Iwasawa-Tate on ζ-functions and L-functions

After the main part, namely, *analytic continuation* and *functional equation* of global zeta integrals:

- The elementary *global* integral is

$$\int_{\mathbb{J}^+/k^\times} |x|^{1-s}\, dx = \frac{|\mathbb{J}^1/k^\times|}{s-1} \qquad \text{(for } \mathrm{Re}(s) > 1)$$

But postpone

$$|\mathbb{J}^1/k^\times| \;=\; \frac{2^{r_1}\,(2\pi)^{r_2}\, h\, R}{D_k^{\frac{1}{2}}\, w}$$

- Vanishing of ramified elementary integrals
- Good finite-prime *local* zeta integrals

$$\int_{k_v^\times \cap \mathfrak{o}_v} |x|^s\, dx = \frac{1}{1-q_v^{-s}} \qquad \text{(for } \mathrm{Re}(s) > 0)$$

- *Local* functional equation

$$\frac{Z_v(s, f_v)}{Z_v(1-s, \widehat{f}_v)} \;=\; \frac{Z_v(s, g_v)}{Z_v(1-s, \widehat{g}_v)} \qquad \text{(for } f, g \in \mathscr{S}(k_v))$$

The elementary global integral :

The poles and residues of zeta integrals are multiples of an elementary integral over $\mathbb{J}^+/k^\times$, which we claim is

$$\int_{\mathbb{J}^+/k^\times} |x|^{1-s}\, d^\times x = \frac{|\mathbb{J}^1/k^\times|}{s-1}$$

Multiplicative measures on $\mathbb{J}$ and $k_v^\times$ are completely determined by giving local units $\mathfrak{o}_v^\times$ measure 1 at *all* finite places, and $d^\times x = \frac{d^+x}{|x|_v}$ at archimedean places. Keep in mind that the product-formula norm $|\cdot|_\mathbb{C}$ is

$$|x|_\mathbb{C} \;=\; |N^\mathbb{C}_\mathbb{R}(x)|_\mathbb{R} \;=\; |x\,\overline{x}|_\mathbb{R} \;=\; \text{square of usual complex norm}$$

That is, the usual complex norm *extends* the norm on $\mathbb{R}$, but for zeta-integrals we must *compose* with the Galois norm.

For *abelian* (hence, *unimodular*) topological groups, the general riff

$$\int_G f(g)\,dg \;=\; \int_{H\backslash G} \Big(\int_H f(h\dot{g})\,dh \Big)\,d\dot{g}$$

applies: fixing any two of the three measures uniquely specifies the normalizing constant for the third so that the equation holds.

Our locally-everywhere normalization of measures on $k_v^\times$ specifies the measure on $\mathbb{J}$. Counting measure on $k^\times$ uniquely specifies the measure on $\mathbb{J}/k^\times$ by *one* instance of the above identity (with the sum being a type of integral, of course)

$$\int_{\mathbb{J}} f(g)\,dg \;=\; \int_{\mathbb{J}/k^\times} \sum_{h\in k^\times} f(h\dot{g})\,d\dot{g}$$

The subgroup $\mathbb{J}^1/k^\times$ of $\mathbb{J}/k^\times$ is *compact*, by Fujisaki, but we do not try to specify its measure directly. Instead, since $\mathbb{J}^1$ is the kernel of $|\cdot|$, $\mathbb{J}^1/k^\times$ fits into an exact sequence

$$1 \longrightarrow \mathbb{J}^1/k^\times \longrightarrow \mathbb{J}/k^\times \longrightarrow \mathbb{R}^+ \longrightarrow 1 \qquad (\mathbb{R}^+ = (0,+\infty))$$

Thus, the usual measure $\frac{dx}{x}$ on $\mathbb{R}^+$ and the measure on $\mathbb{J}/k^\times$ uniquely determine the measure on $\mathbb{J}^1/k^\times$ by

$$\int_{\mathbb{J}/k^\times} f(g)\, dg \;=\; \int_{(\mathbb{J}/k^\times)/(\mathbb{J}^1/k^\times)} \Big(\int_{\mathbb{J}^1/k^\times} f(h\dot{g})\, dh \Big)\, d\dot{g}$$

$$= \int_{\mathbb{R}^+} \Big(\int_{\mathbb{J}^1/k^\times} f(h\dot{g})\, dh \Big)\, d\dot{g}$$

