Iwasawa-Tate on \( \zeta \)-functions and \( L \)-functions

1. Simplest case: Riemann’s zeta [done]
2. Dirichlet \( L \)-functions
3. Dedekind zetas of number fields
4. General case: Hecke \( L \)-functions

Main part of proof of analytic continuation and functional equation for Dirichlet \( L \)-functions, in modern form.

Repeated for Dedekind zetas number fields.

Repeated for Hecke’s (größencharakter) \( L \)-functions.

Some issues are postponed: adelic Poisson summation, evaluation of local integrals, ...

Again, a virtue of the modern (Tate-Iwasawa) viewpoint is that units and class numbers disappear.
Dirichlet $L$-functions

We prove analytic continuation and functional equation for Dirichlet $L$-functions. Few changes are needed.

Dirichlet characters as idele-class characters: For a Dirichlet character $\chi_d$ with conductor $N$. The main adaptation necessary is rewriting $\chi_d$ as a character $\chi$ on $\mathbb{J}/k^\times$.

Given idele $\alpha$, by unique factorization in $\mathbb{Z}$, adjust $\alpha$ by $\mathbb{Q}^\times$ to put its local component inside $\mathbb{Z}_v^\times$ at all finite places. Adjust by $\pm 1$ to make the archimedean component positive. Thus, an idele-class character is completely determined by its values on

$$U = \mathbb{R}^+ \cdot \prod_{v<\infty} \mathbb{Z}_v^\times$$

As the diagonal copy of $\mathbb{Q}^\times$ meets $U$ just at $\{1\}$, there is no risk of ill-definedness. Continuity on $U$ implies continuity on $\mathbb{J}$. 
At finite places $v \sim p$ not dividing $N$, we declare $\chi$ to be trivial on the local units: $\chi(\mathbb{Z}_v^\times) = 1$ for $v \sim p$ not dividing $N$.

For $v \sim p$ with $N = p^e M$ and $p \nmid M$, given $x \in \mathbb{Z}_v^\times$, let $n \in \mathbb{Z}$ such that $n = x \bmod p^e \mathbb{Z}_v$, and $n = 1 \bmod M$, and define $\chi(x) = \chi_d(n)$. Say $\chi$ is unramified at $v$ when $\chi(\mathbb{Z}_v^\times) = 1$. At finite places $v$ where $\chi$ is non-trivial on the local units, $\chi$ is ramified.

Global zeta integrals We consider only idele-class characters $\chi$ trivial on the copy $\{ (t, 1, 1, \ldots, 1) : t > 0 \}$ of positive reals inside $\mathbb{J}$. Define global zeta integrals

$$Z(s, \chi, f) = \int_{\mathbb{J}} |x|^s \chi(x) f(x) d^\times x \quad (f \in \mathcal{S}(\mathbb{A}), \ s \in \mathbb{C}, \ \text{Re} \ s > 1)$$

For suitable $f$, $Z(s, \chi, f)$ is the Dirichlet $L$-function function with its gamma factor, except for complications at ramified primes. Every zeta integral has a meromorphic continuation with poles at worst at $s = 1, 0$, with predictable residues, with functional equation

$$Z(s, \chi, f) = Z(1 - s, \chi^{-1}, \hat{f}) \quad \text{(for arbitrary } f \in \mathcal{S}(\mathbb{A}))$$
Euler products and local zeta integrals

For monomial Schwartz functions $f = \bigotimes f_v$, for $\text{Re } s > 1$,

$$Z(s, \chi, f) = \int \mathbb{J} |x|^s \chi(x) f(x) \, d^\times x = \prod_v \int_{k_v^{\times}} |x_v|^s \chi_v(x) f_v(x) \, d_v^\times x$$

with $\chi_v$ the restriction of $\chi$ to $\mathbb{Q}_v^{\times}$. That is, $Z(s, \chi, f)$ is an infinite product of local integrals. That is, zeta integrals of monomial Schwartz functions have Euler product expansions, in the region of convergence. This motivates defining local zeta integrals to be those local integrals

$$Z_v(s, \chi_v, f_v) = \int_{k_v^{\times}} |x_v|^s \chi_v(x) f_v(x) \, d_v^\times x$$

