Context: Finiteness of class number, Dirichlet’s units theorem, corollaries of Fujisaki (that $\mathbb{J}^1/k^\times$ is compact).

... $\Leftrightarrow$ existence and uniqueness of Haar measure on $\mathbb{A}$ and $\mathbb{A}/k$... compactness of $\mathbb{A}/k$.

... $\Leftrightarrow$ change-of-measure: for idele $\alpha$,

$$\frac{\text{meas}(\alpha E)}{\text{meas}(E)} = |\alpha| \quad \text{(for measurable } E \subset \mathbb{A})$$

Constructed invariant integral on $\mathbb{Q}_p$ by approximating $f$ in $C_c^o(\mathbb{Q}_p)$ by special, continuous simple functions: linear combinations of characteristic functions of sets $p^k\mathbb{Z}_p + y$ for $y \in \mathbb{Q}_p$. 
(Recall) tangible uniqueness: We claim that taking $\text{meas} \left( \mathbb{Z}_p \right) = 1$ and mechanisms as in the construction give the only possible invariant integral/measure on $\mathbb{Q}_p$. Taking advantage of the special features here:

$\mathbb{Z}_p$ is open, so is measurable. It is compact, so its measure is finite. Thus, we can renormalize a given Haar measure $\mu$ so that $\mu(\mathbb{Z}_p) = 1$.

$\mathbb{Z}_p$ is a disjoint union of $p^n$ translates of $p^n \mathbb{Z}_p$, all with the same measure, by translation-invariance, so $\mu(p^n \mathbb{Z}_p) = p^{-n}$. Thus, integrals of the special simple functions are completely determined.

We saw that each $C^\infty_c(p^{-k} \mathbb{Z}_p)$ can be approximated by special simple functions. Positivity/continuity of the invariant integral, this determines integrals of $C^\infty_c(\mathbb{Q}_p)$ completely. \///
Uniqueness by re-usable methods: a topological group $G$ with at least one invariant measure has at most one, up to scalar multiples. The argument is re-usable. For simplicity, suppose $G$ is unimodular, that is, that a left-invariant measure is right-invariant.

Recall that an approximate identity is a sequence $\{\psi_i\}$ of non-negative $\psi_i \in C_c^0(G)$ such that $\int_G \psi_i = 1$ for all $i$, and such that, given a neighborhood $U$ of 1, there is $i_o$ such that for $i \geq i_o$ the support of $\psi_i$ is inside $U$.

**Remark:** This is strictly stronger than requiring that these functions approach the Dirac delta measure in a weak topology.

$R, L$ are the usual right and left translation actions of $G$ on functions $f$ on $G$:

$$R_g f(h) = f(hg) \quad L_g f(h) = f(g^{-1}h)$$
It is a two-epsilon argument, using the *uniform* continuity of continuous functions on compacts, to see that

\[ g \times f \to R_g f \quad g \times f \to L_g f \]

are *continuous* maps \( G \times C_c^\circ(G) \to C_c^\circ(G) \).

*Proof for right translation*: A two-epsilon argument.

The claim is that, given \( \varepsilon > 0 \), there is a neighborhood \( N \) of \( 1 \in G \) and \( \delta > 0 \) such that, for \( g, g' \in G \) with \( g' \in gN \), and

\[ \sup_x |f(x) - f'(x)| < \delta, \]  

we have

\[ \sup_x |f(xg) - f'(xg')| < \varepsilon. \]

\( f \in C_c^\circ(K) \) is *uniformly* continuous, by the same proof as on \( \mathbb{R} \), by the local compactness of \( G \). That is, given \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( 1 \in G \) such that \( |f(x) - f(x')| < \varepsilon \) for all \( x, x' \in G \) with \( x' \in xU \). Let \( U \) be small-enough so that this holds for two \( f, f' \in C_c^\circ(K) \).
Given $x$ in compact $K$, let $g' \in gU$. Then
\[
|f(xg) - f'(xg')| = |f(xg) - f(xg')| + |f(xg') - f'(xg')| < \varepsilon + \varepsilon
\]
since $xg' \in x(gN) = (xg)N$ and $\sup_x |f(x) - f'(x)| < \varepsilon$. This proves the continuity.

**Remark:** This continuity is exactly what is required for the action of $G$ on $C^\infty_c(G)$ to be a representation of $G$.

