Harmonic analysis, on $\mathbb{A}_k/k$, adelic Poisson summation.

**Corollary:** Given non-trivial $\psi \in \mathbb{A}^\vee$, every other element of $\mathbb{A}^\vee$ is of the form $x \rightarrow \psi(\xi \cdot x)$ for some $\xi \in \mathbb{A}$.

The **standard character** $\psi_1$ on $\mathbb{Q}_p$ is as follows: given $x \in \mathbb{Q}_p$, there is $x' \in p^{-k}\mathbb{Z}$ for some $k \in \mathbb{Z}$, such that $x - x' \in \mathbb{Z}_p$, and

$$\psi_1(x) = e^{-2\pi ix'} \quad \text{(sign choice for later)}$$

For $\xi \in \mathbb{Q}_p$, let

$$\psi_\xi(x) = \psi_1(\xi \cdot x) \quad \text{ (for } x, \xi \in \mathbb{Q}_p)$$

For a finite extension $k_v$ of $\mathbb{Q}_p$, the **standard character** is

$$\psi_\xi(x) = \psi_1\left(\operatorname{tr}_{\mathbb{Q}_p}^{k_v}(\xi \cdot x)\right) \quad \text{ (for } x, \xi \in k_v)$$

Probably use these without further comment.
Fourier transform on archimedean or $p$-adic $k_v$ is

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{k_v} \overline{\psi}_\xi(x) f(x) \, dx$$

Fourier inversion

$$f(x) = \int_{k_v} \psi_\xi(x) \hat{f}(\xi) \, d\xi$$

(for nice functions $f$)

The usual space $\mathcal{S}(\mathbb{R})$ of Schwartz functions on $\mathbb{R}$ consists of infinitely-differentiable functions all of whose derivatives are of rapid decay, decaying more rapidly at $\pm \infty$ than every $1/|x|^N$. Its topology is given by semi-norms

$$\nu_{k,N}(f) = \sup_{0 \leq i \leq k} \sup_{x \in \mathbb{R}} \left( (1 + |x|)^N \cdot |f^{(i)}(x)| \right)$$

for $0 \leq k \in \mathbb{Z}$ and $0 \leq N \in \mathbb{Z}$.

**Theorem:** $\mathcal{F}$ is a topological isomorphism $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$. 
$p$-adic Fourier transforms, inversions:

**Claim:** The characteristic function of $\mathbb{Z}_p$ is its own Fourier transform.

**Cancellation Lemma:** For continuous group homomorphism $\psi : K \to \mathbb{C}^\times$ on a compact group $K$,

$$\int_K \psi(x) \, dx = \begin{cases} \text{meas}(K) & (\text{for } \psi = 1) \\ 0 & (\text{for } \psi \neq 1) \end{cases}$$

**Claim:** Characteristic function of $p^k\mathbb{Z}_p$ is $p^{-k}$ times the characteristic function of $p^{-k}\mathbb{Z}_p$.

**Claim:** Characteristic function of $\mathbb{Z}_p + y$ is $\psi_y$ times the characteristic function of $\mathbb{Z}_p$. 

///
Combining the two computations above,
\[
\mathcal{F}\left(\text{char fcn } p^k\mathbb{Z}_p + y\right) = \psi_y \cdot p^{-k} \cdot (\text{char fcn } p^{-k}\mathbb{Z}_p)
\]
Conveniently, products \(\psi_y \cdot (\text{char fcn } p^{-k}\mathbb{Z}_p)\) are in the same class of functions, since \(\psi_y\) has a kernel which is an open (and compact) neighborhood of 0, so \textit{Fourier transform sends this class of functions is mapped to itself under Fourier transform}.

\textbf{Schwartz functions} \(\mathcal{S}(\mathbb{Q}_p)\) on \(\mathbb{Q}_p\) are these \textit{special simple functions}, that is, finite linear combinations of characteristic functions of sets \(p^k\mathbb{Z}_p + y\).

\textit{p-adic Fourier inversion}:
\[
f(x) = \int_{\mathbb{Q}_p} \psi_\xi(x) \hat{f}(\xi) \, d\xi \quad \text{(for } f \in \mathcal{S}(\mathbb{Q}_p))
\]
Thus, \(\mathcal{F} : \mathcal{S}(\mathbb{Q}_p) \to \mathcal{S}(\mathbb{Q}_p)\) is a \textit{bijection}.