Without clarifying the nature of the local integrals, the Euler product assertion is

$$Z(s, \chi, f) = \prod_v Z_v(s, \chi_v, f_v) \quad (\text{Re } s > 1, \text{ with } f = \bigotimes_v f_v)$$
Usual Euler factors, with a complication

We see later that a reasonable choice of $f$ produces the standard factors:

$$Z_v(s, \chi_v, f_v) = \begin{cases} 
\frac{1}{1 - \chi(p)} & (v \sim p, \ p \nmid N) \\
\frac{1}{p^s} & (v \sim \mathbb{R} \text{ and } \chi_d(-1) = 1) \\
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) & (v \sim \mathbb{R} \text{ and } \chi_d(-1) = -1)
\end{cases}$$

There is a complication at finite $v \sim$ with $p|N$: typically there is no Schwartz function $f$ recovering the factor $N^{-s/2}$ in the known functional equations

$$N^\frac{s}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi) = \varepsilon(\chi) N^{(1-s)/2} \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) L(1-s, \chi^{-1})$$

for $\chi$ even, and for $\chi$ odd

$$N^\frac{s}{2} \pi^{-\frac{(s+1)}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi) = \varepsilon(\chi) N^{\frac{1-s}{2}} \pi^{-\frac{(2-s)}{2}} \Gamma\left(\frac{2-s}{2}\right) L(1-s, \chi^{-1})$$
Nevertheless, a reasonable choice will produce $Z(s, \chi, f)$ and $Z(s, \chi^{-1}, \hat{f})$ such that, letting $\Lambda(s, \chi)$ be the $L$-function with its gamma factor and with factor of $N^{s/2}$,

$$Z(s, \chi, f) = N^{-\frac{s}{2}} \Lambda(s, \chi) \quad Z(1 - s, \chi^{-1}, \hat{f}) = \varepsilon N^{-\frac{s}{2}} \Lambda(1 - s, \chi^{-1})$$

with $|\varepsilon| = 1$. Thus, from $Z(s, \chi, f) = Z(1 - s, \chi^{-1}, f)$ the symmetrical functional equation can be obtained.

**Functional equation of a theta function** As before, the theta function attached to a Schwartz function $f$ is

$$\theta_f(x) = \sum_{\alpha \in k} f(\alpha x) \quad \text{(for } x \in \mathbb{J}, f \in \mathcal{S}(\mathbb{A}))$$

Poisson summation gives the functional equation

$$\theta_f(x) = \sum_{\alpha \in k} f(\alpha x) = \frac{1}{|x|} \sum_{\alpha \in k} \hat{f}(\frac{\alpha}{x}) = \frac{1}{|x|} \theta_f(\frac{1}{x})$$
Main argument: analytic continuation and functional equation of global zeta integrals

Again, analytic continuation and functional equation arise from winding up, breaking the integral into two pieces, and applying the functional equation of $\theta$, as in the classical scenario.

For non-trivial $\chi$, the Schwartz function $f$ can be taken so that

$$f(0) = 0 \quad \text{and} \quad \hat{f}(0) = 0$$

relieving us of tracking those values, and giving the simpler presentation

$$\theta_f(x) = \sum_{\alpha \in \mathbb{Q}^\times} f(\alpha x) \quad \text{(for } x \in \mathbb{J} \text{ and } f \in \mathcal{S}(\mathbb{A}))$$