It is not absolutely necessary, but it is easy to identify a *section* $\sigma : \mathbb{R}^+ \to \mathbb{J}$ with

$$|\sigma(t)| \;=\; t$$

For $k = \mathbb{Q}$, just map $t \to (t, 1, 1, \ldots)$, the idele with trivial entries except at $\mathbb{Q}_\infty^\times \approx \mathbb{R}^\times$, where the entry is t. For general number fields k, with r_1, r_2 real-and-complex completions, let

$$\sigma(t) \;=\; (t^{\frac{1}{r_1+r_2}}, \ldots, t^{\frac{1}{r_1+r_2}}, 1, 1, 1, 1, \ldots)$$

with non-trivial entries at archimedean places.

With f being the product of $|\cdot|^{1-s}$ and the characteristic function of $\mathbb{J}^+/k^\times$, this gives

$$\int_{\mathbb{J}^+/k^\times} |g|^{1-s}\, dg \;=\; \int_{(\mathbb{J}^+/k^\times)/(\mathbb{J}^1/k^\times)} \Big(\int_{\mathbb{J}^1/k^\times} |gh|^{1-s}\, dh \Big)\, d\dot{g}$$

$$= \int_{(\mathbb{J}^+/k^\times)/(\mathbb{J}^1/k^\times)} \Big(\int_{\mathbb{J}^1/k^\times} |g|^{1-s}\, dh \Big)\, d\dot{g}$$

$$= \int_{[1,+\infty)} |\dot{g}|^{1-s} \Big(\int_{\mathbb{J}^1/k^\times} 1\, dh \Big)\, d\dot{g} \;=\; |\mathbb{J}^1/k^\times| \cdot \int_1^\infty t^{1-s}\, \frac{dt}{t}$$

$$= |\mathbb{J}^1/k^\times| \cdot \int_1^\infty t^{-s}\, dt \;=\; |\mathbb{J}^1/k^\times| \cdot \Big[\frac{t^{1-s}}{1-s} \Big]_1^\infty \;=\; \frac{|\mathbb{J}^1/k^\times|}{s-1} \qquad ///$$

Remark: We *postpone* the non-elementary computation that

$$|\mathbb{J}^1/k^\times| \;=\; \frac{2^{r_1}\, (2\pi)^{r_2}\, h\, R}{D_k^{\frac{1}{2}}\, w}$$

Vanishing of ramified elementary global integrals:

A character χ_v on $k_v^\times$ is *unramified* if it factors through the *norm*, that is, is of the form $x \to |x|_v^{s_v}$ for some $s_v \in \mathbb{C}$.

For non-archimedean k_v, for χ_v on $k_v^\times$ to be *ramified* means that it is non-trivial on $\mathfrak{o}_v^\times$ for finite places. For $k_v \approx \mathbb{R}$, a ramified character depends on *sign*. For $k_v \approx \mathbb{C}$, a ramified character depends on *argument*.

A character χ on $\mathbb{J}$ is *ramified* if it is ramified on some $k_v^\times$. Ramification is equivalent to *not* being $\mathbb{J}^1$-invariant. The same terminology applies to characters on $\mathbb{J}/k^\times$.

Claim: For *ramified* χ, the elementary global integral vanishes:

$$\int_{J^+/k^\times} |x|^s \chi(x)\, d^\times x \;=\; 0 \qquad\qquad (\text{for } \chi \text{ ramified})$$

Thus, the residues of global zeta integrals $Z(s,\chi,f)$ at $s = 0, 1$ are 0 for ramified χ, so such global zeta integrals are *entire*.

