- **Classfield Theory...**
- Herbrand quotient as Euler-Poincaré characteristic
- Elementary kernel-image relations
- Long exact sequences from short exact of complexes
- Norm index equality for cyclic extensions of local fields

We can produce long exact sequences in (co-) homology from short exact sequences of *complexes*, but have no *general* mechanism to produce short exact sequences of complexes.

Nevertheless, an obvious example is the short exact sequence of *complexes* produced from a short exact sequence $0 \to A \to B \to C \to 0$ of G-modules for *finite cyclic* G, by the *universal* complex construction

$$A \quad \longrightarrow \quad \Big(\cdots \xrightarrow{t} A \xrightarrow{\sigma-1} A \xrightarrow{t} A \xrightarrow{\sigma-1} \cdots \Big)$$

for any G-module A, where $G = \langle \sigma \rangle$ and $t = \sum_{g \in G} g$.

Herbrand quotients: A periodic *complex*

$$\cdots \xrightarrow{f} A \xrightarrow{g} A \xrightarrow{f} A \xrightarrow{g} \cdots$$

has just two (co-) homology groups,

$$\frac{\ker f|_A}{\mathrm{im}\, g_A} \qquad \frac{\ker g|_A}{\mathrm{im}\, f_A}$$

and the Herbrand quotient is of A, f, g is

$$q_{f,g}(A) \;=\; \frac{[\ker f : \mathrm{im}\, g]}{[\ker g : \mathrm{im}\, f]}$$

Key Lemma: For finite A, $q(A) \;=\; 1$. For f-stable, g-stable subgroup $A \subset B$ with $f, g : B \to B$, we have $q(B) = q(A) \cdot q(B/A)$, in the usual sense that if two are finite, so is the third, and the relation holds. ///

With $C = B/A$, the lemma refers to a short exact sequence of complexes

$$\begin{array}{ccccccccc}
 & & \ldots & & \ldots & & \ldots & & \\
 & & \downarrow g & & \downarrow g & & \downarrow g & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow f & & \downarrow f & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow g & & \downarrow g & & \downarrow g & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow f & & \downarrow f & & \\
 & & \ldots & & \ldots & & \ldots & &
\end{array}$$

A special case of the **long exact sequence in (co-) homology** will give a periodic long exact sequence

.

$$
\begin{array}{ccccc}
 & \dfrac{\ker f|_A}{\operatorname{im} g|_A} & \longrightarrow & \dfrac{\ker f|_B}{\operatorname{im} g|_B} & \\
\nearrow & & & & \searrow \\
\dfrac{\ker g|_C}{\operatorname{im} f|_C} & & & & \dfrac{\ker f|_C}{\operatorname{im} g|_C} \\
\nwarrow & & & & \swarrow \\
 & \dfrac{\ker g|_B}{\operatorname{im} f|_B} & \longleftarrow & \dfrac{\ker g|_A}{\operatorname{im} f|_A} &
\end{array}
$$

By *Euler-Poincaré characteristics:*

$$1 \;=\; \frac{[\ker f_A : \operatorname{im} g_A]}{[\ker g_A : \operatorname{im} f_A]} \cdot \frac{[\ker g_B : \operatorname{im} f_B]}{[\ker f_B : \operatorname{im} g_B]} \cdot \frac{[\ker f_C : \operatorname{im} g_C]}{[\ker g_C : \operatorname{im} f_C]} \qquad ///$$

The triviality assertion: For A finite, $\frac{[\ker f_A : \mathrm{im}\, g_A]}{[\ker g_A : \mathrm{im}\, f_A]} = 1$. ///

Another useful, relatively elementary result:

Lemma: For abelian groups $A \supset B$ with a group homomorphism $f : A \to A'$, writing f_A for $f|_A$ and similarly for B,

$$[A : B] \;=\; [\ker f_A : \ker f_B] \;\cdot\; [\mathrm{im}\, f_A : \mathrm{im}\, f_B]$$

in the sense that if two of the indices are *finite*, then the third is, also, and equality holds. ///

