Harmonic analysis, on $\mathbb{A}_k/k$, adelic Poisson summation.

Theorem: Fourier transform is a topological isomorphism $\mathscr{S}(k_v) \to \mathscr{S}(k_v)$ and $\mathscr{S}(\mathbb{A}_k) \to \mathscr{S}(\mathbb{A}_k)$ for number fields k, completions k_v whether archimedean or p-adic, and adeles $\mathbb{A}_k$.

Plancherel: Fourier transform is an L^2-isometry on Schwartz functions.

Then Fourier transforms are extended to $L^2(k_v)$ and $L^2(\mathbb{A})$ *by continuity*, giving the *Fourier-Plancherel* transform, no longer defined literally by the integrals.

Fourier series on $\mathbb{A}/k$: For a unimodular topological group G, let $L^2(G)$ be the *completion* of $C_c^o(G)$ with respect to the usual L^2-norm given by

$$|f|^2 = \int_G |f(g)|^2 \, dg \qquad \text{(for } f \in C_c^o(G)\text{)}$$

and usual *inner product*

$$\langle f, F \rangle = \int_G f \cdot \overline{F}$$

(big) Theorem: For a *compact abelian* group G, with total measure 1, the continuous group homomorphisms (*characters*) $\psi : G \to \mathbb{C}^\times$ form an orthonormal *Hilbert-space basis* for $L^2(G)$. That is,

$$L^2(G) = \text{completion of} \bigoplus_{\psi \in G^\vee} \mathbb{C} \cdot \psi$$

and

$$f = \sum_{\psi \in G^\vee} \langle f, \psi \rangle \cdot \psi \qquad \text{(for } f \in L^2(G)\text{, convergence in } L^2(G)\text{)}$$

Remark: As in the elementary example of the circle $\mathbb{R}/\mathbb{Z}$ and classical Fourier series, convergence in L^2 says little directly about *pointwise* convergence, much less *uniform* pointwise convergence.

Proof of big Theorem: Recap so far: orthonormality follows immediately from the *cancellation lemma.* This is the trivial half.

Completeness requires existence of sufficiently many *eigenvectors* for the action of G on complex-valued functions

$$g \cdot f(x) \;=\; f(xg) \qquad\qquad (\text{for } f \in C_c^o(G) \text{ and } x, g \in G)$$

The eigenvalues $\lambda_f(g)$ are *group homomorphisms*: for $g, h \in G$,

$$\lambda_f(gh) \cdot f \;=\; (gh) \cdot f \;=\; g \cdot (h \cdot f) \;=\; g \cdot (\lambda_f(h)\, f)$$

$$=\; \lambda_f(h)\, g \cdot f \;=\; \lambda_f(h)\, \lambda_f(g)\, f$$

For G *finite*, $L^2(G)$ is finite-dimensional. By finite-dimensional spectral theory for *unitary* operators, [*we saw*]

$$L^2(G) \;=\; \bigoplus_{\psi \in G^\vee} \mathbb{C} \cdot \psi \qquad\qquad (G \text{ finite abelian})$$

We did *not* use the structure theorem for finite abelian groups.

On infinite-dimensional Hilbert spaces, even for *unitary* operators, general spectral theory does *not* guarantee *eigenvectors.*

From a spectral viewpoint, the best operators on infinite-dimensional Hilbert spaces are *self-adjoint compact* operators.

The *self-adjointness* is the usual $\langle Tv, w\rangle = \langle v, Tw\rangle$.

The *compactness* is that the image TB of the unit ball B has *compact closure.* Thus, the image $\{Tv_i\}$ of a *bounded* sequence $\{v_i\}$ has a *convergent subsequence* $\{Tv_{i_k}\}$.

On finite-dimensional vector spaces, *every* linear operator is compact.

