Harmonic analysis, on $\mathbb{A}_k/k$, adelic Poisson summation.

Theorem: Fourier transform is a topological isomorphism $\mathcal{S}(k_v) \to \mathcal{S}(k_v)$ and $\mathcal{S}(\mathbb{A}_k) \to \mathcal{S}(\mathbb{A}_k)$.

Plancherel: Fourier transform is an $L^2$-isometry on Schwartz functions.

(big) Theorem: For a compact abelian group $G$, with total measure 1, the continuous group homomorphisms (characters) $\psi : G \to \mathbb{C}^\times$ form an orthonormal Hilbert-space basis for $L^2(G)$. That is,

$$L^2(G) = \text{completion of } \bigoplus_{\psi \in G^\vee} \mathbb{C} \cdot \psi$$

and

$$f = \sum_{\psi \in G^\vee} \langle f, \psi \rangle \cdot \psi \quad \text{(for } f \in L^2(G), \text{ convergence in } L^2(G))$$
Proof of big Theorem: so far: orthonormality is immediate from cancellation lemma.

Completeness requires existence of sufficiently many eigenvectors... for the translation action of $G$ on complex-valued functions

For $G$ finite, by finite-dimensional spectral theory for unitary operators, [we saw]

$$L^2(G) = \bigoplus_{\psi \in G^\vee} \mathbb{C} \cdot \psi$$

(G finite abelian)

We did not use the structure theorem for finite abelian groups.

The best operators on infinite-dimensional Hilbert spaces are self-adjoint compact operators.

Compactness is that the image $TB$ of the unit ball $B$ has compact closure. Thus, the image $\{Tv_i\}$ of a bounded sequence $\{v_i\}$ has a convergent subsequence $\{Tv_{i_k}\}$.
One of the most useful theorems in the universe:

**Theorem:** Let $R$ be a set of compact, self-adjoint, mutually commuting operators on a Hilbert space $V$. Suppose the action is *non-degenerate* in the sense that for $0 \neq v \in V$ there is $T \in R$ with $Tv \neq 0$. Then $V$ has an orthonormal Hilbert-space basis of *simultaneous eigenvectors* for $R$. The joint eigenspaces are finite-dimensional.

[Simple proof below]

Mostly, compact operators come from integral operators attached to $\eta$ in $C_c^\alpha(G)$, acting on $L^2(G)$ by

$$(\eta \cdot f)(x) = \int_G \eta(g) f(xg) \, dg$$
A change of variables gives
\[(\alpha \cdot f)(x) = \int_G \alpha(y) f(xy) \, dy = \int_G \alpha(x^{-1}y) f(y) \, dy\]

Write \(K(x, y) = \alpha(x^{-1}y)\). It defines a linear operator
\(T : L^2(G) \to L^2(G)\) defined by

\[Tf(x) = (\alpha \cdot f)(x) = \int_G K(x, y) f(y) \, dy \quad \text{for } f \in L^2(G)\]

**Claim:** For locally compact Hausdorff topological spaces \(X, Y\) with nice measures, for \(K(x, y) \in C_c^0(X \times Y)\), the linear operator
\(T : L^2(Y) \to L^2(X)\) by

\[Tf(x) = \int_Y K(x, y) f(y) \, dy\]

is compact. For \(X = Y\) and \(K(y, x) = \overline{K(x, y)}\), \(T\) is self-adjoint.
Proof of spectral theorem for commuting compact self-adjoint operators: The key point is the spectral theorem for a single self-adjoint compact operator $T : V \to V$. We need

**Slightly Clever Lemma:** The operator norm $|T| = \sup_{|v| \leq 1} |Tv|$ of continuous self-adjoint operator $T$ on a Hilbert space $V$ is expressible as

$$|T| = \sup_{|v| \leq 1} |\langle Tv, v \rangle|$$

**Key Lemma:** A compact self-adjoint operator $T$ has largest eigenvalue $\pm |T|$. 

Spectral theorem: for a single self-adjoint compact operator $T$... the non-zero eigenvalues are real, have no accumulation point but $\{0\}$, and multiplicities are finite. For $0 \neq \lambda \in \mathbb{C}$ not among the eigenvalues, $T - \lambda$ is invertible (as continuous linear operator).

