**Harmonic analysis**, on $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$, $\mathbb{Q}_p$, $\mathbb{A}$, and $\mathbb{A}_k/k$, key ingredients in Iwasawa-Tate.

Need the abelian topological group analogue of characters $x \to e^{2\pi i x \xi}$ for $\xi \in \mathbb{R}$, $x \in \mathbb{R}$, and Fourier transforms

$$\hat{f}(\xi) = \mathcal{F} f(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) \, dx$$

and Fourier inversion

$$f(x) = \mathcal{F}^{-1} \hat{f}(x) = \int_{\mathbb{R}} e^{2\pi i \xi x} \hat{f}(\xi) \, d\xi$$

for nice functions $f$ on $\mathbb{Q}_p$ and $\mathbb{A}$. Similarly for all completions $k_v$ and adeles $\mathbb{A}_k$ of number fields. And *adelic Poisson summation*

$$\sum_{x \in k} f(x) = \sum_{x \in k} \hat{f}(x)$$  \hspace{1cm} (for suitable $f$ on $\mathbb{A}_k$)
Recap:

**No small subgroups:** The circle $S^1$ has no small subgroups: there is a neighborhood $U$ of the identity $1 \in S^1$ such that the only subgroup inside $U$ is $\{1\}$.

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**Unitary duals of abelian topological groups:** The unitary dual $G^\vee$ of an abelian topological group $G$ is all continuous group homs $G \to S^1$. For example, $\mathbb{R}^\vee \approx \mathbb{R}$, by $\xi \to (x \to e^{i\xi x})$.

**Theorem:** $\mathbb{Q}_p^\vee \approx \mathbb{Q}_p$ and $\mathbb{A}_p^\vee \approx \mathbb{A}$.

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**Remark:** $\mathbb{Z}_p$ as limit and $\mathbb{Q}_p$ as colimit, and $\mathfrak{o}_v$ and $k_v$ similarly in general, are admirably adapted to determine these duals.

**Remark:** Since our model of the topological group $\mathbb{Q}_p$ implicitly specifies more information, namely, the subgroup $\mathbb{Z}_p$, the isomorphisms are canonical. If we only gave the isomorphism class without specifying a compact-open subgroup, the isomorphism would not be canonical, just as the dual vector space to a finite-dimensional vector space $V$ has the same dimension as $V$, but is not canonically isomorphic to $V$. 
Corollary: Given non-trivial $\psi \in \mathbb{Q}_p^\vee$, every other element of $\mathbb{Q}_p^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in \mathbb{Q}_p$. Similarly, given non-trivial $\psi \in \mathbb{A}^\vee$, every other element of $\mathbb{A}^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in \mathbb{A}$.

Remark: This sort of result is already familiar from the analogue for $\mathbb{R}$, that $x \to e^{i\xi x}$ for $\xi \in \mathbb{R}$ are all the unitary characters of $\mathbb{R}$.

Proof: On one hand, it is clear that, for given continuous group hom $\psi : \mathbb{Q}_p \to S^1$ and $\xi \in \mathbb{Q}_p$, the character $x \to \psi(\xi \cdot x)$ is another. Thus, the dual is a $\mathbb{Q}_p$-vectorspace.

On the other hand, in the proof that $\mathbb{Q}_p^\vee \approx \mathbb{Q}_p$, we chose the pairing $\mathbb{Q}_p \times \mathbb{Q}_p^\vee \to \mathbb{C}^\times$, which would determine the isomorphism. Indeed, given $x \in \mathbb{Q}_p$, there is $x' \in p^{-k}\mathbb{Z}$ for some $k \in \mathbb{Z}$, such that $x - x' \in \mathbb{Z}_p$, the standard character is

$$\psi_1(x) = e^{-2\pi ix'}$$

(sign choice for later purposes)
The character $\psi_1$ is trivial on $\mathbb{Z}_p$. For $\xi \in \mathbb{Q}_p$, let

$$\psi_\xi(x) = \psi_1(\xi \cdot x) \quad \text{(for } x, \xi \in \mathbb{Q}_p)$$

For a finite extension $k_v$ of $\mathbb{Q}_p$ (whether or not we know how $k_v$ arises as a completion of a number field), the standard character is described as

$$\psi_\xi(x) = \psi_1(\text{tr}_{k_v/\mathbb{Q}_p}(\xi \cdot x)) \quad \text{(for } x, \xi \in k_v)$$

Since $\text{tr}(\mathcal{O}_v) \subset \mathbb{Z}_p$, certainly $\ker \psi_\xi \supset \xi^{-1}\mathcal{O}_v$.

Occasionally, the kernel of $\psi_\xi$ can be slightly larger than $\xi^{-1}\mathcal{O}_v$. 
Compact-discrete duality

For abelian topological groups $G$, pointwise multiplication makes $\hat{G}$ an abelian group. A reasonable topology on $\hat{G}$ is the compact-open topology, with a sub-basis

$$U = U_{C,E} = \{ f \in \hat{G} : f(C) \subset E \}$$

for compact $C \subset G$, open $E \subset S^1$.