One of the most useful theorems in the universe:

Theorem: Let R be a set of compact, self-adjoint, mutually commuting operators on a Hilbert space V. Suppose the action is *non-degenerate* in the sense that for $0 \neq v \in V$ there is $T \in R$ with $Tv \neq 0$. Then V has an *orthonormal* Hilbert-space basis of *simultaneous eigenvectors* for R. The joint eigenspaces are finite-dimensional.

[Simple proof is below. Other useful details arise.]

Mostly, compact operators come from *integral operators*: η in $C_c^o(G)$ acts on $L^2(G)$ by the integral operator (*right averaging*)

$$(\eta \cdot f)(x) \;=\; \int_G \eta(g)\, f(xg)\, dg$$

There is the compatibility

$$\alpha \cdot (\beta \cdot f) \;=\; (\alpha * \beta) \cdot f$$

A change of variables gives

$$(\alpha \cdot f)(x) \;=\; \int_G \alpha(y)\, f(xy)\, dy \;=\; \int_G \alpha(x^{-1}y)\; f(y)\, dy$$

Write $K(x,y) = \alpha(x^{-1}y)$ to suggest viewing $\alpha(x^{-1}y)$ as a *kernel* for an *integral operator*, analogous to a *matrix*, but indexed by $x, y \in G$: it defines a linear operator $T : L^2(G) \to L^2(G)$ by

$$Tf(x) \;=\; (\alpha \cdot f)(x) \;=\; \int_G K(x,y)\, f(y)\, dy \qquad\qquad (\text{for } f \in L^2(G))$$

Claim: For locally compact Hausdorff topological spaces X, Y with nice measures, for $K(x,y) \in C_c^o(X \times Y)$, the linear operator $T : L^2(Y) \to L^2(X)$ by

$$Tf(x) \;=\; \int_Y K(x,y)\, f(y)\, dy$$

is *compact*. For $X = Y$ and $K(y,x) = \overline{K(x,y)}$, the operator T is *self-adjoint*.

Remark Invocation of the spectral theory of compact self-adjoint operators applies to compact G that are not necessarily *abelian*, to decompose $L^2(G)$ into *irreducible representations*, although most of the irreducibles are not one-dimensional, *not* spanned by group homomorphisms $G \to \mathbb{C}^\times$. Even for G *non-compact, non-abelian*, for discrete subgroups Γ with $\Gamma\backslash G$ *compact*, the same mechanism decomposes $L^2(\Gamma\backslash G)$.

Specifically, now the *left* and *right* actions of G on itself, and, therefore, on $L^2(G)$,

$$L_g f(x) \;=\; f(g^{-1}x) \qquad\qquad R_g(x) \;=\; f(xg)$$

are not identical. That is, it is really $G \times G$ which acts. The decomposition of $L^2(G)$ for non-commutative but still *compact* G is the natural extension of the classical theorem for finite groups and characteristic 0 representations over algebraically closed fields:

$$L^2(G) \;=\; \text{completion of} \bigoplus_{\text{irreds } \pi \text{ of } G} \pi \otimes \pi^\vee \qquad\qquad (\text{as repns of } G \times G)$$

Proof of spectral theorem for commuting compact self-adjoint operators: The key point is the already-useful spectral theorem for a *single* self-adjoint compact operator $T : V \to V$. To prove this, we need

Slightly Clever Lemma: The *operator norm* $|T| = \sup_{|v|\le 1} |Tv|$ of continuous *self-adjoint* operator T on a Hilbert space V is expressible as

$$|T| \;=\; \sup_{|v|\le 1} |\langle Tv, v\rangle|$$

Proof of Lemma: On one hand, by Cauchy-Schwarz-Bunyakowsky, $|\langle Tv, v\rangle| \le |Tv| \cdot |v|$, giving the easy direction of inequality.