Wind up the zeta integral, use the product formula and $\mathbb{Q}^\times$-invariance of $\chi$, and break the integral into two pieces:
\[
Z(s, \chi, f) = \int_{\mathbb{J}} |x|^s \chi(x) f(x) \, d^\times x
\]

\[
= \int_{\mathbb{J}/\mathbb{Q}^\times} \sum_{\alpha \in \mathbb{Q}^\times} |\alpha x|^s \chi(\alpha x) f(\alpha x) \, d^\times (\alpha x)
\]

\[
= \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s \chi(x) \sum_{\alpha \in \mathbb{Q}^\times} f(\alpha x) \, d^\times x = \int_{\mathbb{J}/\mathbb{Q}^\times} |x|^s \chi(x) \theta_f(x) \, d^\times x
\]

\[
= \int_{\mathbb{J}^+\mathbb{Q}^\times} |x|^s \chi(x) \theta_f(x) \, d^\times x + \int_{\mathbb{J}^-\mathbb{Q}^\times} |x|^s \chi(x) \theta_f(x) \, d^\times x
\]

The integral over \(\mathbb{J}^+\mathbb{Q}^\times\) is \textit{entire}. The functional equation of \(\theta_f\) will give a transformation of the integral over \(\mathbb{J}^-\mathbb{Q}^\times\) into an integral over \(\mathbb{J}^+\mathbb{Q}^\times\). Replace \(x\) by \(1/x\), and simplify:
\[
\int_{\mathbb{J}^-/\mathbb{Q}^\times} |x|^s \chi(x) \theta_f(x) \, d^\times x = \int_{\mathbb{J}^+/\mathbb{Q}^\times} |1/x|^s \chi(1/x) \theta_f(1/x) \, d^\times (1/x)
\]

\[
= \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{-s} \chi^{-1}(x) |x| \theta_{\hat{f}}(x) \, d^\times x = \int_{\mathbb{J}^+/\mathbb{Q}^\times} |x|^{1-s} \chi^{-1}(x) \theta_{\hat{f}}(x) \, d^\times x
\]

The integral of \( \theta_{\hat{f}} \) over \( \mathbb{J}^+/\mathbb{Q}^\times \) is entire. Thus,

\[
Z(s, \chi, f)
\]

\[
= \int_{\mathbb{J}^+/\mathbb{Q}^\times} \left( |x|^s \chi(x) \sum_{\alpha \in \mathbb{Q}^\times} f(\alpha x) + |x|^{1-s} \chi^{-1}(x) \sum_{\alpha \in \mathbb{Q}^\times} \hat{f}(\alpha x) \right) \, d^\times x
\]

The integral is entire, and gives the analytic continuation.

Further, there is visible symmetry \( \chi \leftrightarrow \chi^{-1}, s \leftrightarrow 1 - s, f \leftrightarrow \hat{f} \), so we have the functional equation

\[
Z(s, \chi, f) = Z(1 - s, \chi^{-1}, \hat{f})
\]
Remark: There was no compulsion to track of $|x|^s$ and $\chi(x)$ separately in the above argument. We could rewrite the above to treat an arbitrary $\chi$ on $\mathbb{J}/\mathbb{Q}^\times$, define

$$Z(\chi, f) = \int_{\mathbb{J}} \chi(x) f(x) d^\times x$$

and obtain the slightly cleaner functional equation

$$Z(\chi, f) = Z(|.|^{\chi^{-1}}, \hat{f})$$

That is, rather than $s \to 1 - s$ and $\chi \to \chi^{-1}$, simply replace $\chi$ by $x \to |x| \cdot \chi^{-1}(x)$. 
Dedekind zetas of number fields

The argument is repeated, proving analytic continuation and functional equation for Dedekind zetas of number fields.

