invariance)} \\ f \geq 0 \Longrightarrow \int_G f(g)\,dg \geq 0 & \text{(positivity)} \end{cases}$$

In fact, the positivity condition implies that $f \to \int_G f$ is a *continuous* linear functional on $C^o_c(G)$ in its natural topology, but the arguments here only use the positivity.

The natural topology on $C^o_c(X)$ is *colimit* of Fréchet spaces $C^o(X)_K$, ascending union over $f \in C^o_c(X)$ with supports inside compacts K.

Recap of abstracted argument: with f the characteristic function of E, if there were *no* such x, y, then $\sum_{\gamma\in\Gamma} f(\gamma\cdot x) \le 1$. But then

$$\text{meas}\,(\Gamma\backslash G) \;<\int_G f(g)\,dg \;=\; \int_{\Gamma\backslash G} \Big(\sum_{\gamma\in\Gamma} f(\gamma\cdot g)\Big)\,dg \;\le\; \text{meas}\,(\Gamma\backslash G)$$

Impossible. So there *is* $1 \ne x^{-1}y \notin \Gamma$. ///

Existence of suitable measure on the quotient does not depend on discreteness of Γ, but on the condition $\Delta_G|_H = \Delta_H$, and then, as we proved, there exists a unique measure on $H\backslash G$ such that

$$\int_G f(g)\,dg \;=\; \int_{H\backslash G} \Big(\int_H f(h\,\dot g)\,dh\Big)\,d\dot g \qquad \text{(\textit{unwinding} property)}$$

Example from last time: Finite volume of $\mathbb{R}^n/\mathbb{Z}^n$ is familiar, but ... discrete subgroups Γ in non-abelian G?

Claim: The *arithmetic quotient* $\Gamma\backslash G = SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ has *finite* invariant volume (where $SL_n(R) = n \times n$ matrices with entries in ring R, determinant 1). In fact, in a natural normalization,

$$\text{vol } SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R}) \;=\; \zeta(2)\,\zeta(3)\,\zeta(4)\,\zeta(5)\ldots\zeta(n)$$

Remark: Mysterious ζ(odd) values appear!

Minkowski knew the finiteness, and Siegel computed the value.

The proof does *not* use a fundamental domain, but *does* use Poisson summation, the *unwinding* of integration on quotients, and the *Iwasawa decomposition*

$$SL_n(\mathbb{R}) \;=\; P^+ \cdot K \;=\; \big(\text{upper-triang, diagonal} > 0\big) \cdot \big(\text{rotations}\big)$$

In fact, we can replace $\mathbb{Z}$ by the ring of integers $\mathfrak{o}$ of a number field k, $\mathbb{R}$ by the product $\prod_{v|\infty} k_v = k \otimes_{\mathbb{Q}} \mathbb{R}$ of archimedean completions of k, and by the same argument prove (up to reasonable normalization)

$$\text{vol } SL_n(\mathfrak{o})\backslash SL_n(k \otimes_{\mathbb{Q}} \mathbb{R}) = \zeta_k(2)\,\zeta_k(3)\,\zeta_k(4)\,\zeta_k(5)\ldots\zeta_k(n)$$

Over number fields, the best *proof* is in modern (adelic) terms:

$$\text{vol } SL_n(k)\backslash SL_n(\mathbb{A}_k) = \zeta_k(2)\,\zeta_k(3)\,\zeta_k(4)\,\zeta_k(5)\ldots\zeta_k(n)$$

Modern (adelic) Poisson summation and presentation of $\zeta_k(s)$ also appear in the Iwasawa-Tate modernization of Hecke's continuation and functional equation of Dedekind zeta functions, and of grossencharacter L-functions of number fields.

Comments on knowability of $\zeta_k(n)$, etc.

Over $k = \mathbb{Q}$, at even integers $\pi^{-2n}\zeta(2n) \in \mathbb{Q}$, with an explicit formula in terms of Bernouilli numbers.

$\zeta(3)$ was proven *irrational* by Apery [1978], which is not the same as asserting that $\pi^{-3}\zeta(3)$ is irrational. Rivoal [2000] gave a similar result, expanded to address $\zeta(3), \zeta(5), \zeta(7), \ldots$.

The volume computation shows the values ζ(odd) are not mere sums-of-series, being volumes of natural, canonical objects.

Physical occurrence of zeta values was taken up by A. Borel in his study of *regulators*, and also by Bloch, Kato, and Beilinson.

Knowability of ζ(even) fits into a conjecture of Deligne (1978) on *motivic* L-functions. All known special-value results fit Deligne's conjecture, although verification of compatibility is often difficult.

The larger conjectures *may* subsume Deligne's, ...

[... *on knowability of special values...*]

Half the values of Dirichlet L-functions over $\mathbb{Q}$ are knowable:

$$\pi^{-n} L(n, \chi) = \text{algebraic} \qquad \text{(for } n, \chi \text{ of equal parity)}$$

with explicit Galois behavior. Recall: χ is *odd* when $\chi(-1) = -1$. There are explicit formulas in terms of generalized Bernoulli numbers, from Fourier series expansions of polynomials. (See notes from 2005-6.)

