Iwasawa-Tate on $\zeta$-functions and $L$-functions

1. Simplest case: Riemann’s zeta
2. Dirichlet $L$-functions
3. Dedekind zetas of number fields
4. General case: Hecke $L$-functions

Proof of analytic continuation and functional equation of Riemann’s zeta, in modern form.

Repeated for Dedekind zeta functions of number fields, noting complications.

Repeated for Hecke’s (größencharakter) $L$-functions, noting complications.

Some issues are postponed: adelic Poisson summation, evaluation of local integrals, ...

A virtue of the modern (Tate-Iwasawa) viewpoint is that concern about units and class numbers evaporates completely, and all number fields are treated in a fashion scarcely different from Riemann’s treatment of zeta.
**Simplest case: Riemann’s zeta**

The modern argument is completely parallel to Riemann’s. Let $d^x x$ be a Haar measure on $\mathbb{J}$. Define **global zeta integrals**

$$Z(s, f) = \int_{\mathbb{J}} |x|^s f(x) d^x x \quad (f \in \mathcal{S}(\mathbb{A}), \ s \in \mathbb{C}, \ \text{Re} \ s > 1)$$

We will see that, for suitable choice of $f$, the zeta integral is the zeta function with its gamma factor. We prove that every such global zeta integral has a meromorphic continuation with poles at worst at $s = 1, 0$, with predictable residues, with functional equation

$$Z(s, f) = Z(1 - s, \hat{f}) \quad (\text{for arbitrary } f \in \mathcal{S}(\mathbb{A}))$$

Part of the point is that meromorphic continuation and functional equation of $Z(s, f)$ follow for all $f$, without worrying about best choice of Schwartz function $f$. 

Euler products and local zeta integrals

Let $d_v^\times x$ be a Haar measure on $\mathbb{Q}_v^\times$ with $d^\times x = \prod_v d_v^\times x$. For monomial Schwartz functions $f = \bigotimes_v f_v$, for $\text{Re } s > 1$, the zeta integral is

$$Z(s, f) = \int \lvert x \rvert^s f(x) d^\times x = \prod_v \int_{\mathbb{Q}_v^\times} \lvert x \rvert_v^s f_v(x) d_v^\times x$$

an infinite product of local integrals. That is, zeta integrals of monomial Schwartz functions have Euler product expansions in the region of convergence. This motivates defining local zeta integrals to be those local integrals

$$Z_v(s, f_v) = \int_{\mathbb{Q}_v^\times} \lvert x \rvert_v^s f_v(x) d_v^\times x$$

and

$$Z(s, f) = \prod_v Z_v(s, f_v) \quad (\text{for } \text{Re } s > 1, \text{ with } f = \bigotimes_v f_v)$$
The usual Euler factors appear

We see later that a reasonable choice for $f$, with $\hat{f} = f$ produces the standard factors:

$$Z_v(s, f_v) = \begin{cases} 
\frac{1}{1 - \frac{1}{p^s}} & \text{(for finite } v \sim p) \\
\pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) & \text{(for } v = \infty) 
\end{cases}$$

That is, for reasonable choices, in this situation,

$$Z(s, f) = \xi(s) = \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s)$$
Functional equation of a theta function

The analogue of the theta function appearing in Riemann’s and Hecke’s classical arguments is

$$\theta_f(x) = \sum_{\alpha \in \mathbb{Q}} f(\alpha x) \quad \text{(for } x \in \mathbb{J}, \ f \in \mathcal{S}(\mathbb{A}))$$

Adelic Poisson summation will give the functional equation of the theta function. From the obvious change of variables,

$$\int_{\mathbb{A}} \overline{\psi}(\xi \alpha) \ f(\alpha x) \ d\alpha = \int_{\mathbb{A}} \overline{\psi}(\xi \alpha/x) \ f(\alpha) \ d(\alpha/x)$$