Global zeta integrals

\[ Z(s, f) = \int_{\mathbb{R}} |x|^s f(x) \, dx \quad (f \in \mathcal{S}(\mathbb{R}), s \in \mathbb{C}, \text{Re} \, s > 1) \]

We will see that, for suitable choice of \( f \), the zeta integral is the Dedekind zeta function with its gamma factors. Just below, we prove that every such global zeta integral has a meromorphic continuation with poles at worst at \( s = 1, 0 \), with predictable residues, with functional equation

\[ Z(s, f) = Z(1 - s, \hat{f}) \quad (\text{for arbitrary } f \in \mathcal{S}(\mathbb{R})) \]
**Euler products and local zeta integrals** For *monomial* Schwartz functions $f = \bigotimes f_v$, for $\text{Re } s > 1$, the zeta integral factors over primes as a product of local integrals

$$Z(s, f) = \int \|x\|^s f(x) \, d^\times x = \prod_v \int_{k_v^\times} |x|^s f_v(x) \, d_v^\times x$$

Letting

$$Z_v(s, f_v) = \int_{k_v^\times} |x|^s f_v(x) \, d_v^\times x$$

and without clarifying the nature of the local integrals, the Euler product assertion is

$$Z(s, f) = \prod_v Z_v(s, f_v) \quad (\text{Re } s > 1, \text{ monomial } f = \bigotimes_v f_v)$$
the usual Euler factors, with a complication We see later that a reasonable choice of $f$ (and measures $d\chi x$) produces the standard factors at all but finitely-many primes: with $q_v$ the cardinality of the residue field for non-archimedean $v$,

$$Z_v(s, f_v) = \begin{cases} 
\frac{1}{1 - \frac{1}{q^s_v}} & \text{(for } k_v \text{ unramified over } \mathbb{Q}_w) \\
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) & \text{(for } v \approx \mathbb{R}) \\
(2\pi)^{-s} \Gamma(s) & \text{(for } v \approx \mathbb{C})
\end{cases}$$

However, there is a complication due to finite $v$ with $k_v/\mathbb{Q}_w$ ramified. The Dedekind zeta function of $k$ is

$$\zeta_k(s) = \prod_{v<\infty} \frac{1}{1 - \frac{1}{q^s_v}}$$
Let
\[ \Gamma_R(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \quad \Gamma_C(s) = (2\pi)^{-s} \Gamma(s) \]
and let \( r_1, r_2 \) be the number or real and complex places. Hecke found that the functional equation of the Dedekind zeta function \( \zeta_k(s) \) involves the discriminant \( D_k \) of \( \mathfrak{o}_k \) over \( \mathbb{Z} \), with symmetrical form
\[
\Gamma_R(s)^{r_1} \Gamma_C(s)^{r_2} \cdot |D_k|^{-\frac{s}{2}} \cdot \zeta_k(s) = \Gamma_R(1-s)^{r_1} \Gamma_C(1-s)^{r_2} \cdot |D_k|^{-\frac{1-s}{2}} \cdot \zeta_k(1-s)
\]
The discriminant is
\[
D_k = \text{vol} \left( k \otimes_{\mathbb{Q}} \mathbb{R} / \mathfrak{o} \right)^2 = |\det \begin{pmatrix}
\sigma_1(\alpha_1) & \sigma_2(\alpha_1) & \ldots & \sigma_r(\alpha_1) \\
\vdots & \vdots & & \vdots \\
\sigma_1(\alpha_r) & \sigma_2(\alpha_r) & \ldots & \sigma_r(\alpha_r)
\end{pmatrix}|^2
\]
where \( \sigma_j \) are the topologically distinct imbeddings \( k \to \mathbb{C} \).
The factor $|D_k|^{-\frac{s}{2}}$ is a product of local contributions, as follows. The absolute value of the discriminant is the ideal-norm of the absolute different

$$\mathfrak{d}_o/\mathbb{Z} = \{\alpha \in k : \text{tr}_k^k(\alpha \mathfrak{o}) \subset \mathbb{Z}\}^{-1}$$ (fractional ideal inverse)