Earlier, we proved that \(\mathcal{S}(\mathbb{Q}_p)\) is \textit{dense} in \(C^0_c(\mathbb{Q}_p)\).
Schwartz functions $\mathcal{S}(\mathbb{A})$ on the adeles are finite linear combinations of monomial functions

$$\left( \bigotimes_{v \leq \infty} f_v \right)(\{x_v\}) = \prod_v f_v(x_v)$$

with $f_v \in \mathcal{S}(\mathbb{Q}_v)$, and where for all but finitely-many $v$ the local function $f_v$ is the characteristic function of $\mathbb{Z}_v$.

Fourier transform on $\mathcal{S}(\mathbb{A})$ is the product of all the local Fourier transforms, and Fourier inversion follows for $\mathcal{S}(\mathbb{A})$ because it holds for each $\mathcal{S}(\mathbb{Q}_v)$.

Identical definitions and properties apply to all number fields $k$, their completions $k_v$, and adeles $\mathbb{A} = \mathbb{A}_k$, with nearly identical proofs.
The harmonic analysis on $\mathbb{R}$ really is parallel to that on $\mathbb{Q}_p$ and $\mathbb{A}$ in many regards. For example, 

**Plancherel theorem:** As on $\mathbb{R}$, $\int_{\mathbb{Q}_p} \hat{f} \cdot \hat{g} = \int_{\mathbb{Q}_p} f \cdot g$ for $f, g \in \mathcal{S}(\mathbb{Q}_p)$.

**Proof:** The key point is the surjectivity of $\mathcal{F} : \mathcal{S}(\mathbb{Q}_p) \to \mathcal{S}(\mathbb{Q}_p)$:

\[
\int_{\mathbb{Q}_p} f \cdot g = \int_{\mathbb{Q}_p} f \cdot \mathcal{F}^{-1} \hat{g} = \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} f(x) \cdot \psi_1(-\xi x) \cdot \hat{g}(\xi) \, d\xi \, dx
\]

\[
= \int_{\mathbb{Q}_p} \left( \int_{\mathbb{Q}_p} f(x) \cdot \psi_1(-\xi x) \, dx \right) \cdot \hat{g}(\xi) \, d\xi = \int_{\mathbb{Q}_p} \hat{f} \cdot \hat{g}
\]

This is the same proof as for $\mathbb{R}$, and also applies to $\mathbb{A}$. \[///\]

Then $\mathcal{F}$ is extended to $L^2(\mathbb{Q}_p)$ by continuity, giving the *Fourier-Plancherel* transform, no longer defined literally by the integrals.
**Fourier series on** $\mathbb{A}/k$: For a unimodular topological group $G$, let $L^2(G)$ be the completion of $C_c^\infty(G)$ with respect to the usual $L^2$-norm given by

$$|f|^2 = \int_G |f(g)|^2 \, dg \quad \text{(for } f \in C_c^\infty(G))$$

**Remark:** The measurable-function version of $L^2(G)$ contains this completion, and is provably equal, but we only need integrals of continuous compactly-supported functions.

The usual *inner product* is

$$\langle f, F \rangle = \int_G f \cdot \overline{F}$$

As usual, the completeness makes $L^2(G)$ a *Hilbert space*.

**Remark:** Defining or characterizing $L^2(G)$ as the completion of $C_c^\infty(G)$ makes it complete. In contrast, giving $L^2(G)$ as the collection of measurable functions meeting a condition leaves us needing to prove completeness.
(big) **Theorem:** For a *compact abelian* group $G$, with total measure 1, the continuous group homomorphisms (*characters*) $\psi: G \to \mathbb{C}^\times$ form an orthonormal *Hilbert-space basis* for $L^2(G)$. That is,

$$L^2(G) = \text{completion of } \bigoplus_{\psi \in G^\vee} \mathbb{C} \cdot \psi$$

and

$$f = \sum_{\psi \in G^\vee} \langle f, \psi \rangle \cdot \psi \quad \text{(for } f \in L^2(G), \text{ convergence in } L^2(G))$$

**Remark:** This applies to the circle $\mathbb{R}/\mathbb{Z}$!

**Remark:** Recall that a *Hilbert-space* basis of a Hilbert space $V$ is not a *vector-space* basis for $V$, but for a dense subspace.

**Remark:** For *finite* abelian groups, this follows from the spectral theorem for commuting *unitary* operators on finite-dimensional $\mathbb{C}$-vectorspaces. (See 2010-11 notes.)

**Remark:** As in the elementary example of the circle $\mathbb{R}/\mathbb{Z}$, convergence in $L^2$ says nothing directly about *pointwise* convergence, much less *uniform* pointwise convergence.
Proof: Orthonormality is easy: for $\psi \neq \varphi$ characters,

$$\langle \psi, \varphi \rangle = \int_G \psi(g) \cdot \overline{\varphi(g)} \, dg = \int_G \psi \varphi^{-1}(g) \, dg$$

By the cancellation lemma, this is 0 for $\psi \neq \varphi$.