Proof of theorem for single operator: In part, this is similar to the proof for self-adjoint operators on finite-dimensional spaces.

If $|T| = 0$, then $T = 0$. Otherwise, the key lemma gives a non-zero eigenvalue. The orthogonal complement of the corresponding eigenvector $v$ is $T$-stable: for $w \perp v$,

$$\langle v, Tw \rangle = \langle Tv, w \rangle = \lambda \langle v, w \rangle = 0 \quad \text{(for } Tv = \lambda v \text{ and } \langle v, w \rangle = 0)$$

The restriction of $T$ to that orthogonal complement is still compact (!), so unless that restriction is 0, $T$ has a non-zero eigenvalue there, too. Continue...
For $\lambda \neq 0$, the $\lambda$-eigenspace being infinite-dimensional would contradict the compactness of $T$: the unit ball in an infinite-dimensional inner-product space is not compact, as any infinite orthonormal set is a sequence with no convergent subsequence.

Similarly, for $c > 0$, the set of eigenvalues (counting multiplicities) larger than $c$ being infinite would contradict compactness.

Thus, $0$ is the only limit-point of eigenvalues.

Finally, the restriction of $T$ to the orthogonal complement of the sum of all its non-zero eigenspaces is still compact. If its operator norm were positive, there would be a further non-zero eigenvalue, contradiction. Thus, that restriction has $0$ norm, so is $0$. This proves the spectral theorem for a single self-adjoint compact operator.
For the commuting family of operators: as usual, the commutativity ensures that the operators stabilize each others’ eigenspaces: for $v$ a $\lambda$-eigenvalue for $T$, for another operator $S$,

$$T(Sv) = (TS)v = (ST)v = S(Tv) = S(\lambda v) = \lambda \cdot Sv$$

The non-degeneracy ensures that the orthogonal complement of all the joint eigenspaces is \{0\}.

\textbf{Remark:} For proving existence of eigenfunctions, there really is no alternative to self-adjoint compact operators. Meanwhile, compact operators have been understood, in terms appropriate for the time, for at least 120 years.

\textbf{Claim:} Hilbert-Schmidt operators $K(x, y) \in C_c(X \times Y)$ give compact operators $T : L^2(Y) \to L^2(X)$ by

$$Tf(x) = \int_Y K(x, y) f(y) \, dy$$
Remark: The class of Hilbert-Schmidt operators often is taken to include not only operators with kernels in $C^0(X \times Y)$, but also kernels in $L^2(X \times Y)$. In practice, usually kernels are in $L^2$ because they are in $C^0$.

Remark: In fact, the Schwartz Kernel Theorem shows that continuous operators from $C^\infty_c(\mathbb{R}^n)$ to distributions on $\mathbb{R}^n$ are given by kernels $K(x, y)$, themselves distributions on $\mathbb{R}^n \times \mathbb{R}^n$. Pseudo-differential operators, singular integral operators, and Fourier integral operators are important, non-trivial examples.

Proof: We show that $T$ is an operator-norm limit of finite-rank operators, that is, operators with finite-dimensional images. Fix $\varepsilon > 0$, find a finite collection of functions $f_i, F_i$ such that

$$\sup_{x, y} \left| K(x, y) - \sum_i f_i \otimes F_i \right| < \varepsilon$$
For each \((x, y)\) in the support of \(K\), let \(U_x \times V_y\) be a neighborhood of \((x, y)\) such that \(|K(x, y) - K(x', y')| < \varepsilon\) for \(x' \in U_x\) and \(y' \in V_y\), where \(U_x\) and \(V_y\) are neighborhoods of \(x, y\).

By compactness of the support of \(K(x, y)\), there are finitely-many \(x_j, y_j\) such that \(U_j \times V_j\) (abbreviating \(U_{x_j} \times V_{y_j}\)) cover the support of \(K(x, y)\). Let

\[
\varphi_j = \text{char fcn } U_j \quad \text{and} \quad \Phi_j = K(x_j, y_j) \cdot (\text{char fcn } U_j)
\]

The sets \(U_j \times V_j\) overlap, so \(K \neq \sum_j \varphi_j \otimes \Phi_j\), necessitating minor adjustments.