Remark: The reasonable-ness of this topology is utilitarian. For a compact topological space $X$, $C^o(X)$ with the sup-norm is a Banach space. The compact-open topology is the analogue for $C^o(X,Y)$ when $X,Y$ are topological groups. More aspects of this will become clear later.
Granting for now that the compact-open topology makes $\hat{G}$ an abelian (locally-compact, Hausdorff) topological group,

**Theorem:** The unitary dual of a *compact* abelian group is *discrete*. The unitary dual of a *discrete* abelian group is *compact*.

**Proof:** Let $G$ be compact. Let $E$ be a small-enough open in $S^1$ so that $E$ contains no non-trivial subgroups of $G$. Using the compactness of $G$ itself, let $U \subset \hat{G}$ be the open

$$U = \{f \in \hat{G} : f(G) \subset E\}$$

Since $E$ is small, $f(G) = \{1\}$. That is, $f$ is the trivial homomorphism. This proves discreteness of $\hat{G}$ for compact $G$. 
For $G$ discrete, every group homomorphism to $S^1$ is continuous. The space of all functions $G \to S^1$ is the cartesian product of copies of $S^1$ indexed by $G$. By Tychonoff’s theorem, this product is compact. For discrete $X$, the compact-open topology on the space $C^o(X,Y)$ of continuous functions from $X \to Y$ is the product topology on copies of $Y$ indexed by $X$.

The set of functions $f$ satisfying the group homomorphism condition

$$f(gh) = f(g) \cdot f(h) \quad \text{ (for } g, h \in G)$$

is closed, since the group multiplication $f(g) \times f(h) \to f(g) \cdot f(h)$ in $S^1$ is continuous. Since the product is also Hausdorff, $\hat{G}$ is also compact.

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Theorem: \((\mathbb{A}/k)^\sim \approx k\). In particular, given any non-trivial character \(\psi\) on \(\mathbb{A}/k\), all characters on \(\mathbb{A}/k\) are of the form \(x \to \psi(\alpha \cdot x)\) for some \(\alpha \in k\).

Proof: For a (discretely topologized) number field \(k\) with adeles \(\mathbb{A}\), \(\mathbb{A}/k\) is compact, and \(\mathbb{A}\) is self-dual.

Because \(\mathbb{A}/k\) is compact, \((\mathbb{A}/k)^\sim\) is discrete. Since multiplication by elements of \(k\) respects cosets \(x + k\) in \(\mathbb{A}/k\), the unitary dual has a \(k\)-vectorspace structure given by

\[
(\alpha \cdot \psi)(x) = \psi(\alpha \cdot x) \quad (\text{for } \alpha \in k, \ x \in \mathbb{A}/k)
\]

There is no topological issue in this \(k\)-vectorspace structure, because \((\mathbb{A}/k)^\sim\) is discrete. The quotient map \(\mathbb{A} \to \mathbb{A}/k\) gives a natural injection \((\mathbb{A}/k)^\sim \to \hat{\mathbb{A}}\).
Given non-trivial \( \psi \in (\mathbb{A}/k)^\wedge \), the \( k \)-vectorspace \( k \cdot \psi \) inside \( (\mathbb{A}/k)^\wedge \) injects to a copy of \( k \cdot \psi \) inside \( \hat{\mathbb{A}} \approx \mathbb{A} \). Assuming for a moment that the image in \( \mathbb{A} \) is essentially the same as the diagonal copy of \( k \), \( (\mathbb{A}/k)^\wedge/k \) injects to \( \mathbb{A}/k \). The topology of \( (\mathbb{A}/k)^\wedge \) is discrete, and the quotient \( (\mathbb{A}/k)^\wedge/k \) is still discrete. These maps are continuous group homs, so the image of \( (\mathbb{A}/k)^\wedge/k \) in \( \mathbb{A}/k \) is a discrete subgroup of a compact group, so is finite. Since \( (\mathbb{A}/k)^\wedge \) is a \( k \)-vectorspace, \( (\mathbb{A}/k)^\wedge/k \) is a singleton. Thus, \( (\mathbb{A}/k)^\wedge \approx k \), if the image of \( k \cdot \psi \) in \( \mathbb{A} \approx \hat{\mathbb{A}} \) is the usual diagonal copy.

To see how \( k \cdot \psi \) is imbedded in \( \mathbb{A} \approx \hat{\mathbb{A}} \), fix non-trivial \( \psi \) on \( \mathbb{A}/k \), and let \( \psi \) be the corresponding character on \( \mathbb{A} \). The self-duality of \( \mathbb{A} \) is that the action of \( \mathbb{A} \) on \( \hat{\mathbb{A}} \) by \( (x \cdot \psi)(y) = \psi(xy) \) gives an isomorphism. The subgroup \( x \cdot \psi \) with \( x \in k \) is certainly the usual diagonal copy.