Completeness is more serious. We must prove existence of sufficiently many eigenvectors for the action of $G$ on complex-valued functions

$$g \cdot f(x) = f(xg) \quad \text{(for } f \in C_c^0(G) \text{ and } x, g \in G)$$

For $f$ to be an eigenfunction means that

$$g \cdot f = \lambda_f(g) \cdot f \quad \text{(for all } g \in G, \text{ with } \lambda_f(g) \in \mathbb{C})$$

The unitariness is

$$\langle g \cdot f, g \cdot F \rangle = \int_G f(xg) \overline{\varphi(xg)} \, dx = \int_G f(x) \overline{\varphi(x)} \, dx = \langle f, F \rangle$$
The eigenvalues $\lambda_f(g)$ cannot be unrelated: for $g, h \in G$,

$$\lambda_f(gh) \cdot f = (gh) \cdot f = g \cdot (h \cdot f) = g \cdot (\lambda_f(h) f)$$

$$= \lambda_f(h) g \cdot f = \lambda_f(h) \lambda_f(g) f$$

so $\lambda_f : G \to \mathbb{C}^\times$ is a group homomorphism.

For $G$ finite, $L^2(G)$ is finite-dimensional. By finite-dimensional spectral theory for unitary operators, $L^2(G)$ is a direct sum of eigenspaces $V_\lambda$, for group homomorphism $\lambda : G \to \mathbb{C}^\times$.

Each eigenfunction $f$ is itself a constant multiple of a group homomorphism $G \to \mathbb{C}^\times$:

$$f(x) = f(1 \cdot x) = \lambda_f(x) f(1)$$

If $\lambda_f = \lambda_F$, with normalization $f(1) = 1 = F(1)$,

$$f(x) = f(1 \cdot x) = \lambda_f(x) f(1) = \lambda_F(x) F(1) = F(x)$$

That is, each $\lambda_f$ occurs with multiplicity one.
Certainly every group homomorphism $G \rightarrow \mathbb{C}^\times$ is a complex-valued function on finite $G$, so

$$L^2(G) = \bigoplus_{\psi \in G^\vee} \mathbb{C} \cdot \psi \quad (G \text{ finite abelian})$$

We did not use the structure theorem for finite abelian groups.

On infinite-dimensional Hilbert spaces, even for unitary operators, general spectral theory does not guarantee eigenvectors.

From a spectral viewpoint, the best operators on infinite-dimensional Hilbert spaces are self-adjoint compact operators.

The self-adjointness is the usual $\langle Tv, w \rangle = \langle v, Tw \rangle$.

The compactness is that the image $TB$ of the unit ball $B$ has compact closure. Thus, the image $\{Tv_i\}$ of a bounded sequence $\{v_i\}$ has a convergent subsequence $\{Tv_{i_k}\}$.

On finite-dimensional vector spaces, every linear operator is compact.
One of the most useful theorems in the universe:

**Theorem:** Let $R$ be a set of compact, self-adjoint, mutually commuting operators on a Hilbert space $V$. Suppose the action is *non-degenerate* in the sense that for $0 \neq v \in V$ there is $T \in R$ with $Tv \neq 0$. Then $V$ has an *orthonormal* Hilbert-space basis of *simultaneous eigenvectors* for $R$. The joint eigenspaces are finite-dimensional.

[The simple proof is below. Other useful details arise.]

*Where do compact operators come from?*

From *integral operators*, sometimes misleadingly called *convolution operators*. This misnomer is understandable, but does make less intelligible what’s going on.

A function $\eta \in C^\infty_c(G)$ acts on $L^2(G)$ by the integral operator

$$(\eta \cdot f)(x) = \int_G \eta(g) f(xg) \, dg$$
There is the compatibility
\[
\alpha \cdot (\beta \cdot f)(x) = \int_G \int_G \alpha(h) \beta(g) f(xhg) \, dg \, dh
\]
\[
= \int_G \left( \int_G \alpha(hg^{-1}) \beta(g) \, dg \right) f(xh) \, dh
\]
\[
= \int_G (\alpha * \beta)(h) f(xh) \, dh = ((\alpha * \beta) \cdot f)(x)
\]

That \( \alpha * \beta \) is convolution, indeed, but the action on a vector space on which \( G \) acts is much more general than convolution. Further, there is a discrepancy of inverse-or-not if we try to force the action of \( C_c^\circ(G) \) on \( L^2(G) \) to be convolution.

