• **Classfield Theory...**
• Herbrand quotient as Euler-Poincaré characteristic
• Elementary kernel-image relations
• Long exact sequences from short exact of complexes
• Norm index equality for cyclic extensions of local fields

We can produce long exact sequences in (co-) homology from short exact sequences of *complexes*, but have no *general* mechanism to produce short exact sequences of complexes.

Nevertheless, an obvious example is the short exact sequence of *complexes* produced from a short exact sequence $0 \to A \to B \to C \to 0$ of $G$-modules for *finite cyclic* $G$, by the *universal* complex construction

$$A \rightarrow \left( \cdots \xrightarrow{t} A \xrightarrow{\sigma^{-1}} A \xrightarrow{t} A \xrightarrow{\sigma^{-1}} \cdots \right)$$

for any $G$-module $A$, where $G = \langle \sigma \rangle$ and $t = \sum_{g \in G} g$.  


**Herbrand quotients:** A periodic complex

\[ \cdots \xrightarrow{f} A \xrightarrow{g} A \xrightarrow{f} A \xrightarrow{g} \cdots \]

has just two (co-) homology groups,

\[ \frac{\ker f}{\operatorname{im} g} \quad \frac{\ker g}{\operatorname{im} f} \]

and the Herbrand quotient is of \( A, f, g \) is

\[ q_{f,g}(A) = \frac{[\ker f : \operatorname{im} g]}{[\ker g : \operatorname{im} f]} \]

**Key Lemma:** For finite \( A \), \( q(A) = 1 \). For \( f \)-stable, \( g \)-stable subgroup \( A \subset B \) with \( f, g : B \rightarrow B \), we have \( q(B) = q(A) \cdot q(B/A) \), in the usual sense that if two are finite, so is the third, and the relation holds.

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With $C = B/A$, the lemma refers to a short exact sequence of complexes

\[
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
\downarrow g & \downarrow g & \downarrow g \\
0 & \rightarrow A & \rightarrow B & \rightarrow C & \rightarrow 0 \\
\downarrow f & \downarrow f & \downarrow f \\
0 & \rightarrow A & \rightarrow B & \rightarrow C & \rightarrow 0 \\
\downarrow g & \downarrow g & \downarrow g \\
0 & \rightarrow A & \rightarrow B & \rightarrow C & \rightarrow 0 \\
\downarrow f & \downarrow f & \downarrow f \\
\cdots & \cdots & \cdots 
\end{array}
\]

A special case of the **long exact sequence in (co-) homology** will give a periodic long exact sequence
By Euler-Poincaré characteristics:

\[
1 = \frac{\ker f_A : \im g_A}{\ker g_A : \im f_A} \cdot \frac{\ker g_B : \im f_B}{\ker f_B : \im g_B} \cdot \frac{\ker f_C : \im g_C}{\ker g_C : \im f_C}
\]
The triviality assertion: For $A$ finite, $\frac{[\ker f_A : \text{im } g_A]}{[\ker g_A : \text{im } f_A]} = 1$. ///

Another useful, relatively elementary result:

**Lemma:** For abelian groups $A \supset B$ with a group homomorphism $f : A \to A'$, writing $f_A$ for $f|_A$ and similarly for $B$,

\[
[A : B] = [\ker f_A : \ker f_B] \cdot [\text{im } f_A : \text{im } f_B]
\]

in the sense that if two of the indices are finite, then the third is, also, and equality holds.

///
Avoiding a general discussion of origins of complexes, Hilbert’s Theorem 90 suggests a (non-topological) source: with finite cyclic $G = \langle \sigma \rangle$, with $t = \sum_{g \in G} g$, an exact sequence $0 \to A \to B \to C \to 0$ of $G$-modules gives a short exact sequence of complexes as in the Herbrand quotient situation:

\[
\begin{array}{c}
0 \\[-2em]
\downarrow \\
A \\[-2em]
\downarrow \\
B \\[-2em]
\downarrow \\
C \\[-2em]
\downarrow \\
0
\end{array}
\]

\[
\begin{array}{ccc}
\sigma^{-1} & \sigma^{-1} & \sigma^{-1} \\
\downarrow & \downarrow & \downarrow \\
A & B & C \\
\sigma^{-1} & \sigma^{-1} & \sigma^{-1} \\
\downarrow & \downarrow & \downarrow \\
A & B & C
\end{array}
\]
and the Herbrand quotient lemma gives