The adelic change of measure is the idele norm, and

$$\int_{\mathbb{A}} \overline{\psi}(\xi \alpha/x) \ f(\alpha) \ d(\alpha/x) = \frac{1}{|x|} \int_{\mathbb{A}} \overline{\psi}(\xi \alpha/x) \ f(\alpha) \ d\alpha = \frac{1}{|x|} \hat{f}(\frac{\xi}{x})$$

Then Poisson summation gives the functional equation

$$\theta_f(x) = \sum_{\alpha \in \mathbb{Q}} f(\alpha x) = \frac{1}{|x|} \sum_{\alpha \in \mathbb{Q}} \hat{f}\left(\frac{\alpha}{x}\right) = \frac{1}{|x|} \theta_{\hat{f}}\left(\frac{1}{x}\right)$$
Main argument: analytic continuation and functional equation of global zeta integrals

The analytic continuation and functional equation arise from winding up, breaking the integral into two pieces, and applying the functional equation of $\theta$’s, as in the classical scenario. Let

$$J^+ = \{ x \in J : |x| \geq 1 \} \quad J^- = \{ x \in J : |x| \leq 1 \}$$

and $J^1 = \{ x \in J : |x| = 1 \}$. Let

$$\theta_f^+(x) = \theta_f(x) - f(0) = \sum_{\alpha \in \mathbb{Q}^\times} f(\alpha x) \quad (x \in J \text{ and } f \in \mathcal{S}(A))$$

Wind up the zeta integral, use the product formula, and break the integral into two pieces:
$Z(s, f) = \int_\mathbb{J} |x|^s f(x) \, dx = \int_{\mathbb{J}/Q^\times} \sum_{\alpha \in Q^\times} |\alpha x|^s f(\alpha x) \, d\times(\alpha x)$

$= \int_{\mathbb{J}/Q^\times} |x|^s \sum_{\alpha \in Q^\times} f(\alpha x) \, d\times x = \int_{\mathbb{J}/Q^\times} |x|^s \theta_f^*(x) \, d\times x$

$= \int_{\mathbb{J}^+ / Q^\times} |x|^s \theta_f^*(x) \, d\times x + \int_{\mathbb{J}^- / Q^\times} |x|^s \theta_f^*(x) \, d\times x$

just like classical

$\xi(s) = \int_1^\infty \frac{y^{s/2} \theta(iy) - 1}{2} \, dy + \int_0^1 \frac{y^{s/2} \theta(iy) - 1}{2} \, dy$

The integral over $\mathbb{J}^+ / Q^\times$ is *entire*. (Proof!?!)

The functional equation of $\theta_f$ will transforms the integral over $\mathbb{J}^+ / \mathbb{Q}^\times$ into an integral over $\mathbb{J}^- / \mathbb{Q}^\times$ plus two elementary terms describing the poles.

Replace $x$ by $1/x$, and simplify:

$$\int_{\mathbb{J}^- / \mathbb{Q}^\times} |x|^s \, \theta_f^*(x) \, d^\times x = \int_{\mathbb{J}^+ / \mathbb{Q}^\times} |1/x|^s \, \theta_f^*(1/x) \, d^\times (1/x)$$

$$= \int_{\mathbb{J}^+ / \mathbb{Q}^\times} |x|^{-s} \cdot \left[ |x| \theta_f^*(x) - f(0) \right] \, d^\times x$$

$$= \int_{\mathbb{J}^+ / \mathbb{Q}^\times} |x|^{1-s} \, \theta_f^*(x) \, d^\times x + \hat{f}(0) \int_{\mathbb{J}^+ / \mathbb{Q}^\times} |x|^{1-s} \, d^\times x$$

$$- f(0) \int_{\mathbb{J}^+ / \mathbb{Q}^\times} |x|^{-s} \, d^\times x$$

The integral of $\theta_f^*$ over $\mathbb{J}^+ / \mathbb{Q}^\times$ is entire. The elementary integrals can be evaluated:
\[
\int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{1-s} \, d^\times x = |J^1/\mathbb{Q}^\times| \cdot \int_1^\infty x^{1-s} \frac{dx}{x} = \frac{|J^1/\mathbb{Q}^\times|}{s-1}
\]