For $F$ a continuous $C^\infty_c(G)$-valued function on $G$, such as $F(g) = R_g f$, and for $\psi \in C^\infty_c(G)$, the function-valued integral
\[
F \mapsto \int_G \psi(g) F(g) \, dg
\]
is characterized by
\[
\lambda \left( \int_G \psi(g) F(g) \, dg \right) = \int_G \psi(g) \lambda(F(g)) \, dg \quad \text{(for all } \lambda \in C^\infty_c(G)^*)
\]
By Hahn-Banach, there is at most one such integral: the continuous linear functionals separate points.
Further, granting *existence* of the integral, Hahn-Banach in fact shows that

\[ \int_G \psi(g) F(g) \, dg \in \text{closure of convex hull of } \{ F(g) : g \in \text{spt}\psi \} \]

**Proposition:**

\[ \int_G \psi_i(g) F(g) \, dg \longrightarrow F(1) \quad \text{(in the } C_c^o(G) \text{ topology)} \]

**Proof:** given \( \varepsilon > 0 \) and \( F \), let \( U \ni 1 \) be small-enough so that

\[ |F(x) - F(1)| < \varepsilon, \quad \text{where } |*| \text{ is sup-norm on a particular } C_c^o(K). \]

Let \( i \) be large enough so that the support of \( \psi_i \) is inside \( U \). Then

\[
F(1) - \int_G \psi_i(g) F(g) \, dg = F(1) \int_G \psi_i(g) \, dg - \int_G \psi_i(g) F(g) \, dg \\
= \int_G \psi_i(g) (F(1) - F(g)) \, dg
\]
The absolute value estimate, with $| * |$ sup-norm on $K$, gives

$$\left| F(1) - \int_G \psi_i(g) F(g) \, dg \right| \leq \int_G \psi_i(g) \left| F(1) - F(g) \right| \, dg$$

$$< \int_G \psi_i(g) \cdot \varepsilon \, dg \quad = \quad \varepsilon$$

This is the proposition.  

Returning to the main thread of the proof, with $F(h) = f(gh)$, for invariant $u$ in $C_c^0(G)^*$, by continuity of $u$,

$$u(f) = \lim_i u \left( g \rightarrow \int_G \psi_i(h) f(gh) \, dh \right)$$

which is

$$\lim_i u \left( g \rightarrow \int_G f(h) \psi_i(g^{-1}h) \, dh \right)$$

replacing $h$ by $g^{-1}h$. 

///
Moving the functional $u$ inside the integral the above becomes

$$u(f) = \lim_i \int_G f(h) u(g \to \psi_i(g^{-1}h)) \, dh$$

By left invariance of $u$,

$$u(f) = \lim_i \int_G f(h) u(g \to \psi_i(g)) \, dh = \lim_i u(\psi_i) \cdot \int_G f(h) \, dh$$

Thus, for $f$ with $\int_G f \neq 0$, $\lim_i u(\psi_i)$ exists. We conclude that $u(f)$ is a constant multiple of the indicated integral with given invariant measure.

**Remark:** A nearly identical argument proves that $G$-invariant distributions on Lie groups $G$ are unique up to constants, assuming existence.
In summary: On \( \mathbb{R} \) and \( \mathbb{Q}_p \) and tangible topological groups \( G \) it is often easy to give explicit constructions of invariant (Haar) integrals, especially on \( C_c^0(G) \). Often, those constructions give uniqueness.

The general construction/proof-of-existence is reasonable, but the ideas are less re-usable than some.

In contrast, the general uniqueness argument is an instance of an important, re-usable proof mechanism, above.

In any case, what was used in Fujisaki’s lemma was existence, uniqueness, and the winding-unwinding property that there is a unique measure on \( H \setminus G \) such that

\[
\int_G f(g) \, dg = \int_{H \setminus G} \left( \int_H f(h \dot{g}) \, dh \right) \, d\dot{g}
\]

(for \( f \in C_c^0(G) \)) under the reasonable hypothesis \( \Delta_H = \Delta_G|_H \).
Next: This adelic harmonic analysis is also exactly what is used in Iwasawa-Tate’s modernization of Hecke’s treatment of zeta functions of all number fields, and all $L$-functions for $GL(1)$.

In addition to invariant measures, we need the general abelian topological group analogue of characters $x \to e^{2\pi ix}\xi$ for $\xi \in \mathbb{R}$, on $\mathbb{R}$, and Fourier transforms and inversion

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}} e^{-2\pi ix}\xi f(x) \, dx \quad \text{and} \quad \mathcal{F} \mathcal{F} f(x) = f(-x)$$

for nice functions $f$ on $\mathbb{Q}_p$ and $\mathbb{A}$. Naturally, we need the same for all completions $k_v$ and adeles $\mathbb{A}_k$ of number fields. And adelic Poisson summation

$$\sum_{x \in k} f(x) = \sum_{x \in k} \mathcal{F} f(x) \quad \text{(for suitable } f \text{ on } \mathbb{A}_k)$$

Granting this and Fujisaki’s lemma, the argument will be identical to Riemann’s!