\[
\frac{[\ker t\mid_A : \im (\sigma - 1)\mid_A]}{[\ker(\sigma - 1)\mid_A : \im t\mid_A]} \times \frac{[\ker(\sigma - 1)\mid_B : \im t\mid_B]}{[\ker t\mid_B : \im (\sigma - 1)\mid_B]}
\times \frac{[\ker t\mid_C : \im (\sigma - 1)\mid_C]}{[\ker(\sigma - 1)\mid_C : \im t\mid_C]} = 1
\]

for an exact sequence \(0 \to A \to B \to C \to 0\) of modules for a finite cyclic group \(G = \langle \sigma \rangle\).

**Local cyclic norm index theorem:** (Also, see Lang, p. 187 ff.)

For a cyclic extension \(K/k\) of degree \(n\) of local fields, with Galois group \(G = \langle \sigma \rangle\) and ramification index \(e\), integers \(\mathfrak{o} \subset k\) and \(\mathcal{O} \subset K\),

\[
[k^\times : N_k^K k^\times] = n \quad [\mathfrak{o}^\times : N_k^K \mathcal{O}^\times] = e
\]
**Proof:** Apply the Herbrand quotient lemma to

\[ 0 \rightarrow \mathfrak{O}^\times \rightarrow K^\times \rightarrow \mathbb{Z} \rightarrow 0 \]

where \( \text{ord} : K^\times \rightarrow \mathbb{Z} \). Since Galois preserves \( | \cdot |_K \), the action of \( G \) on this copy of \( \mathbb{Z} \) is *trivial*. Thus,

\[
\ker t|_\mathbb{Z} = \{0\} \quad \text{im} t|_\mathbb{Z} = n \cdot \mathbb{Z}
\]

and

\[
\ker(\sigma - 1)|_\mathbb{Z} = \mathbb{Z} \quad \text{im} (\sigma - 1)|_\mathbb{Z} = 0
\]

so

\[
q_{\sigma - 1,t}(\mathbb{Z}) = \frac{[\ker(\sigma - 1)|_\mathbb{Z} : \text{im} t|_\mathbb{Z}]}{[\ker t|_\mathbb{Z} : \text{im} (\sigma - 1)|_\mathbb{Z}]} = \frac{[\mathbb{Z} : n \cdot \mathbb{Z}]}{[\{0\} : \{0\}]} = n
\]
Theorem 90 is $\ker t|_{K^\times} = \text{im} (\sigma - 1)|_{K^\times}$. Thus,

$$q_{\sigma - 1, t}(K^\times) = \frac{[\ker (\sigma - 1)|_{K^\times} : \text{im } t|_{K^\times}]}{[\ker t|_{K^\times} : \text{im} (\sigma - 1)|_{K^\times}]}$$

$$= \frac{[k^\times : N_k^K K^\times]}{1} = [k^\times : N_k^K K^\times]$$

Thus, the Herbrand Lemma gives

$$n = q_{\sigma - 1, t}(\mathbb{Z}) = \frac{q_{\sigma - 1, t}(K^\times)}{q_{\sigma - 1, t}(\mathcal{O}^\times)} = \frac{[k^\times : N_k^K K^\times]}{q_{\sigma - 1, t}(\mathcal{O}^\times)}$$

We show that $q_{\sigma - 1, t}(\mathcal{O}^\times) = 1$. 

There is a normal basis $x_1, \ldots, x_n$ for $K/k$, that is, $G$ acts transitively on the $x_i$. Multiply every $x_i$ by the same sufficiently high power of a local parameter in $k$ to preserve the normal basis property, and to put all the $x_i$ inside a sufficiently high power of the maximal ideal in $\mathcal{O}$ such that the exponential map is defined on $V = \sum_i ax_i$, and inverse given by logarithm is defined on its image $U = \exp V$.

