• **Interlude:** Calculus on spheres: invariant integrals, invariant $\Delta = \Delta^S$, integration-by-parts, etc.

Decomposition of $L^2(S^{n-1})$ into $\Delta^S$-eigenfunctions.

Representation theory of orthogonal groups $O(n, \mathbb{R})$ or $SO(n, \mathbb{R})$.

... combine to prove

**Hecke’s identity:** For a homogeneous, degree $d$ harmonic polynomial $P$ on $\mathbb{R}^n$, $P(x) e^{-\pi|x|^2}$ is a Fourier transform eigenfunction with eigenvalue $i^{-d}$:

$$\left( P(x) e^{-\pi|x|^2} \right) \hat{\cdot}(\xi) = i^{-d} \cdot P(\xi) e^{-\pi|\xi|^2}$$

We need the representation theory of $SO(n, \mathbb{R})$. 

A little representation theory: Given a (topological) group $G$, a group homomorphism $T : G \to \text{Aut}_C^0(V)$ to the group of continuous $C$ linear automorphisms of a complex vector space $V$ is a representation of $G$ (on $V$). The representation is finite-dimensional when $V$ is. When the group homomorphism $G \to \text{Aut}_C^0(V)$ is understood, the standard abuse is to say that $V$ itself is the representation, not naming the homomorphism. In that case, rather than writing $(Tg)(v)$ for $g \in G$ and $v \in V$, write simply $gv$ for the action of $g$ on $v$.

There is a continuity requirement, that

$$G \times V \longrightarrow V$$

is continuous

For finite-dimensional $V$, there is a (provably) unique (topological vector space) topology, so need not be specified explicitly. Topologies on infinite-dimensional $V$ must be specified. The topology on $G$ should be clear from context.
A **$G$-subrepresentation** of a representation $V$ of $G$ is a complex vector subspace $W$ of $V$ stable under the action of $G$... and when $V$ is infinite-dimensional $W$ must be topologically *closed*.

A representation $V$ of $G$ is **irreducible** if there is no *proper* $G$-subrepresentation, that is, if there is no $G$-subrepresentation of $V$ other than $\{0\}$ and $V$ itself.

A $G$-homomorphism from one $G$-representation $V$ to another $G$-representation $W$ is a complex-linear map $\varphi : V \to W$ commuting with the action of $G$ in the sense that

$$\varphi(gv) = g \varphi(v) \quad \text{(for } g \in G, v \in V)$$

Such $G$-homomorphisms are also called $G$-morphism, $G$-maps, or also **$G$-intertwinings**, or **$G$-intertwining operators**. The collection of all $G$-intertwinings of $V$ to $W$ is $\text{Hom}_G(V, W)$. 
**Theorem:** *(instance of Schur’s Lemma)* For a finite-dimensional irreducible representation $V$ of a group $G$, any $G$-intertwining $\varphi : V \rightarrow V$ of $V$ to itself is *scalar*.

**Proof:** First, claim that the collection $\text{Hom}_G(V, V)$ of all $G$-intertwinings of finite-dimensional $V$ to itself is a *division ring*. Indeed, given $\varphi \neq 0$ in the ring $\text{Hom}_G(V, V)$, the image $\varphi(V)$ is readily seen to be a $G$-subrepresentation of $V$. For $\varphi$ not the zero map, since $V$ is irreducible, the image is either $\{0\}$ or $V$, so must be $V$ since $\varphi \neq 0$, and $\varphi$ is *surjective*. Similarly, the kernel of $\varphi$ is a $G$-subrepresentation, so is either the whole $V$, impossible since $\varphi$ is not the zero map, or is $\{0\}$. Thus, $\varphi$ is injective. Thus, $\varphi$ is a bijection, and therefore has an inverse (easily seen to be a $G$-map). Thus, non-zero elements of the ring $\text{Hom}_G(V, V)$ have multiplicative inverses, and $\text{Hom}_G(V, V)$ is a division ring.
For $V$ finite-dimensional the whole collection of complex-linear endomorphisms of $V$ is finite-dimensional. Certainly $\mathbb{C}$ naturally lies inside the center of $\text{Hom}_G(V, V)$. For $\varphi \in \text{Hom}_G(V, V)$, the collection of rational expressions $\mathbb{C}(\varphi)$ is a field, and is finite-dimensional as a vector space over the copy of $\mathbb{C}$ lying in the center of $\text{Hom}_G(V, V)$, so is algebraic. But $\mathbb{C}$ is algebraically closed (by Liouville’s theorem), so $\varphi \in \mathbb{C}$.

