Context: Finiteness of class number, *Dirichlet's units theorem*, corollaries of *Fujisaki* (that $\mathbb{J}^1/k^\times$ is *compact*).

... $\Leftarrow$ existence and uniqueness of Haar measure on $\mathbb{A}$ and $\mathbb{A}/k$... compactness of $\mathbb{A}/k$.

... $\Leftarrow$ *change-of-measure:* for *idele* α,

$$\frac{\text{meas}\,(\alpha E)}{\text{meas}\,(E)} = |\alpha| \qquad\qquad (\text{for measurable } E \subset \mathbb{A})$$

Constructed invariant *integral* on $\mathbb{Q}_p$ by approximating f in $C_c^o(\mathbb{Q}_p)$ by *special, continuous* simple functions: linear combinations of characteristic functions of sets $p^k\mathbb{Z}_p + y$ for $y \in \mathbb{Q}_p$.

(Recall) tangible uniqueness: We claim that taking meas $(\mathbb{Z}_p) = 1$ and mechanisms as in the construction give the only possible invariant integral/measure on $\mathbb{Q}_p$. Taking advantage of the special features here:

$\mathbb{Z}_p$ is *open*, so is measurable. It is compact, so its measure is *finite*. Thus, we can renormalize a given Haar measure μ so that $\mu(\mathbb{Z}_p) = 1$.

$\mathbb{Z}_p$ is a disjoint union of p^n *translates* of $p^n\mathbb{Z}_p$, all with the same measure, by translation-invariance, so $\mu(p^n\mathbb{Z}_p) = p^{-n}$. Thus, integrals of the *special* simple functions are completely determined.

We saw that each $C_c^o(p^{-k}\mathbb{Z}_p)$ can be approximated by special simple functions. Positivity/continuity of the invariant integral, this determines integrals of $C_c^o(\mathbb{Q}_p)$ completely. ///

Uniqueness by re-usable methods: a topological group G with *at least* one invariant measure has *at most* one, up to scalar multiples. The argument is *re-usable.* For simplicity, suppose G is *unimodular*, that is, that a left-invariant measure is right-invariant.

Recall that an *approximate identity* is a sequence $\{\psi_i\}$ of non-negative $\psi_i \in C_c^o(G)$ such that $\int_G \psi_i = 1$ for all i, and such that, given a neighborhood U of 1, there is i_o such that for $i \geq i_o$ the support of ψ_i is inside U.

Remark: This is strictly stronger than requiring that these functions approach the Dirac delta measure in a weak topology.

R, L are the usual right and left translation actions of G on functions f on G:

$$R_g f(h) \;=\; f(hg) \qquad\qquad L_g f(h) = f(g^{-1}h)$$

It is a two-epsilon argument, using the *uniform* continuity of continuous functions on compacts, to see that

$$g \times f \to R_g f \qquad\qquad g \times f \to L_g f$$

are *continuous* maps $G \times C_c^o(G) \to C_c^o(G)$.

Proof for right translation: A two-epsilon argument.

The claim is that, given $\varepsilon > 0$, there is a neighborhood N of $1 \in G$ and $\delta > 0$ such that, for $g, g' \in G$ with $g' \in gN$, and $\sup_x |f(x) - f'(x)| < \delta$, we have $\sup_x |f(xg) - f'(xg')| < \varepsilon$.

$f \in C_c^o(K)$ is *uniformly* continuous, by the same proof as on $\mathbb{R}$, by the local compactness of G. That is, given $\varepsilon > 0$, there is a neighborhood U of $1 \in G$ such that $|f(x) - f(x')| < \varepsilon$ for all $x, x' \in G$ with $x' \in xU$. Let U be small-enough so that this holds for two $f, f' \in C_c^o(K)$.

Given x in compact K, let $g' \in gU$. Then

$$|f(xg) - f'(xg')| \;=\; |f(xg) - f(xg')| + |f(xg') - f'(xg')| \;<\; \varepsilon + \varepsilon$$

since $xg' \in x(gN) = (xg)N$ and $\sup_x |f(x) - f'(x)| < \varepsilon$. This proves the continuity.

Remark: This continuity is exactly what is required for the action of G on $C_c^o(G)$ to be a *representation* of G.

For F a continuous $C_c^o(G)$-valued function on G, such as $F(g) = R_g f$, and for $\psi \in C_c^o(G)$, the function-valued integral

$$F \longrightarrow \int_G \psi(g)\, F(g)\, dg$$

is characterized by

$$\lambda\Big(\int_G \psi(g)\, F(g)\, dg\Big) = \int_G \psi(g)\, \lambda\big(F(g)\big)\, dg \qquad \text{(for all } \lambda \in C_c^o(G)^*\text{)}$$

By Hahn-Banach, there is *at most one* such integral: the continuous linear functionals separate points.