This is essentially the product of local differentials

$$\mathfrak{d}_v = \mathfrak{d}_{ov}/\mathbb{Z}_w = \{\alpha \in k_v : \text{tr}_v^v(\alpha \mathfrak{o}_v) \subset \mathbb{Z}_v\}^{-1}$$

Thus, to have the functional equation, the local factor at ramified $v$ should be

$$\frac{[\mathfrak{o}_v : \mathfrak{d}_{ov}/\mathbb{Z}_w]^{-\frac{s}{2}}}{1 - \frac{1}{q_v^s}}$$

However, typically, there is no choice of $f$ or local component $f_v$ to produce this Euler factor as a local zeta integral!
In fact, typically, there is no choice of \( f \) such that \( \hat{f} = f \), because, typically, at ramified \( v \) there is no \( f_v \in \mathcal{S}(k_v) \) with \( \hat{f}_v = f_v \). That is, there is no choice of Schwartz function to make the local zeta functions \( Z_v(s, f_v) \) and \( Z_v(s, \hat{f}_v) \) the same.

That is, while the functional equation

\[
Z(s, f) = Z(1 - s, \hat{f})
\]

holds, there is simply no choice of \( f \) to make the functional equation obviously relate a zeta integral to itself.

However, there are other options. A reasonable choice of \( f = \bigotimes_v f_v \) will produce the expected factors at archimedean and unramified finite places, and at ramified finite \( v \) will produce

\[
Z_v(s, f_v) = \frac{[\mathcal{O}_v^*: \mathcal{O}_v]^{-\frac{1}{2}}}{1 - \frac{1}{q^s}} \quad Z_v(s, \hat{f}_v) = \frac{[\mathcal{O}_v^*: \mathcal{O}_v]^{s - \frac{1}{2}}}{1 - \frac{1}{q^s}}
\]
Thus,

\[ Z(s, f) = |D_k|^\frac{1}{2} \cdot \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{R}}(s)^{r_2} \cdot \zeta_k(s) \]

\[ Z(s, \hat{f}) = |D_k|^{s-\frac{1}{2}} \cdot \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{R}}(s)^{r_2} \cdot \zeta_k(s) \]

From \( Z(s, f) = Z(1-s, \hat{f}) \),

\[ |D_k|^\frac{1}{2} \cdot \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{R}}(s)^{r_2} \zeta_k(s) \]

\[ = |D_k|^{(1-s)-\frac{1}{2}} \cdot \Gamma_{\mathbb{R}}(1-s)^{r_1} \Gamma_{\mathbb{R}}(1-s)^{r_2} \cdot \zeta_k(1-s) \]

Divide through by \(|D_k|^{s/2}\) to obtain the symmetrical form of the functional equation for \( \zeta_k(s) \).

**Remark:** Asymmetry in zeta integrals cannot be avoided, in general. Thus, zeta *functions*, including optimized gamma factors and powers of discriminants, are *not exactly* given by zeta *integrals*. Nevertheless, the zeta integrals *are* inevitably correct at all but finitely-many places.
Functional equation of a theta function

The analogue of the theta function appearing in Riemann’s and Hecke’s classical arguments is

$$\theta_f(x) = \sum_{\alpha \in k} f(\alpha x) \quad \text{(for } x \in \mathbb{J}, f \in \mathcal{S}(\mathbb{A}))$$

Poisson summation gives the functional equation

$$\theta_f(x) = \sum_{\alpha \in k} f(\alpha x) = \frac{1}{|x|} \sum_{\alpha \in k} \hat{f}(\frac{\alpha}{x}) = \frac{1}{|x|} \theta_{\hat{f}}(\frac{1}{x})$$

Analytic continuation and functional equation arise from winding up, and breaking the integral into two pieces, and applying the functional equation of \(\theta\)’s.
Notation for \( \theta_f \) with its constant removed:

\[
\theta^*_f(x) = \theta_f(x) - f(0) = \sum_{\alpha \in k^\times} f(\alpha x) \quad (x \in J, f \in \mathcal{H}(A))
\]