\textbf{Theorem:} The space $H_d$ of harmonic homogeneous total-degree $d$ polynomials is an irreducible $SO(n, \mathbb{R})$-representation.

\textit{Proof:} Suppose, to the contrary, that $X$ is a proper $SO(n, \mathbb{R})$-subspace of $H_d$. Then the orthogonal complement $Y$ of $X$ inside $H_d$ with respect to the $SO(n, \mathbb{R})$-invariant inner product is also $SO(n, \mathbb{R})$-stable.
The subspace $X$ consists of continuous functions, so, for $x \in S^{n-1}$, the functional $f \to f(x)$ is a linear functional on $X$. By an especially easy case of Riesz-Fischer, there is $F_x \in X$ such that $f(x) = \langle f, F_x \rangle$ for all $f \in X$. Since $X$ is rotation-invariant and not $\{0\}$, the functional $f \to f(x)$ cannot be 0 on all of $X$, so $F_x \neq 0 \in X$.

Similarly, there is $0 \neq \Phi_x \in Y$ such that $f(x) = \langle f, \Phi_x \rangle$ for all $f \in Y$.

By rotating, without loss of generality $x = e_n$. Then the following funny lemma proves that $F_x$ and $\Phi_x$ must be scalar multiples of each other, contradiction.
Let $SO(n - 1)$ be the smaller orthogonal group which is the isotropy group of the point $e_n = (0, \ldots, 0, 1)$.

**Lemma:** On $\mathbb{R}^n$, for each fixed $d$, in $H_d$ there is a unique (up to constant multiples) $SO(n - 1, \mathbb{R})$-fixed vector $f$, that is, so that $h \cdot f = f$ for every $h \in SO(n - 1, \mathbb{R})$.

**Proof:** A function invariant under $SO(n - 1)$ must be of the form

$$f_o(x) = F(\rho^2, t) \quad \text{(where } \rho^2 = x_1^2 + \ldots + x_{n-1}^2 \text{ and } t = x_n)$$

Computing,

$$\Delta f_0 = \sum_i \frac{\partial}{\partial x_i}(2x_i F_1) + F_{22} = \sum_i (2F_1 + 4x_i^2 F_{11}) + F_{22}$$

$$= 2(n - 1)F_1 + 4\rho^2 F_{11} + F_{22}$$

where subscripts denote derivatives.
Write the function as a polynomial in $t$ with coefficients functions of $\rho$. These coefficients are necessarily homogeneous functions of $\rho$, so are powers of $\rho$. Write these as powers of $\rho^2$.

$$f_\circ(x) = F(\rho^2, t)$$

$$= c_d t^d + c_{d-1} t^{d-1} (\rho^2)^{1/2} + c_{d-2} t^{d-2} (\rho^2)^1 + \ldots + c_0 (\rho^2)^{d/2}$$

The harmonic-ness $0 = 2(n - 1)F_1 + 4\rho^2 F_{11} + F_{22}$ rewritten in powers of $t$ gives a recurrence for the coefficients $c_i$. Explicitly, looking at the $(i - 2)^{th}$ power of $t$ in the harmonic-ness condition, ...

\[
0 = 2(n-1) \left( \frac{\partial}{\partial (\rho^2)} \right) \left( c_{i-2} (\rho^2)^{(d-i+2)/2} \right) \\
+ 4\rho^2 \left( \frac{\partial}{\partial (\rho^2)} \right)^2 \left( c_{i-2} (\rho^2)^{(d-i+2)/2} \right) i(i-1)c_i (\rho^2)^{(d-i)/2} \\
= \left[ 2(n-1) \frac{1}{2} (d-i+2) + 4 \cdot \frac{1}{2} (d-i+2) \cdot \frac{1}{2} (d-i) \right] (\rho^2)^{(d-i)/2} c_{i-2} \\
+ i(i-1)(\rho^2)^{(d-i)/2} c_i 
\]
Thus,
\[
i(i-1)c_i = -(d-i+2)(n-1+d-i)c_{i-2} 
\]
and the \(c_i\) are determined completely from \(c_1\) and \(c_0\).
On the other hand, looking at the $t^{d-1}$ term in the harmonic-ness relation,

$$0 = [2(n-1)\frac{1}{2}(\rho^2)^{-1/2} + 4\rho^2 \frac{1}{2}(-\frac{1}{2})(\rho^2)^{-3/2}]c_1 = [(n-1) - 1] \rho$$

So unless $n = 2$ we have $c_1 = 0$, and all coefficients are determined by $c_o$, giving the desired uniqueness result.