On the other hand, let $\sigma \;=\; \sup_{|v|\le 1} |\langle Tv, v\rangle|$. A polarization identity gives

$$2\langle Tv, w\rangle + 2\langle Tw, v\rangle \;=\; \langle T(v+w), v+w\rangle - \langle T(v-w), v-w\rangle$$

With $w \;=\; t \cdot Tv$ with $t \;>\; 0$, since $T \;=\; T^*$, both $\langle Tv, w\rangle$ and $\langle Tw, v\rangle$ are non-negative real. Taking absolute values, we have

$$4\langle Tv, t\cdot Tv\rangle \;=\; \Big|\langle T(v+t\cdot Tv), v+t\cdot Tv\rangle - \langle T(v-t\cdot Tv), v-t\cdot Tv\rangle\Big|$$

$$\le\; \sigma\cdot|v+t\cdot Tv|^2 + \sigma\cdot|v-t\cdot Tv|^2 \;=\; 4\sigma\cdot\big(|v|^2+t^2\cdot|Tv|^2\big)$$

Divide through by $4t$ and set $t = |v|/|Tv|$ to minimize the right-hand side, obtaining

$$|Tv|^2 \;\le\; \sigma\cdot|v|\cdot|Tv|$$

giving the other inequality, proving the Lemma. ///

Key Lemma: A compact self-adjoint operator T has largest eigenvalue $\pm|T|$.

Proof of Key Lemma: Take $|T| > 0$, or else $T = 0$. Using the re-characterization of operator norm, let v_i be a sequence of unit vectors such that $|\langle Tv_i, v_i\rangle| \to |T|$. Let λ be $\pm|T|$ such that there is an infinite subsequence with $\langle Tv_{i_k}, v_{i_k}\rangle \to \lambda$, and replace v_i by this subsequence. On one hand, using $\langle Tv, v\rangle = \langle v, Tv\rangle$,

$$0 \le |Tv_i - \lambda v_i|^2 = |Tv_i|^2 - 2\lambda\langle Tv_i, v_i\rangle + \lambda^2|v_i|^2$$

$$\le \lambda^2 - 2\lambda\langle Tv_i, v_i\rangle + \lambda^2$$

By assumption, the right-hand side goes to 0. Using compactness, replace v_i with a subsequence such that Tv_i has limit w. Then the inequality shows that $\lambda v_i \to w$, so $v_i \to \lambda^{-1}w$. Thus, by continuity of T, $Tw = \lambda w$. This proves the key lemma. ///

Spectral theorem: for a *single* self-adjoint compact operator T... the non-zero eigenvalues are *real*, have no accumulation point but $\{0\}$, and multiplicities are finite. For $0 \ne \lambda \in \mathbb{C}$ not among the eigenvalues, $T - \lambda$ is *invertible* (as continuous linear operator).

Remark: The latter point is that indispensable, since in general $T - \lambda$ could fail to be invertible *without* λ being an eigenvalue. This would entail some trouble, since there could not possibly be a basis of eigenvectors.

Proof of theorem for single operator: In part, this is similar to the proof for self-adjoint operators on *finite*-dimensional spaces.

If $|T| = 0$, then $T = 0$. Otherwise, the key lemma gives a non-zero eigenvalue. The orthogonal complement of the corresponding eigenvector v is T-stable: for $w \perp v$,

$$\langle v, Tw \rangle = \langle Tv, w \rangle = \lambda \langle v, w \rangle = 0 \qquad (\text{for } Tv = \lambda v \text{ and } \langle v, w \rangle = 0)$$

The restriction of T to that orthogonal complement is still compact (!), so unless that restriction is 0, T has a non-zero eigenvalue there, too. Continue...

For $\lambda \neq 0$, the λ-eigenspace being infinite-dimensional would contradict the compactness of T: the unit ball in an infinite-dimensional inner-product space is not compact, as any infinite orthonormal set is a sequence with no convergent subsequence.

Similarly, for $c > 0$, the set of eigenvalues (counting multiplicities) larger than c being infinite would contradict compactness.

Thus, 0 is the only limit-point of eigenvalues. ...