• **Interlude:** Calculus on spheres: invariant integrals, invariant \( \Delta = \Delta^S \), integration-by-parts, etc.

Decomposition of \( L^2(S^{n-1}) \) into \( \Delta^S \)-eigenfunctions.

Representation theory of orthogonal groups \( O(n, \mathbb{R}) \) or \( SO(n, \mathbb{R}) \).

... combine to prove

**Hecke’s identity:** For a homogeneous, degree \( d \) harmonic polynomial \( P \) on \( \mathbb{R}^n \), \( P(x) e^{-\pi |x|^2} \) is a Fourier transform eigenfunction with eigenvalue \( i^{-d} \):

\[
\left( P(x) e^{-\pi |x|^2} \right) \hat{\cdot}(\xi) = i^{-d} \cdot P(\xi) e^{-\pi |\xi|^2}
\]
Proof recap: Whether or not $P$ is harmonic,

$$\left( P(x) e^{-\pi |x|^2} \right) \hat{\cdot}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi, x \rangle} P(x) e^{-\pi |x|^2} \, dx$$

$$= P\left( \frac{1}{-2\pi i} \frac{\partial}{\partial \xi} \right) \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi, x \rangle} e^{-\pi |x|^2} \, dx$$

$$= P\left( \frac{1}{-2\pi i} \frac{\partial}{\partial \xi} \right) e^{-\pi |\xi|^2} = P^\#(\xi) e^{-\pi |\xi|^2}$$

for a polynomial $P^\#$ of total degree at most that of $P$. Since Fourier transform commutes with the action of $O(n, \mathbb{R})$ on functions,

$$\left( (P \circ g)(x) e^{-\pi |x|^2} \right) \hat{\cdot}(\xi) = P^\#(g\xi) e^{-\pi |\xi|^2}$$

Thus, $P \to P^\#$ is an $O(n, \mathbb{R})$-map: $(P \circ g)^\# = P^\# \circ g$ for $g \in O(n, \mathbb{R})$. 
Write $\Delta^S$ for a/the rotation-invariant second-order differential operator (Laplacian) on functions on $S = S^{n-1}$, and $\int_S f$ the rotation-invariant integral. Two characterizing properties are

$$\int_S (\Delta^S f) \cdot \varphi = \int_S f \cdot (\Delta^S \varphi) \quad \text{(self-adjointness)}$$

$$\int_S (\Delta^S f) \cdot \overline{f} \leq 0 \quad \text{(definiteness)}$$

with equality only for $f$ constant. Assume also that $\Delta^S$ has real coefficients, in the sense that $\Delta^S f = \Delta^S \overline{f}$.

There is the natural complex hermitian inner product

$$\langle f, g \rangle = \int_S f \cdot \overline{g} \quad \text{(for differentiable functions $f, g$ on $S$)}$$
Corollary: $\Delta^S$-eigenvectors $f, g$ with distinct eigenvalues are orthogonal. Eigenvalues are non-positive real.

Claim: The action of $SO(n)$ on $S^{n-1}$ is transitive.

The isotropy group $SO(n)_{e_n}$ of the last standard basis vector $e_n = (0, \ldots, 0, 1)$ is
\[
\{ \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} : A \in SO(n-1) \} \approx SO(n-1)
\]

By transitivity, as $SO(n)$-spaces $S^{n-1} \approx SO(n-1) \setminus SO(n)$

The action of $k \in SO(n)$ on functions $f$ on the sphere $S = S^{n-1}$ (or on the ambient $\mathbb{R}^n$) is $(k \cdot f)(x) = f(xk)$. The rotation invariance conditions are
\[
\int_S k \cdot f = \int_S f \quad \Delta^S(k \cdot f) = k \cdot (\Delta^S f) \quad \text{(for } k \in SO(n))
\]
The spherical Laplacian For $f$ on $S$, create a function $F$ on $\mathbb{R}^n - 0$ by $F(x) = f(x/|x|)$, and define

$$\Delta^S f = (\Delta F)|_S$$

Then $\Delta^S \overline{f} = \overline{\Delta^S f}$ and $\Delta^S$ is $SO(n)$-invariant.

Claim: For $f$ positive-homogeneous of degree $s$ on $\mathbb{R}^n - 0$

$$\Delta(|x|^{-s} f) = -s(s + n - 2)|x|^{-(s+2)} f + |x|^{-s} \Delta f$$

Corollary: For $f$ positive-homogeneous of degree $s$ and harmonic, the restriction $f|_S$ of $f$ to $S^{n-1}$ is an eigenfunction for $\Delta^S$,

$$\Delta^S (f|_S) = -s(s + n - 2) \cdot (f|_S)$$
The proof is a direct computation, except for one interesting fact, Euler’s identity:

\[ \sum_i x_i f_i(x) = s \cdot f \quad \text{(f positive-homogeneous degree s)} \]

Euler’s identity is proven by considering the function \( g(t) = f(tx) \) for \( t > 0 \), differentiating with respect to \( t \), and evaluating at \( t = 1 \).