Further, granting *existence* of the integral, Hahn-Banach in fact shows that

$$\int_G \psi(g)\, F(g)\, dg \;\; \in \;\; \text{closure of convex hull of } \{F(g) : g \in \text{spt}\psi\}$$

Proposition:

$$\int_G \psi_i(g)\, F(g)\, dg \longrightarrow F(1) \qquad\qquad \text{(in the } C_c^o(G) \text{ topology)}$$

Proof: given $\varepsilon > 0$ and F, let $U \ni 1$ be small-enough so that $|F(x) - F(1)| < \varepsilon$, where $|*|$ is sup-norm on a particular $C_c^o(K)$. Let i be large enough so that the support of ψ_i is inside U. Then

$$F(1) - \int_G \psi_i(g)\, F(g)\, dg \;=\; F(1) \int_G \psi_i(g)\, dg - \int_G \psi_i(g)\, F(g)\, dg$$

$$= \int_G \psi_i(g) \big(F(1) - F(g)\big)\, dg$$

The absolute value estimate, with $|*|$ sup-norm on K, gives

$$\Big|F(1) - \int_G \psi_i(g)\, F(g)\, dg\Big| \;\le\; \int_G \psi_i(g)\, \Big|F(1) - F(g)\Big|\, dg$$

$$< \; \int_G \psi_i(g) \cdot \varepsilon\, dg \;\; = \;\; \varepsilon$$

This is the proposition. ///

Returning to the main thread of the proof, with $F(h) = f(gh)$, for invariant u in $C_c^o(G)^*$, by *continuity* of u,

$$u(f) \;=\; \lim_i \, u\left(g \to \int_G \, \psi_i(h)\, f(gh)\, dh\right)$$

which is

$$\lim_i u\left(g \to \int_G \, f(h)\, \psi_i(g^{-1}h)\, dh\right)$$

replacing h by $g^{-1}h$.

Moving the functional u inside the integral the above becomes

$$u(f) \;=\; \lim_i \int_G f(h)\, u\left(g \to \psi_i(g^{-1}h)\right)\, dh$$

By *left* invariance of u,

$$u(f) \;=\; \lim_i \int_G f(h)\, u(g \to \psi_i(g))\, dh = \lim_i u(\psi_i) \cdot \int_G f(h)\, dh$$

Thus, for f with $\int_G f \neq 0$, $\lim_i u(\psi_i)$ *exists.* We conclude that $u(f)$ is a constant multiple of the indicated integral with *given* invariant measure. ///

Remark: A nearly identical argument proves that G-invariant *distributions* on Lie groups G are unique up to constants, assuming existence.

In summary: On $\mathbb{R}$ and $\mathbb{Q}_p$ and tangible topological groups G it is often easy to give explicit constructions of invariant (Haar) integrals, especially on $C_c^o(G)$. Often, those constructions give *uniqueness.*

The *general* construction/proof-of-existence is reasonable, but the ideas are less re-usable than some.

In contrast, the general *uniqueness* argument is an instance of an important, re-usable proof mechanism, above.

In any case, what was used in Fujisaki's lemma was *existence*, *uniqueness*, and the winding-unwinding property that there is a unique measure on $H\backslash G$ such that

$$\int_G f(g)\,dg \;=\; \int_{H\backslash G} \Big(\int_H f(h\dot{g})\,dh \Big)\,d\dot{g} \qquad\qquad (\text{for } f \in C_c^o(G))$$

under the reasonable hypothesis $\Delta_H = \Delta_G|_H$.

Next: This adelic harmonic analysis is also exactly what is used in Iwasawa-Tate's modernization of Hecke's treatment of zeta functions of all number fields, and *all* L-functions for $GL(1)$.

In addition to invariant measures, we need the general abelian topological group analogue of *characters* $x \to e^{2\pi i x\xi}$ for $\xi \in \mathbb{R}$, on $\mathbb{R}$, and *Fourier transforms* and *inversion*

$$\mathscr{F} f(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i x\xi}\, f(x)\, dx \qquad \text{and} \qquad \mathscr{F}\mathscr{F} f(x) \;=\; f(-x)$$

for nice functions f on $\mathbb{Q}_p$ and $\mathbb{A}$. Naturally, we need the same for all completions k_v and adeles $\mathbb{A}_k$ of number fields. And *adelic Poisson summation*

$$\sum_{x \in k} f(x) \;=\; \sum_{x \in k} \mathscr{F} f(x) \qquad\qquad \text{(for suitable } f \text{ on } \mathbb{A}_k)$$

Granting this and Fujisaki's lemma, the argument will be identical to Riemann's!