**Claim:** $\ker(\sigma - 1)|_V = \text{im } t|_V$ and $\ker t|_V = \text{im } (\sigma - 1)|_V$.

**Proof of claim:** With $(\sigma - 1) \sum c_i x_i = 0$ with $c_i \in k$, all coefficients are the same, by transitivity. On the other hand, if the coefficients are the same, certainly the element is in $\ker(\sigma - 1)$. Application of $t$ to $c_1 x_1$ produces all elements with identical coefficients. Thus, $\ker(\sigma - 1)|_V = \text{im } t|_V$. This is one equality.
For the other equality, index so that \( x_i = \sigma^{i-1}(x_1) \).

Vanishing \( t \cdot \sum_i c_i x_i = 0 \) is exactly \( (\sum_i c_i)(\sum_i x_i) = 0 \), which is \( \sum_i c_i = 0 \). Then

\[
\sum_i c_i x_i = \sum_i c_i (x_i - x_1) = \sum_i c_i (\sigma^{i-1} - 1)x_1 \in (\sigma - 1) \sum_i \sigma x_i
\]

so \( \ker t|_V = \im (\sigma - 1)|_V \).

Since the Galois action is \textit{continuous} on \( K \), it commutes with exp and log where the series converge. Thus, \( V = \exp U \) is a \( G \)-module with the same Herbrand-related quotients as \( U \), namely

\[
\ker (\sigma - 1)|_U = \im t|_U \quad \text{and} \quad \ker t|_U = \im (\sigma - 1)|_U
\]
Since \([\mathcal{O}^\times : U] < \infty\), by the Lemma \(q_{\sigma-1,t}(\mathcal{O}^\times / U) = 1\), and, again by the Herbrand Lemma,

\[
1 = q_{\sigma-1,t}(V) = q_{\sigma-1,t}(U) = \frac{q_{\sigma-1,t}(\mathcal{O}^\times)}{q_{\sigma-1,t}(\mathcal{O}^\times / U)} = q_{\sigma-1,t}(\mathcal{O}^\times)
\]

From this,

\[
1 = q_{\sigma-1,t}(\mathcal{O}^\times) = \frac{[\sigma^\times : N^K_k \mathcal{O}^\times]}{[\ker t|_{\mathcal{O}^\times} : \text{im } (\sigma - 1)|_{\mathcal{O}^\times}]}
\]

Since \(|\sigma x/x| = 1\), by Theorem 90, \(\ker t|_{\mathcal{O}^\times} = \text{im } (\sigma - 1)|_{K^\times}\). Thus,
\[
[\ker t|_{\mathcal{O}^\times} : \im (\sigma - 1)|_{\mathcal{O}^\times}] = [\im (\sigma - 1)|_{K^\times} : \im (\sigma - 1)|_{\mathcal{O}^\times}]
\]
\[
= [\im (\sigma - 1)|_{K^\times} : \im (\sigma - 1)|_{k^\times\mathcal{O}^\times}]
\]
Using \([A : B] = [\ker f|_A : \ker f|_B] \cdot [\im f|_A : \im f|_B]\) for \(A \supset B\), this is
\[
\frac{[K^\times : k^\times\mathcal{O}^\times]}{[\ker(\sigma - 1)|_{K^\times} : \ker(\sigma - 1)|_{k^\times\mathcal{O}^\times}]}
\]
Essentially by definition, \([K^\times : k^\times\mathcal{O}^\times]\) = e, and
\[
[\ker(\sigma - 1)|_{K^\times} : \ker(\sigma - 1)|_{k^\times\mathcal{O}^\times}] = [k^\times : k^\times] = 1
\]
so \([\ker t|_{\mathcal{O}^\times} : \im (\sigma - 1)|_{\mathcal{O}^\times}] = e\).
Thus,

$$1 = \frac{[\mathfrak{o}^\times : N^K_k \mathfrak{O}^\times]}{[\ker t|_{\mathfrak{O}^\times} : \text{im } (\sigma - 1)|_{\mathfrak{O}^\times}]} = \frac{[\mathfrak{o}^\times : N^K_k \mathfrak{O}^\times]}{e}$$

and the cyclic local norm index theorem is done. ///

Elementary abelian group theory and induction give

**Corollary:** For finite *abelian* extension $K/k$ of local fields,

$$[k^\times : N^K_k K^\times] \leq [K : k] \quad \text{and} \quad [\mathfrak{o}^\times : N^K_k \mathfrak{O}^\times] \leq e \quad ///$$

**Remark:** Local classfield theory asserts *equalities* here for all finite abelian extensions, not only cyclic.