In this case, the natural measure of \(J^1/\mathbb{Q}^\times\) is 1, so

\[
Z(s, f) = \int_{\mathbb{J}^+/\mathbb{Q}^\times} \left( |x|^s \sum_{\alpha \in \mathbb{Q}^\times} f(\alpha x) + |x|^{1-s} \sum_{\alpha \in \mathbb{Q}^\times} \hat{f}(\alpha x) \right) d^\times x
\]

\[
+ \frac{\hat{f}(0)}{s-1} - \frac{f(0)}{s}
\]

The integral is entire, so the latter expression gives the analytic continuation. There is visible symmetry under \(s \leftrightarrow 1-s\) and \(f \leftrightarrow \hat{f}\), so we have the functional equation

\[
Z(s, f) = Z(1-s, \hat{f})
\]
Dirichlet $L$-functions

We adapt the argument to prove analytic continuation and functional equation for Dirichlet $L$-functions. One should observe how few changes are needed.

Dirichlet characters as idele-class characters For a Dirichlet character $\chi_d$ with conductor $N$. The main adaptation necessary is rewriting $\chi_d$ as a character $\chi$ on $\mathbb{J}/k^\times$.

Given idele $\alpha$, by unique factorization in $\mathbb{Z}$, adjust $\alpha$ by $\mathbb{Q}^\times$ to put its local component inside $\mathbb{Z}_v^\times$ at all finite places. Adjust by $\pm1$ to make the archimedean component positive. Thus, an idele-class character is completely determined by its values on

$$U = \mathbb{R}^+ \cdot \prod_{v<\infty} \mathbb{Z}_v^\times$$

As the diagonal copy of $\mathbb{Q}^\times$ meets $U$ just at $\{1\}$, there is no risk of ill-definedness. Continuity on $U$ implies continuity on $\mathbb{J}$. 
At finite places \( v \sim p \) not dividing \( N \), we declare \( \chi \) to be trivial on the local units: \( \chi(\mathbb{Z}_v^\times) = 1 \) for \( v \sim p \) not dividing \( N \).

For \( v \sim p \) with \( N = p^e M \) and \( p \nmid M \), given \( x \in \mathbb{Z}_v^\times \), let \( n \in \mathbb{Z} \) such that \( n = x \mod p^e \mathbb{Z}_v \), and \( n = 1 \mod M \), and define \( \chi(x) = \chi_d(n) \). Say \( \chi \) is unramified at \( v \) when \( \chi(\mathbb{Z}_v^\times) = 1 \). At finite places \( v \) where \( \chi \) is non-trivial on the local units, \( \chi \) is ramified.

**Global zeta integrals** We consider only idele-class characters \( \chi \) trivial on the copy \( \{(t, 1, 1, \ldots, 1) : t > 0\} \) of positive reals inside \( \mathbb{J} \). Define **global zeta integrals**

\[
Z(s, \chi, f) = \int_{\mathbb{J}} |x|^s \chi(x) f(x) \, d^\times x \quad (f \in \mathcal{S}(\mathbb{A}), \ s \in \mathbb{C}, \ \text{Re} \ s > 1)
\]

For suitable \( f \), \( Z(s, \chi, f) \) is the Dedekind zeta function with its gamma factor, except for complications at ramified primes. Every zeta integral has a meromorphic continuation with poles at worst at \( s = 1, 0 \), with predictable residues, with functional equation

\[
Z(s, \chi, f) = Z(1 - s, \chi^{-1}, \hat{f}) \quad \text{(for arbitrary } f \in \mathcal{S}(\mathbb{A}))
\]
Euler products and local zeta integrals