*Wind up* the zeta integral, use the product formula, and break the integral into two pieces:

\[
Z(s, f) = \int_J |x|^s f(x) \, d^\times x = \int_{J/k^\times} |x|^s \sum_{\alpha \in k^\times} f(\alpha x) \, d^\times (\alpha x)
\]

\[
= \int_{J/k^\times} |x|^s \sum_{\alpha \in k^\times} f(\alpha x) \, d^\times x = \int_{J/k^\times} |x|^s \theta^*_f(x) \, d^\times x
\]

\[
= \int_{J^+/k^\times} |x|^s \theta^*_f(x) \, d^\times x + \int_{J^-/k^\times} |x|^s \theta^*_f(x) \, d^\times x
\]

The integral over \( J^+/k^\times \) is *entire*. The functional equation of \( \theta_f \) will give a transformation of the integral over \( J^-/k^\times \) into an integral over \( J^+/k^\times \) plus two elementary terms describing the poles.
Replace $x$ by $1/x$, and simplify:

$$\int_{\mathbb{J}^{-}/k^\times} |x|^s \, \theta^*_f(x) \, d^x x = \int_{\mathbb{J}^{+}/k^\times} |1/x|^s \, \theta^*_f(1/x) \, d^x(1/x)$$

$$= \int_{\mathbb{J}^{+}/k^\times} |x|^{-s} \cdot [|x| \tilde{\theta}_f(x) - f(0)] \, d^x x = \int_{\mathbb{J}^{+}/k^\times} |x|^{1-s} \, \theta^*_f(x) \, d^x x$$

$$+ \tilde{f}(0) \int_{\mathbb{J}^{+}/k^\times} |x|^{1-s} \, d^x x - f(0) \int_{\mathbb{J}^{+}/k^\times} |x|^{-s} \, d^x x$$

The integral of $\theta^*_f$ over $\mathbb{J}^{+}/k^\times$ is *entire*. The elementary integrals can be evaluated:

$$\int_{\mathbb{J}^{+}/k^\times} |x|^{1-s} \, d^x x = \text{meas} \left( \mathbb{J}^{1}/k^\times \right) \cdot \int_1^\infty x^{1-s} \frac{dx}{x} = \frac{|\mathbb{J}^{1}/k^\times|}{s - 1}$$
In this case, the natural measure of \( \mathbb{J}^1/k^\times \) is

\[
\text{meas}(\mathbb{J}^1/k^\times) = \frac{2^{r_1} (2\pi)^{r_2} h R}{|D_k|^{\frac{1}{2}} w}
\]

where \( r_1, r_2 \) are the numbers of real and complex places, respectively, \( h \) is the class number of \( \mathfrak{o} \), \( R \) is the regulator

\[
R = \text{vol}\left(\{\alpha \in k \otimes \mathbb{Q} \otimes \mathbb{R} : \prod_{v|\infty} |\alpha|_v = 1\}/\mathfrak{o}^\times\right)
\]

\( D_k \) is the discriminant, and \( w \) is the number of roots of unity in \( k \).

Thus,

\[
Z(s, f) = \int_{\mathbb{J}^1/k^\times} \left(|x|^s \sum_{\alpha \in k^\times} f(\alpha x) + |x|^{1-s} \sum_{\alpha \in k^\times} \hat{f}(\alpha x)\right) d^x x
\]

\[
+ \frac{|J^1/k^\times| \cdot \hat{f}(0)}{s-1} - \frac{|J^1/k^\times| \cdot f(0)}{s}
\]

The integral is entire, so the latter expression gives the analytic continuation. Further, there is visible symmetry under \( s \leftrightarrow 1 - s \) and \( f \leftrightarrow \hat{f} \) and so we have the functional equation

\[
Z(s, f) = Z(1-s, \hat{f})
\]