The case $n = 2$ can be treated more directly, from the easily demonstrable fact that

$$H_d = \mathbb{C} \cdot (x + iy)^d \oplus \mathbb{C} \cdot (x - iy)^d \quad \text{(on } \mathbb{R}^2)$$

This proves the uniqueness lemma, and the irreducibility of $H_d$ as $SO(n, \mathbb{R})$ representation.
The irreducibility of $H_d$ is the key point, but there are a few other small requirements before Hecke’s identity is completed.

As expected, two $G$-representations $V, W$ are *isomorphic* when there is a vector space isomorphism $V \to W$ that is a $G$-hom. When the vector spaces are infinite-dimensional, the map is required to be a *topological* vector space isomorphism, as expected.

Similar to Schur’s Lemma:

**Proposition:** For non-isomorphic irreducible finite-dimensional $G$-representations $V, W$, the only $G$-hom $\varphi : V \to W$ is the zero map.

**Proof:** The kernel and image of $\varphi$ are $G$-subrepresentations, so are either the whole space or $\{0\}$. The case that $\varphi : V \to W$ is a vector space isomorphism is excluded by the non-isomorphism assumption.
**Proposition:** (Complete Reducibility) A finite-dimensional $G$ representation $V$ with a $G$-invariant inner product $\langle \cdot, \cdot \rangle$ is a finite orthogonal direct sum of irreducible $G$-subrepresentations.

**Remark:** A representation space with a $G$-invariant inner product is said to be *unitary*.

**Proof:** Induction on dimension. For $V$ irreducible, we are done. Otherwise, $V$ has a proper subrepresentation $W$, which is a direct sum of irreducibles, by induction. The orthogonal complement $W^\perp$ of $W$ is immediately $G$-stable, so is a $G$-subrepresentation, and also is a direct sum of irreducibles, by induction. 

**Remark:** There is no straightforward infinite-dimensional analogue of the previous, even for Hilbert spaces. The orthogonality argument still succeeds, but induction fails.
The idea is that the \( SO(n, \mathbb{R}) \)-map

\[
\# : H_d \longrightarrow \mathbb{C}[x]_{\leq d} \approx H_d \oplus \text{(other irreducibles)}
\]

must map \( H_d \) to the other copy of \( H_d \), and, by Schur’s lemma, be a scalar.

This idea is correct, but has not quite been proven so far.

Specifically, although we easily showed that, for irreducibles \( \sigma, \tau \), the space of \( G \)-homs \( \text{Hom}_G(\sigma, \tau) \) is either 0 or \( \mathbb{C} \) depending on whether \( \sigma \not\cong \tau \) or \( \sigma \cong \tau \), this does not instantly address the question of \( G \)-maps to sums of irreducibles.

It is best to examines some clarifying structure.
Isotypes and co-isotypes A direct sum of a number of copies of an irreducible $\pi$ is denoted

$$m \cdot \pi = \pi \oplus \ldots \oplus \pi$$

Given an irreducible $\pi$ of $G$, we want to specify a $G$-sub $V^\pi$ of a $G$-rep’n $V$ such that any map $m \cdot \pi \to V$ factors (uniquely) through $V^\pi$, that is, $m \cdot \pi \to V^\pi \subset V$, the $\pi$-isotype of $V$.

Dually, the $\pi$-co-isotype $V_\pi$ of $V$ is the quotient of $V$ such that any map $V \to m \cdot \pi$ factors through $V_\pi$: $V \to V_\pi \to m \cdot \pi$.

A priori, existence is unclear, but on categorical grounds they are unique up to unique isomorphism if they do exist at all.

For unitary representations, the kernel of the map to the co-isotype has an orthogonal complement, so the co-isotype is naturally isomorphic to a sub-object, ... but in general we should not expect this simplicity.
Happily, for finite-dimensional irreducibles $\pi$ of \textit{compact} $G$, there is a natural \textit{projector} to the $\pi$-isotype.

It is not obvious, but the history of these issues does reasonably lead to the following. For a finite-dimensional irreducible $\pi$ of compact $G$, the \textit{character} $\chi_\pi$ of $G$ is a function on $G$ defined by

$$\chi_\pi(g) = \text{trace} \pi(g)$$

**Proposition:** For any $G$-representation $V$, the map

$$v \longrightarrow \chi_\pi \cdot v = \int_G \chi_\pi(g) \cdot g \cdot v \, dg$$

is a $G$-hom \textit{projecting} $V \rightarrow V^\pi$, where $\pi^\vee$ is the \textit{contragredient (dual) representation} of $\pi$. 