Avoiding a general discussion of origins of complexes, Hilbert's Theorem 90 suggests a (non-topological) source: with finite cyclic $G = \langle \sigma \rangle$, with $t = \sum_{g \in G} g$, an exact sequence $0 \to A \to B \to C \to 0$ of G-modules gives a short exact sequence of complexes as in the Herbrand quotient situation:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma-1} & & \downarrow{\scriptstyle \sigma-1} & & \downarrow{\scriptstyle \sigma-1} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma-1} & & \downarrow{\scriptstyle \sigma-1} & & \downarrow{\scriptstyle \sigma-1} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

and the Herbrand quotient lemma gives

$$\frac{[\ker t|_A : \mathrm{im}\,(\sigma-1)|_A]}{[\ker(\sigma-1)|_A : \mathrm{im}\,t|_A]} \times \frac{[\ker(\sigma-1)|_B : \mathrm{im}\,t|_B]}{[\ker t|_B : \mathrm{im}\,(\sigma-1)|_B]}$$

$$\times \frac{[\ker t|_C : \mathrm{im}\,(\sigma-1)|_C]}{[\ker(\sigma-1)|_C : \mathrm{im}\,t|_C]} \quad = \quad 1$$

for an exact sequence $0 \to A \to B \to C \to 0$ of modules for a finite cyclic group $G = \langle\sigma\rangle$.

Local cyclic norm index theorem: (Also, see Lang, p. 187 ff.) For a cyclic extension K/k of degree n of local fields, with Galois group $G = \langle\sigma\rangle$ and ramification index e, integers $\mathfrak{o} \subset k$ and $\mathfrak{O} \subset K$,

$$[k^\times : N_k^K K^\times] = n \qquad [\mathfrak{o}^\times : N_k^K \mathfrak{O}^\times] = e$$

Proof: Apply the Herbrand quotient lemma to

$$0 \longrightarrow \mathfrak{O}^\times \longrightarrow K^\times \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $\mathrm{ord} : K^\times \to \mathbb{Z}$. Since Galois preserves $|\cdot|_K$, the action of G on this copy of $\mathbb{Z}$ is *trivial.* Thus,

$$\ker t|_{\mathbb{Z}} \;=\; \{0\} \qquad\qquad \mathrm{im}\, t|_{\mathbb{Z}} \;=\; n\cdot\mathbb{Z}$$

and

$$\ker(\sigma-1)|_{\mathbb{Z}} \;=\; \mathbb{Z} \qquad\qquad \mathrm{im}\,(\sigma-1)|_{\mathbb{Z}} \;=\; 0$$

so

$$q_{\sigma-1,t}(\mathbb{Z}) \;=\; \frac{[\ker(\sigma-1)|_{\mathbb{Z}} : \mathrm{im}\, t|_{\mathbb{Z}}]}{[\ker t|_{\mathbb{Z}} : \mathrm{im}\,(\sigma-1)|_{\mathbb{Z}}]} \;=\; \frac{[\mathbb{Z} : n\cdot\mathbb{Z}]}{[\{0\} : \{0\}]} \;=\; n$$

Theorem 90 is $\ker t|_{K^\times} = \mathrm{im}\,(\sigma - 1)|_{K^\times}$. Thus,

$$q_{\sigma-1,t}(K^\times) \;=\; \frac{[\ker(\sigma-1)|_{K^\times} : \mathrm{im}\, t|_{K^\times}]}{[\ker t|_{K^\times} : \mathrm{im}\,(\sigma-1)|_{K^\times}]}$$

$$= \frac{[k^\times : N_k^K K^\times]}{1} \;=\; [k^\times : N_k^K K^\times]$$

Thus, the Herbrand Lemma gives

$$n \;=\; q_{\sigma-1,t}(\mathbb{Z}) \;=\; \frac{q_{\sigma-1,t}(K^\times)}{q_{\sigma-1,t}(\mathfrak{O}^\times)} \;=\; \frac{[k^\times : N_k^K K^\times]}{q_{\sigma-1,t}(\mathfrak{O}^\times)}$$

We show that $q_{\sigma-1,t}(\mathfrak{O}^\times) = 1$.

There is a *normal basis* $x_1, \ldots, x_n$ for K/k, that is, G acts transitively on the x_i. Multiply every x_i by the same sufficiently high power of a local parameter in k to preserve the normal basis property, and to put all the x_i inside a sufficiently high power of the maximal ideal in $\mathfrak{O}$ such that the exponential map is defined on $V = \sum_i \mathfrak{o} x_i$, and inverse given by logarithm is defined on its image $U = \exp V$.

Claim: $\ker(\sigma - 1)|_V = \mathrm{im}\, t|_V$ and $\ker t|_V = \mathrm{im}\,(\sigma - 1)|_V$.

Proof of claim: With $(\sigma - 1) \sum_i c_i x_i = 0$ with $c_i \in k$, all coefficients are the same, by transitivity. On the other hand, if the coefficients are the same, certainly the element is in $\ker(\sigma - 1)$. Application of t to $c_1 x_1$ produces all elements with identical coefficients. Thus, $\ker(\sigma - 1)|_V = \mathrm{im}\, t|_V$. This is one equality.