Proof: For lighter notation, absorb the $|x|^s$ into χ. Use the integration riff

$$\int_G f(g)\, dg \;=\; \int_{H\backslash G} \Big(\int_H f(h\dot{g})\, dh \Big)\, d\dot{g}$$

to obtain

$$\int_{\mathbb{J}^+/k^\times} \chi(g)\, dg \;=\; \int_{(\mathbb{J}^+/k^\times)/(\mathbb{J}^1/k^\times)} \Big(\int_{\mathbb{J}^1/k^\times} \chi(\dot{g}h)\, dh \Big)\, dg$$

This invites a variant of the cancellation lemma: to be clear, we give the very slightly modified argument... let $h_o \in \mathbb{J}^1$ be such that $\chi(h_o) \neq 1$. Then, replacing h by hh_o,

$$\int_{\mathbb{J}^1/k^\times} \chi(\dot{g}h)\, dh \;=\; \int_{\mathbb{J}^1/k^\times} \chi(\dot{g}hh_o)\, dh \;=\; \chi(h_o) \cdot \int_{\mathbb{J}^1/k^\times} \chi(\dot{g}h)\, dh$$

Thus, the inner integral cancels, so the whole integral is 0. ///

Good finite-prime local integrals: $\int_{k_v^\times \cap \mathfrak{o}_v} |x|_v^s \, d^\times x$

Good includes the assertion that the local Schwartz function f_v in the local zeta integral expression

$$Z_v(s, f_v) \;=\; \int_{k_v^\times} |x|_v^s \, f_v(x) \, d^\times x$$

is the *characteristic function* of the local integers $\mathfrak{o}_v$.

The *good prime* assumption also includes less obvious, important points. By convention, *archimedean* primes are *never* good.

The good prime assumption includes the assertion that k_v is *absolutely unramified*, meaning k_v is unramified over the corresponding completion $\mathbb{Q}_p$, meaning p *stays prime* in $\mathfrak{o}_v$.

We will show that unramifiedness entails that the natural measure is $|\mathfrak{o}_v| \;=\; 1$, and the Fourier transform of the characteristic function of $\mathfrak{o}_v$ is *itself*. But these points do not affect the local *multiplicative* computation.

First, at finite primes, *always* normalize the multiplicative Haar measure so that $|\mathfrak{o}_v^\times| = 1$. Then the usual

$$\int_G f(g)\,dg \;=\; \int_{G/H} \int_H f(\dot{g}h)\;dh\;\;dg$$

with f the product of $|\cdot|_v^s$ and the characteristic function of $\mathfrak{o}_v$ gives

$$\int_{k_v^\times} f(g)\,dg \;=\; \int_{k_v^\times/\mathfrak{o}_v^\times} \int_{\mathfrak{o}_v^\times} f(\dot{g}h)\;dh\;\;dg$$

$$= \int_{(k_v^\times \cap \mathfrak{o}_v)/\mathfrak{o}_v^\times} \int_{\mathfrak{o}_v^\times} |\dot{g}h|_v^s\;dh\;\;dg \;=\; \int_{(k_v^\times \cap \mathfrak{o}_v)/\mathfrak{o}_v^\times} |\dot{g}|_v^s \Big(\int_{\mathfrak{o}_v^\times} 1\;dh \Big)\;dg$$

$$= \int_{(k_v^\times \cap \mathfrak{o}_v)/\mathfrak{o}_v^\times} |\dot{g}|_v^s\;dg \;=\; \sum_{n=0}^{\infty} |p^n|_v^s \;=\; \frac{1}{1-|p|_v^{-s}} \;=\; \frac{1}{1-q_v^{-s}}$$

where $q_v \;=\; |p|_v^{-1}$ is the residue field cardinality. ///

The same computation applies to the *seemingly* more general

$$Z_v(s, \chi_v, f_v) \;=\; \int_{k_v^\times} |x|_v^s \, \chi_v(x) \, f_v(x) \; d^\times x$$

with f_v the characteristic function of $\mathfrak{o}_v$ and χ_v *unramified*, meaning that χ_v is trivial on $\mathfrak{o}_v^\times$. That is, the group homomorphism χ_v is $\mathfrak{o}_v$-invariant, so is inescapably of the form

$$\chi_v(x) \;=\; |x|_v^{it_\chi} \qquad\qquad \text{(for some } t_\chi \in \mathbb{R} \text{ depending on } \chi_v\text{)}$$

Then the unramified non-archimedean local zeta factor is

$$Z_v(s, \chi_v, f_v) \;=\; \int_{k_v^\times} |x|_v^s \, \chi_v(x) \, f_v(x) \; d^\times x$$

$$= \int_{k_v^\times} |x|_v^{s+it_\chi} \, f_v(x) \; d^\times x \;=\; \frac{1}{1 - q_v^{-s-it_\chi}}$$

This kind of shifting occurs for all kinds of L-functions...