Values for *mismatched* parity are presumed of the same nature as $\zeta(3)$, though this seems not known.

These results have been known for about 200 years, and are essentially elementary.

Zetas of abelian extension of $\mathbb{Q}$: with $k = \mathbb{Q}(\sqrt{D})$, quadratic reciprocity for $\chi(p) = (D/p)_2$ gives

$$\zeta_k(s) \;=\; \zeta(s)\cdot L(s,\chi) \;=\; \begin{cases} \zeta(s)\cdot L(s,\mathrm{even}) & \text{for } D>0 \\ \\ \zeta(s)\cdot L(s,\mathrm{odd}) & \text{for } D<0 \end{cases}$$

The trivial character in $\zeta(s)$ is *even*. Thus, the results for Dirichlet L-functions give

$$(\pi^{-n})^2\,\zeta_k(n) \;=\; \begin{cases} \text{algebraic} & (\text{for } n \text{ even, } D>0) \\ \\ \text{unknown} & (\text{otherwise}) \end{cases}$$

Similarly, using reciprocity laws, zetas of *totally real* (archimedean completions are *real*, not *complex*) subfields of cyclotomic fields $\mathbb{Q}(\zeta_n)$ are products of Dirichlet L-functions with *even* χ, so are knowable at positive even integer arguments.

No analogous special values over not-totally-real number fields.

Beyond the reach of reciprocity: by the 1960s, Siegel and Klingen had gotten around the condition of abelian-ness over $\mathbb{Q}$, proving

$$\pi^{-2n[k:\mathbb{Q}]}\,\zeta_k(2n) \;=\; \text{algebraic} \qquad\qquad (2n \text{ even, } k \text{ totally real})$$

In fact, Klingen showed that, for *totally even* or *totally odd* finite-order Hecke-character χ on totally real k, values at integers n of matching parity were knowable:

$$\pi^{-n[k:\mathbb{Q}]}\,L(n,\chi) \;=\; \text{algebraic} \qquad\qquad (\text{parities of } n, \chi \text{ match})$$

In modern (adelic) terms, a grossencharacter χ is *totally even* when $\chi(k_v^\times) = 1$ for all archimedean v. It is *totally odd* when it takes both ± 1 values on $k_v^\times$ for all archimedean v.

Proofs use Eisenstein series on *Hilbert modular groups* $SL_2(\mathfrak{o}_k)$.

Example: Siegel also computed the volume of another family of *arithmetic quotients* $\Gamma\backslash G$. Apart from interest in the *possibility* of the computation, it is noteworthy that only *known* (knowable?) values of $\zeta(s)$ appear.

Let $J = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$, and define the *symplectic group*

$$Sp_n \;=\; \{g \in GL_{2n} \;:\; g^\top J g \;=\; J\}$$

Then (Siegel)

$$\text{vol}\; Sp_n(\mathbb{Z})\backslash Sp_n(\mathbb{R}) \;=\; \zeta(2)\zeta(4)\zeta(6)\ldots\zeta(2n-2)\zeta(2n)$$

Just as with SL_n, the analogous result holds over number fields.

Next: More about Haar measure...

Change-of-measure and Haar measure on $\mathbb{A}$ and k_v:

Another thing used in the proof of Fujisaki's lemma was that, for *idele* α, the change-of-measure on $\mathbb{A}$ is

$$\frac{\text{meas}\,(\alpha E)}{\text{meas}\,(E)} = |\alpha| \qquad \text{(for measurable } E \subset \mathbb{A})$$

Naturally, this should be examined...

This is the full, modern version of the familiar fact for Lebesgue measure on the real line: for $\alpha \in \mathbb{R}^\times$ and measurable $E \subset \mathbb{R}$, written exactly the same way

$$\frac{\text{meas}\,(\alpha E)}{\text{meas}\,(E)} = |\alpha| \qquad \text{(for measurable } E \subset \mathbb{R})$$

Local version for p-adic completions, first.

What is the measure on $\mathbb{Q}_p$?

We could describe how to integrate functions in $C_c^o(\mathbb{Q}_p)$, and invoke Riesz' theorem.

In any case, of course we want the *regularity* promised by Riesz on locally compact, Hausdorff, countably-based topological spaces: the measure of a set is the *inf* of measure of opens containing it, and *sup* of measure of compacts contained in it.

For (locally compact...) *totally disconnected abelian groups* such as $\mathbb{Q}_p$, there is a local basis $U_n = p^n\mathbb{Z}_p$ at 0 consisting of open *subgroups*. Since Z_p is also *compact* (and closed), so is each U_n. Since Z_p is the disjoint union of p^n of these cosets,

$$\text{meas}\,(p^n\mathbb{Z}_p) \;=\; p^{-n}\cdot\text{meas}\,(\mathbb{Z}_p)$$

Probably normalize $\text{meas}\,(\mathbb{Z}_p) = 1$. Even without *uniqueness* of Haar measure, this specifies a regular measure on $\mathbb{Q}_p$.