For the other equality, index so that $x_i = \sigma^{i-1}(x_1)$.

Vanishing $t \cdot \sum_i c_i x_i = 0$ is exactly $(\sum_i c_i)(\sum_i x_i) = 0$, which is $\sum_i c_i = 0$. Then

$$\sum_i c_i x_i \;=\; \sum_i c_i(x_i - x_1) \;=\; \sum_i c_i(\sigma^{i-1} - 1)x_1 \;\in\; (\sigma - 1)\sum_i \mathfrak{o} x_i$$

so $\ker t|_V = \mathrm{im}\,(\sigma - 1)|_V$. ///

Since the Galois action is *continuous* on K, it commutes with exp and log where the series converge. Thus, $V = \exp U$ is a G-module with the same Herbrand-related quotients as U, namely

$$\ker(\sigma - 1)|_U = \mathrm{im}\, t|_U \qquad \text{and} \qquad \ker t|_U = \mathrm{im}\,(\sigma - 1)|_U$$

Since $[\mathfrak{O}^\times : U] < \infty$, by the Lemma $q_{\sigma-1,t}(\mathfrak{O}^\times/U) = 1$, and, again by the Herbrand Lemma,

$$1 \;=\; q_{\sigma-1,t}(V) \;=\; q_{\sigma-1,t}(U) \;=\; \frac{q_{\sigma-1,t}(\mathfrak{O}^\times)}{q_{\sigma-1,t}(\mathfrak{O}^\times/U)} \;=\; q_{\sigma-1,t}(\mathfrak{O}^\times)$$

From this,

$$1 \;=\; q_{\sigma-1,t}(\mathfrak{O}^\times) \;=\; \frac{[\mathfrak{o}^\times : N_k^K \mathfrak{O}^\times]}{[\ker t|_{\mathfrak{O}^\times} : \mathrm{im}\,(\sigma-1)|_{\mathfrak{O}^\times}]}$$

Since $|\sigma x/x| = 1$, by Theorem 90, $\ker t|_{\mathfrak{O}^\times} = \mathrm{im}\,(\sigma - 1)|_{K^\times}$. Thus,

$$[\ker t|_{\mathfrak{O}^\times} : \mathrm{im}\,(\sigma - 1)|_{\mathfrak{O}^\times}] \;=\; [\mathrm{im}\,(\sigma - 1)|_{K^\times} : \mathrm{im}\,(\sigma - 1)|_{\mathfrak{O}^\times}]$$

$$=\; [\mathrm{im}\,(\sigma - 1)|_{K^\times} : \mathrm{im}\,(\sigma - 1)|_{k^\times\mathfrak{O}^\times}]$$

Using $[A : B] = [\ker f|_A : \ker f|_B] \;\cdot\; [\mathrm{im}\, f|_A : \mathrm{im}\, f|_B]$ for $A \supset B$, this is

$$\frac{[K^\times : k^\times\mathfrak{O}^\times]}{[\ker(\sigma - 1)|_{K^\times} : \ker(\sigma - 1)|_{k^\times\mathfrak{O}^\times}]}$$

Essentially by definition, $[K^\times : k^\times\mathfrak{O}^\times] = e$, and

$$[\ker(\sigma - 1)|_{K^\times} : \ker(\sigma - 1)|_{k^\times\mathfrak{O}^\times}] \;=\; [k^\times : k^\times] \;=\; 1$$

so $[\ker t|_{\mathfrak{O}^\times} : \mathrm{im}\,(\sigma - 1)|_{\mathfrak{O}^\times}] = e$.

Thus,

$$1 \;=\; \frac{[\mathfrak{o}^\times : N_k^K \mathfrak{O}^\times]}{[\ker t|_{\mathfrak{O}^\times} : \mathrm{im}\,(\sigma - 1)|_{\mathfrak{O}^\times}]} \;=\; \frac{[\mathfrak{o}^\times : N_k^K \mathfrak{O}^\times]}{e}$$

and the cyclic local norm index theorem is done. ///

Elementary abelian group theory and induction give

Corollary: For finite *abelian* extension K/k of local fields,

$$[k^\times : N_k^K K^\times] \le [K : k] \qquad \text{and} \qquad [\mathfrak{o}^\times : N_k^K \mathfrak{O}^\times] \le e \qquad ///$$

Remark: Local classfield theory asserts *equalities* here for all finite abelian extensions, not only cyclic.