For monomial Schwartz functions $f = \bigotimes f_v$, for $\text{Re } s > 1$,

$$Z(s, f) = \int J |x|^s \chi(x) f(x) d^\infty x = \prod_v \int_{k_v^\times} |x|^s \chi_v(x) f_v(x) d_v^\times x$$

with $\chi_v$ the restriction of $\chi$ to $Q_v^\times$. That is, $Z(s, f)$ is an infinite product of local integrals. That is, zeta integrals of monomial Schwartz functions have Euler product expansions, in the region of convergence. This motivates defining local zeta integrals to be those local integrals

$$Z_v(s, \chi_v, f_v) = \int_{k_v^\times} |x|^s \chi_v(x) f_v(x) d_v^\times x$$

Without clarifying the nature of the local integrals, the Euler product assertion is

$$Z(s, f) = \prod_v Z_v(s, \chi_v, f_v) \quad \text{(Re } s > 1, \text{ with } f = \bigotimes_v f_v)$$
Usual Euler factors, with a complication

We see later that a reasonable choice of \( f \) produces the standard factors:

\[
Z_v(s, \chi_v, f_v) = \begin{cases} 
\frac{1}{1 - \frac{\chi(p)}{p^s}} & (v \sim p, \ p \nmid N) \\
\pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) & (v \approx \mathbb{R} \text{ and } \chi_d(-1) = 1) \\
\pi^{-\frac{s+1}{2}} \Gamma \left( \frac{s+1}{2} \right) & (v \approx \mathbb{R} \text{ and } \chi_d(-1) = -1)
\end{cases}
\]

There is a complication at finite \( v \sim \) with \( p|N \): typically there is no Schwartz function \( f \) recovering the factor \( N^{-s/2} \) in the known functional equations

\[
N^{\frac{s}{2}} \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) L(s, \chi) = \varepsilon(\chi) \frac{N^{(1-s)/2}}{\pi^{-s/2}} \Gamma \left( \frac{1-s}{2} \right) L(1-s, \chi^{-1})
\]

for \( \chi \) even, and for \( \chi \) odd

\[
N^{\frac{s}{2}} \pi^{-\frac{(s+1)}{2}} \Gamma \left( \frac{s+1}{2} \right) L(s, \chi) = \varepsilon(\chi) \frac{N^{(1-s)}\pi^{-\frac{2-s}{2}}}{\pi^{-\frac{(2-s)}{2}}} \Gamma \left( \frac{2-s}{2} \right) L(1-s, \chi^{-1})
\]
Nevertheless, a reasonable choice will produce $Z(s, \chi, f)$ and $Z(s, \chi^{-1}, \hat{f})$ such that, letting $\Lambda(s, \chi)$ be the $L$-function with its gamma factor and with factor of $N^{s/2}$,

$$Z(s, \chi, f) = N^{-s/2} \cdot \Lambda(s, \chi)$$

$$Z(1 - s, \chi^{-1}, \hat{f}) = \varepsilon \cdot N^{-s/2} \cdot \Lambda(1 - s, \chi^{-1})$$

with $|\varepsilon| = 1$. Thus, from $Z(s, \chi, f) = Z(1 - s, \chi^{-1}, f)$ the symmetrical functional equation can be obtained.

**Functional equation of a theta function** As before, the theta function attached to a Schwartz function $f$ is

$$\theta_f(x) = \sum_{\alpha \in k} f(\alpha x) \quad \text{(for } x \in \mathbb{J}, \ f \in \mathcal{S}(\mathbb{A}))$$

and Poisson summation gives the functional equation

$$\theta_f(x) = \sum_{\alpha \in k} f(\alpha x) = \frac{1}{|x|} \sum_{\alpha \in k} \hat{f}(\frac{\alpha}{x}) = \frac{1}{|x|} \theta_{\hat{f}}(\frac{1}{x})$$
Main argument: analytic continuation and functional equation of global zeta integrals

Again, analytic continuation and functional equation arise from \textit{winding up}, breaking the integral into two pieces, and applying the functional equation of $\theta$, as in the classical scenario.