Define complex-hermitian \((, )\) on \( \mathbb{C}[x_1, \ldots, x_n] \) by

\[(P, Q) = \overline{Q(\partial)}(P(x)) \mid_{x=0}\]

where \( Q(\partial) \) means to replace \( x_i \) by \( \partial/\partial x_i \) in a polynomial, and \( R \mid_{x=0} \) means to evaluate \( R \) at \( x = 0 \).
Multiplication by $r^2$ is adjoint to application of $\Delta$:

$$(\Delta f, g) = (f, r^2 g) \quad \text{(with } r^2 = x_1^2 + \ldots + x_n^2)$$

\textbf{Claim:} $\Delta : \mathbb{C}[x_1, \ldots, x_n]^{(d)} \rightarrow \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$ is surjective. Harmonic polynomials $f$ in $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$ are orthogonal to polynomials $r^2 h$ with $h \in \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$.

\textbf{Proof:} For $h \in \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$, if $(\Delta f, h) = 0$ for all $f$ in $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$, then

$$0 = (\Delta f, h) = (f, r^2 h) \quad \text{(for all } f)$$

so $r^2 h = 0$, so $h = 0$, by the positive-definiteness of $(,)$.

This also proves the second assertion.

\textbf{Corollary:} $\mathbb{C}[x_1, \ldots, x_n]^{(d)} = H_d \oplus r^2 H_{d-2} \oplus r^4 H_{d-4} + \ldots$
Corollary: Polynomials restricted to the $n$-sphere are equal to linear combinations of harmonic polynomials.

Proof: Use the observation

$$\mathbb{C}[x_1, \ldots, x_n]^{(d)} = H_d \oplus r^2 H_{d-2} \oplus r^4 H_{d-4} + \ldots$$

to write a homogeneous polynomial as

$$f = f_0 + r^2 f_2 + r^4 f_4 + \ldots$$

with each $f_i$ harmonic. Restricting to the sphere,

$$f|_S = (f_0 + r^2 f_2 + r^4 f_4 + \ldots)|_S = (f_0 + f_2 + f_4 + \ldots)|_S$$

since $r^2 = 1$ on the sphere.
Remark: From computations above,

$$\Delta^S f = -d(d + n - 2) \cdot f \quad \text{(for } f \in H_d)$$

Since $d \geq 0$,

$$\lambda_d = -d(d + n - 2) = -\left(d + \frac{n-2}{2}\right)^2 + \left(\frac{n-2}{2}\right)^2 \leq 0$$

The eigenvalues $\lambda_d = -d(d + n - 2)$ are strictly decreasing as $d \to +\infty$, so the spaces $H_d$ are distinguished by their eigenvalues for the spherical Laplacian.

Remark: For $S^1$, the 0-eigenspace is 1-dimensional and for $d > 0$ the $(-d^2)$-eigenspace is 2-dimensional, with basis $(x \pm iy)^d$. In contrast, for $n > 1$ the dimensions of eigenspaces are unbounded as the degree $d$ goes to $+\infty$. Specifically, ...
**Claim:** \( \dim_\mathbb{C} H_d = \dim \mathbb{C}[x_1, \ldots, x_n]^{(d)} - \dim \mathbb{C}[x_1, \ldots, x_n]^{(d-2)} \)

\[
= \binom{n+d-1}{n-1} - \binom{n+d-3}{n-1} \sim \text{constant} \cdot d^{n-2}
\]

**Proof:** From above, \( \Delta : \mathbb{C}[x_1, \ldots, x_n]^{(d)} \longrightarrow \mathbb{C}[x_1, \ldots, x_n]^{(d-2)} \) is surjective, so \( \dim H_d \) is the difference of dimensions.

The dimension of total-degree \( d \) polynomials in \( n \) variables is the number of monomials \( x_1^{e_1} \cdots x_n^{e_n} \) with \( \sum_i e_i = d \). Imagine each exponent as the corresponding number of marks, with \( n - 1 \) additional marks to separate the marks corresponding to the \( n \) distinct variables \( x_i \), for a total of \( n + d - 1 \). The choice of location of the separating marks is the binomial coefficient. 

**Corollary** (instance of Weyl’s Law) The dimension of the direct sum of (polynomial) \( \Delta^S \)-eigenspaces with \( |\lambda| < T \) grows like \( T^{\frac{n-1}{2}} = T^{\frac{1}{2}} \dim S^{n-1} \).
Invariant integrals on spheres, integration by parts for $\Delta^S$.

We have used an $SO(n)$-invariant integral on $S^{n-1}$ to show that eigenvalues for the spherical Laplacian $\Delta^S$ are non-positive, in determining all eigenvectors, using integration by parts on $S^{n-1}$.