An innocent change of variables gives
\[
(\alpha \cdot f)(x) = \int_G \alpha(y) f(xy) \, dy = \int_G \alpha(x^{-1}y) f(y) \, dy
\]
Write \( K(x, y) = \alpha(x^{-1}y) \) to suggest viewing \( \alpha(x^{-1}y) \) as a kernel for an integral operator, analogous to a matrix, but indexed by \( x, y \in G \).
Claim: For topological spaces $X, Y$ with nice measures, for $K(x, y) \in C_c^0(X \times Y)$, the linear operator $T : L^2(Y) \to L^2(X)$ by

$$Tf(x) = \int_Y K(x, y) f(y) \, dy$$

is compact. For $X = Y$ and $K(y, x) = \overline{K(x, y)}$, the operator $T$ is self-adjoint.

Remark: Fredholm, Volterra, Hilbert, Riesz, and others inverted certain ordinary differential operators (Sturm-Liouville problems) to integral operators, which happened to be compact, thus giving a basis of eigenfunctions, enabling solution of such problems.

Remark This same strategy applies to compact $G$ that are not necessarily abelian, to decompose $L^2(G)$ into irreducible representations, although most of the irreducibles are not one-dimensional, not spanned by group homomorphisms $G \to \mathbb{C}^\times$. Even for $G$ non-compact, non-abelian, for discrete subgroups $\Gamma$ with $\Gamma \backslash G$ compact, the same mechanism decomposes $L^2(\Gamma \backslash G)$. 
Proof of spectral theorem for compact self-adjoint operators: The key point of the theorem above is the spectral theorem for a single self-adjoint compact operator $T : V \to V$.

**Lemma:** A continuous self-adjoint operator $T$ on a Hilbert space $V$ has operator norm $|T| = \sup_{|v| \leq 1} |Tv|$ expressible as

$$|T| = \sup_{|v| \leq 1} |\langle Tv, v \rangle|$$

Proof of Lemma: On one hand, certainly $|\langle Tv, v \rangle| \leq |Tv| \cdot |v|$, giving the easy direction of inequality.

On the other hand, let $\sigma = \sup_{|v| \leq 1} |\langle Tv, v \rangle|$. A polarization identity gives

$$2\langle Tv, w \rangle + 2\langle Tw, v \rangle = \langle T(v + w), v + w \rangle − \langle T(v − w), v − w \rangle$$

With $w = t \cdot Tv$ with $t > 0$, since $T = T^*$, both $\langle Tv, w \rangle$ and $\langle Tw, v \rangle$ are non-negative real. Taking absolute values,
\[
4\langle Tv, t \cdot Tv \rangle = \sigma \cdot |v + t \cdot Tv|^2 + \sigma \cdot |v - t \cdot Tv|^2
\]

\[
= \left| \langle T(v + t \cdot Tv), v + t \cdot Tv \rangle - \langle T(v - t \cdot Tv), v - t \cdot Tv \rangle \right|
\]

\[
\leq \sigma \cdot |v + t \cdot Tv|^2 + \sigma \cdot |v - t \cdot Tv|^2 = 4\sigma \cdot (|v|^2 + t^2 \cdot |Tv|^2)
\]

Divide through by 4t and set \( t = \frac{|v|}{|Tv|} \) to minimize the right-hand side, obtaining

\[
|Tv|^2 \leq \sigma \cdot |v| \cdot |Tv|
\]

giving the other inequality, proving the Lemma. ///

**Key Lemma:** A compact self-adjoint operator \( T \) has largest eigenvalue \( \pm |T| \).
**Proof of Key Lemma:** Take $|T| > 0$, or else $T = 0$. Using the characterization of operator norm, let $v_i$ be a sequence of unit vectors such that $|\langle Tv_i, v_i \rangle| \to |T|$. On one hand, using $\langle Tv, v \rangle = \langle v, Tv \rangle = \langle Tv, v \rangle$,

$$0 \leq |Tv_i - \lambda v_i|^2 = |Tv_i|^2 - 2\lambda \langle Tv_i, v_i \rangle + \lambda^2 |v_i|^2$$

$$\leq \lambda^2 - 2\lambda \langle Tv_i, v_i \rangle + \lambda^2$$

By assumption, the right-hand side goes to 0. Using compactness, replace $v_i$ with a subsequence such that $Tv_i$ has limit $w$. Then the inequality shows that $\lambda v_i \to w$, so $v_i \to \lambda^{-1}w$. Thus, by continuity of $T$, $Tw = \lambda w$. ///

The commutativity of the set $R$ of operators ensures that the operators stabilize each others’ eigenspaces. The non-degeneracy ensures that the orthogonal complement of all the joint eigenspaces is $\{0\}$. ///

Garrett 02-22-2012 17
Next, prove that $K(x, y) \in C_c^o(X \times Y)$ gives a compact operator...