For non-trivial $\chi$, the Schwartz function $f$ can be taken so that

$$f(0) = 0 \quad \text{and} \quad \hat{f}(0) = 0$$

relieving us of tracking those values, and giving the simpler presentation

$$\theta_f(x) = \sum_{\alpha \in \mathbb{Q}^\times} f(\alpha x) \quad \text{(for } x \in \mathbb{J} \text{ and } f \in \mathcal{S}(\mathbb{A}))$$

Wind up the zeta integral, use the product formula and $\mathbb{Q}^\times$-invariance of $\chi$, and break the integral into two pieces:
\[ Z(s, \chi, f) = \int_{\mathbb{J}} |x|^s \chi(x) f(x) \, d^\times x \]

\[ = \int_{\mathbb{J}/\mathbb{Q}^\times} \sum_{\alpha \in \mathbb{K}^\times} |\alpha x|^s \chi(\alpha x) f(\alpha x) \, d^\times(\alpha x) \]

\[ = \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s \chi(x) \sum_{\alpha \in \mathbb{K}^\times} f(\alpha x) \, d^\times x = \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s \chi(x) \theta_f(x) \, d^\times x \]

\[ = \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^s \chi(x) \theta_f(x) \, d^\times x + \int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s \chi(x) \theta_f(x) \, d^\times x \]

The integral over \( \mathbb{J}^+/\mathbb{Q}^\times \) is entire. The functional equation of \( \theta_f \) will give a transformation of the integral over \( \mathbb{J}^-/\mathbb{Q}^\times \) into an integral over \( \mathbb{J}^+/\mathbb{Q}^\times \). Replace \( x \) by \( 1/x \), and simplify:
\[
\int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s \chi(x) \theta_f(x) \, d^\times x = \int_{\mathbb{J}^+/\mathbb{Q}^\times} |1/x|^s \chi(1/x) \theta_f(1/x) \, d^\times (1/x)
\]

\[
= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{-s} \chi^{-1}(x) |x| \hat{\theta}_f(x) \, d^\times x = \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{1-s} \chi^{-1}(x) \theta_{\hat{f}}(x) \, d^\times x
\]

The integral of \( \theta_{\hat{f}} \) over \( \mathbb{J}^+/\mathbb{Q}^\times \) is entire. Thus,

\[
Z(s, \chi, f)
\]

\[
= \int_{\mathbb{J}^+/\mathbb{Q}^\times} \left( |x|^s \chi(x) \sum_{\alpha \in \mathbb{Q}^\times} f(\alpha x) + |x|^{1-s} \chi^{-1}(x) \sum_{\alpha \in \mathbb{Q}^\times} \hat{f}(\alpha x) \right) \, d^\times x
\]

The integral is entire, and gives the analytic continuation. Further, there is visible symmetry \( \chi \leftrightarrow \chi^{-1}, \, s \leftrightarrow 1 - s, \, f \leftrightarrow \hat{f} \), so we have the functional equation

\[
Z(s, \chi, f) = Z(1 - s, \chi^{-1}, \hat{f})
\]
Remark: There was no compulsion to track of \(|x|^s\) and \(\chi(x)\) separately in the above argument. We could rewrite the above to treat an arbitrary \(\chi\) on \(J/Q^\times\), define

\[
Z(\chi, f) = \int_J \chi(x) f(x) d^\times x
\]

and obtain the slightly cleaner functional equation

\[
Z(\chi, f) = Z(|.| \chi^{-1}, \hat{f})
\]

That is, rather than \(s \to 1 - s\) and \(\chi \to \chi^{-1}\), simply replace \(\chi\) by \(x \to |x| \cdot \chi^{-1}(x)\)