Instead of invoking Haar measure, we could write a formula as follows, using $SO(n)$-invariance of the measure on $\mathbb{R}^n$. For continuous $f$ on $S$, define

$$
\int_S f = \int_{\mathbb{R}^n_{0}} \gamma(|x|^2) \, f(x/|x|) \, dx
$$

where $\gamma$ is a fixed smooth non-negative function on $[0, \infty)$ with

$$
\int_{\mathbb{R}^n} \gamma(|x|^2) \, dx = 1
$$

For convenience, we may at some moments suppose that $\gamma$ has compact support and vanishes identically on a neighborhood of 0.
For $k \in SO(n)$ we have the $SO(n)$-invariance of this integral:

$$\int_S k \cdot f = \int_{\mathbb{R}^n - 0} \gamma(|x|^2) f\left(\frac{xk}{|x|}\right) dx = \int_{\mathbb{R}^n - 0} \gamma(|xk^{-1}|^2) f\left(\frac{x}{|x|}\right) dx$$

$$= \int_{\mathbb{R}^n - 0} \gamma(|x|^2) f\left(\frac{x}{|x|}\right) dx = \int_S f$$

by changing variables to replace $x$ by $xk^{-1}$, and using $|xk^{-1}| = |x|$. Less trivial is proof of the desired integration-by-parts-twice result from this clunky viewpoint:

**Proposition:** For differentiable functions $f, \varphi$ on $S^n$,

$$\int_S (\Delta^S f) \cdot \varphi = \int_S f \cdot \Delta^S \varphi$$

Further, $\Delta^S$ is negative-definite in the sense that $\int_S (\Delta^S f) \cdot \bar{f} \leq 0$ with equality only for $f$ constant.
Proof: Let $F(x) = f(x/r)$ and $\Phi(x) = \varphi(x/r)$. By definition,

$$
\int_S (\Delta^S f) \cdot \varphi = \int_{\mathbb{R}^n_0} \gamma(r^2) r^2 \cdot (\Delta F)(x) \Phi(x) \, dx
$$

where $r^2$ is inserted so $r^2 \Delta F$ is positive-homogeneous of degree 0 as required. Integrating by parts on $\mathbb{R}^n$, this becomes

$$
-\int_{\mathbb{R}^n_0} \sum_i \frac{\partial F}{\partial x_i} \frac{\partial}{\partial x_i} \left( r^2 \cdot \gamma(r^2) \Phi(x) \right) \, dx
$$

With $\beta(r^2) = r^2 \gamma(r^2)$, the derivative $\frac{\partial}{\partial x_i} \left[ r^2 \cdot \gamma(r^2) \Phi(x) \right]$ is

$$
\frac{\partial}{\partial x_i} \left[ \beta(r^2) \Phi(x) \right] = 2x_i \beta'(r^2) \Phi(x) + \beta(r^2) \frac{\partial \Phi}{\partial x_i}
$$
Thus, the whole is

\[- \int_{\mathbb{R}^n - 0} \sum_i \frac{\partial F}{\partial x_i} \left[ 2x_i \beta'(r^2)\Phi(x) + \beta(r^2) \frac{\partial \Phi}{\partial x_i} \right] \, dx\]

\[= - \int_{\mathbb{R}^n - 0} \sum_i \frac{\partial F}{\partial x_i} \beta(r^2) \frac{\partial \Phi}{\partial x_i} \, dx\]

since by Euler’s identity \(\sum_i x_i \frac{\partial F}{\partial x_i} = (\text{degree } F) \cdot F = 0\). The last expression for the integral is symmetric in \(F\) and \(\Phi\). And with \(\Phi = \Phi \) the last expression is non-positive, and 0 only for \(\partial F/\partial x_i = 0\) for all \(i\), only if \(F\) is constant, only if \(f\) is constant.

Remark: A more persuasive argument will be given later.
Spectral decomposition of $L^2(S^{n-1})$ Functions on the sphere should be sums of eigenfunctions for $\Delta^S$, with convergence in $L^2$. $L^2$ convergence does not imply pointwise convergence, but for smooth functions eventually prove convergence in $C^\infty(S^{n-1})$.

**Theorem:**

\[ L^2(S^{n-1}) = \text{completion } \bigoplus_{d\geq 0} H_d|_{S^{n-1}} \quad \text{(orthogonal direct sum)} \]

**Proof:** For completeness, we will prove that restrictions to the sphere of harmonic polynomials are dense in $C^\infty(S^{n-1})$, which is dense in $L^2(S^{n-1})$.

With $S^{n-1} \subset \mathbb{R}^n$, a short-cut is available: invoke Weierstrass approximation to know that polynomials are sup-norm dense in $C^\infty(E)$ on any compact subset $E$ of $\mathbb{R}^n$. From above, polynomials restricted to $S^{n-1}$ are equal to harmonic polynomials. //
Thus, every $L^2$ function $f$ on $S^{n-1}$ has an $L^2$ Fourier-Laplace expansion

$$f = \sum_{d=0}^{\infty} f_d$$

(in $L^2(S^{n-1})$)

where $f_d$ is the orthogonal projection of $f$ in $L^2(S^{n-1})$ to the space $H_d$ of homogeneous degree $d$ harmonic polynomials restricted to the sphere.

The $d^{th}$ component $f_d$ is an eigenfunction for $\Delta^S$ with eigenvalue $\lambda_d = -d(d + n - 2)$.

**Note:** the $\Delta^S$-eigenvalues $\lambda_d = -d(d + n - 2)$ on $H_d$ are distinct.

Next, we look at this decomposition of $L^2(S^{n-1})$ in terms of the representation theory of $SO(n, \mathbb{R})$. 