One way to compensate for the overlaps is by subtracting two-fold overlaps, adding back three-fold overlaps, subtracting four-fold, and so on: let ...
\[ Q = \sum_i \varphi_i \otimes \Phi_i - \sum_{i_1 < i_2} \min (\varphi_{i_1}, \varphi_{i_2}) \otimes \min (\Phi_{i_1}, \Phi_{i_2}) \]

\[ + \sum_{i_1 < i_2 < i_3} \min (\varphi_{i_1}, \varphi_{i_2}, \varphi_{i_3}) \otimes \min (\Phi_{i_1}, \Phi_{i_2}, \Phi_{i_3}) - \ldots \]

Because the subcover is finite, \( Q \) is a finite linear combination
\[ Q = \sum_j f_j \otimes F_j. \]

By construction, \( \sup_{x,y} |K(x, y) - Q(x, y)| < \varepsilon. \)

The operator
\[ f \rightarrow \int_G Q(x, y) f(y) \, dy \]

is finite-rank, because the image is in the span of the finitely-many \( f_i \) appearing in the definition of \( Q(x, y) \).
Let $\chi$ be the characteristic function of the closure $\overline{U}$ of a compact-closure open $U$ containing the support of $K$. For every $\varepsilon > 0$, the opens $U_x$ and $U_y$ can be chosen inside $U$. Then

$$\left| \int_G Q(x, y) f(y) \, dy - \int_G K(x, y) f(y) \, dy \right|$$

$$\leq \int_G |Q(x, y) - K(x, y)| \cdot |f(y)| \, dy$$

$$\leq \varepsilon \int_G |\chi(x, y)| \cdot |f(y)| \, dy \leq \varepsilon \cdot |\chi|_{L^2} \cdot |f|_{L^2}$$

Thus, the operator norm of the difference can be made arbitrarily small, proving that the operator $T$ given by $K(x, y) \in C^0_c(X \times Y)$ is an operator-norm limit of finite-rank operators. ///
Prove operator-norm limits of finite-rank operators are compact:

**Remark:** on Hilbert spaces, the converse is true, that compact operators are operator-norm limits of finite-rank ones. On Banach spaces, the converse is false, by counter-examples due to P. Enflo.

Let \( T = \lim_i T_i \), where \( T_i : X \to Y \) is finite-rank \( X \to Y \). Let \( B \) be the unit ball in \( X \). We show that \( TB \) has compact closure by showing that it is *totally bounded*, that is, for every \( \varepsilon > 0 \) it can be covered by finitely-many \( \varepsilon \)-balls.

Given \( \varepsilon > 0 \), let \( i \) be large-enough so that \( |T - T_i| < \varepsilon \). Since \( T_i \) is finite-rank, \( T_iB \) is covered by finitely-many \( \varepsilon \)-balls \( B_1, \ldots, B_n \) in \( Y \) with respective centers \( y_1, \ldots, y_n \). For \( x \in B \), with \( T_i x \in B_j \),

\[
|Tx - y_j| \leq |Tx - T_i x| + |T_i x - y_j| < \varepsilon + \varepsilon
\]

Thus, \( TB \) is covered by a finite number of \( 2\varepsilon \)-balls. This holds for every \( \varepsilon > 0 \), so \( TB \) is *totally bounded.*
Recall the proof that total boundedness of a set $E$ in a complete metric space implies compact closure:

Since metric spaces have countable local bases, it suffices to show sequential compactness. That is, a sequence $\{v_i\}$ in $E$, exhibit a convergent subsequence.

Cover $E$ by finitely-many $2^{-1}$-balls, choose one, call it $B_1$, with infinitely-many $v_i$ in $E \cap B_1$, and let $w_1$ be one of those infinitely-many $v_i$.

Next, cover $E$ by finitely-many $2^{-2}$-balls. Certainly $E \cap B_1$ is covered by these, and $E \cap B_1 \cap B_2$ contains infinitely-many $v_i$ for at least one of these, call it $B_2$. Let $w_2 \in E \cap B_1 \cap B_2$ be one of these $v_i$, other than $w_1$.

Inductively, find an infinite subsequence $w_n$ of distinct points, with $w_n \in E \cap B_1 \cap \ldots \cap B_n$, where $B_n$ is of radius $2^{-n}$. The sequence $w_i$ is Cauchy. //