- **Classfield Theory...**
- Non-homological statement of main theorems
- Abrupt, insufficiently prepared, homological statement
- Parsing some notions of group (co-) homology

In fact, we did not yet complete *any* proof of local or global classfield theory, although the Herbrand quotient discussion and discussion of long exact sequences from short exact sequences of *complexes* are essential for almost any sensible proof.

For a beginner, the homological *statements* are more difficult to understand than the non-homological. However, the price is worth paying, since the classical arguments of Takagi and Artin are complicated and somewhat idiosyncratic. An overtly homological approach is (potentially) better organized, easier to understand in the long run, and connects more obviously to standard mathematics.

Non-homological statements: *recall... Local* classfield theory asserts that Galois groups of finite abelian extensions K of a local field k are exactly the quotients $k^\times/N_k^K(K^\times) \xrightarrow[\approx]{\alpha_{L/k}} \mathrm{Gal}(K/k)$.

Existence: *every* open finite-index subgroup of $k^\times$ is of the form $N_k^K K^\times$. The **Artin/reciprocity law** maps to Galois groups are *natural*: for finite abelian extensions $L \supset K \supset k$ there is a commutative diagram

$$\begin{array}{ccc} k^\times/N_k^L(L^\times) & \xrightarrow{\alpha_{L/k}} & \mathrm{Gal}(L/k) \\ \Big\downarrow \text{quot} & & \Big\downarrow \text{quot} \\ k^\times/N_k^K(K^\times) & \xrightarrow[\alpha_{K/k}]{} & \mathrm{Gal}(K/k) \end{array}$$

For an abelian extension of number fields K/k, the *global* Artin/reciprocity map $\alpha_{K/k} : \mathbb{J} \to \mathrm{Gal}(K/k)$ is *essentially the product of the local ones...*

Remark: So far, with considerable effort, we have only managed to prove $[K : k] = [k^\times : N_k^K K^\times]$ for *cyclic* local field extensions.

Recall: For $\mathfrak{p}$ in $\mathfrak{o}_k$ and $\mathfrak{P}|\mathfrak{p}$ in $\mathfrak{o}_K$ unramified in abelian K/k, the inertia subgroup of the decomposition group $G_\mathfrak{p} \subset \mathrm{Gal}(K/k)$ is trivial, $G_\mathfrak{p}$ is generated by the Artin element $(\mathfrak{p}, K/k)$.

The corresponding unramified extension of completions K_w/k_v is *cyclic* with Galois group generated by the local Artin element $(\mathfrak{m}_v, K_w/k_v)$ with $\mathfrak{m}_v$ the unique non-zero prime in $\mathfrak{o}_v$. The local Artin/reciprocity map $\alpha_{w/v} : k_v^\times \to \mathrm{Gal}(K_w/k_v)$ is

$$\alpha_{w/v}(x) \;=\; (\mathfrak{m}_v, K_w/k_v)^{\mathrm{ord}_v x} \qquad\qquad (\text{unramified } K_w/k_v)$$

Identifying the two cyclic groups $\mathrm{Gal}(K_w/k_v) \approx G_\mathfrak{p}$ by identifying their corresponding Artin elements $(\mathfrak{m}_v, K_w/k_v) \longleftrightarrow (\mathfrak{p}, K/k)$, we can consider the local Artin map as mapping to $G_\mathfrak{p}$, and

$$\alpha_{w/v} \;:\; k_v^\times \;\longrightarrow\; \mathrm{Gal}(K_w/k_v) \;\approx\; G_\mathfrak{p} \;\subset\; \mathrm{Gal}(K/k)$$

With the identification $\mathrm{Gal}(K_w/k_v) \approx G_\mathfrak{p} \subset \mathrm{Gal}(K/k)$ at unramified places, define the *global* Artin/reciprocity map $\alpha_{K/k} : \mathbb{J} \longrightarrow \mathrm{Gal}(K/k)$ by

$$\alpha_{K/k}(x) = \prod_v \prod_{w|v} \alpha_{w/v}(x_v) \qquad \text{(for } x = \{x_v\} \in \mathbb{J}_k\text{)}$$

Remark: For the moment, we seem not to know how to define local Artin/reciprocity maps at *ramified* primes.

The *critical* part of the assertion of global classfield theory is that the global $\alpha_{K/k}$ *factors through* the idele class group $\mathbb{J}_k/k^\times$ and $\alpha_{K/k} : \mathbb{J}_k/k^\times N_k^K \mathbb{J}_K \longrightarrow \mathrm{Gal}(K/k)$ is an *isomorphism*.

Remark: The significance of factoring through $\mathbb{J}/k^\times$ and $\mathbb{J}/k^\times N_k^K \mathbb{J}_K$ comes in two parts:

Since norms in unramified extensions of non-archimedean fields are *surjective* to local units, and norms on archimedean fields are open maps, the image $N_k^K \mathbb{J}_K$ is *open* in $\mathbb{J}_k$. Thus, the local and global Artin maps are *continuous*.

The latter open-ness/continuity reformulates *part* of the classical assertion that the Artin map **has a conductor**. But the difficult part is proving $k^\times$-invariance.

Recall that such assertions, such as we considered for the quadratic Hilbert symbol, imply classical-looking reciprocity laws.

Homological assertions: In contrast to our usual, we give statements and explain them afterward. Write $G_{K/k} = \mathrm{Gal}(K/k)$.

For an extension K/k of local fields with $[K : k] = n$, $H^2(G_{K/k}, K^\times)$ is *cyclic* of order n, and contains a unique generator, the **canonical class** $u_{K/k}$ which under the *Brauer invariant* map $H^2(G_{K/k}, K^\times) \to \mathbb{Q}/\mathbb{Z}$ maps to $1/n$.

Theorem: For $q \in \mathbb{Z}$, the *cup-product* $\alpha \to \alpha \cdot u_{K/k}$ on Tate cohomology $\widehat{H}^q(G_{K/k}, \mathbb{Z}) \to \widehat{H}^{q+2}(G_{K/k}, K^\times)$ is an *isomorphism.*

In particular, for $q = -2$, $\widehat{H}^{-2}(G, \mathbb{Z})$ is ordinary group homology $H_1(G, \mathbb{Z})$, and $H_1(G, \mathbb{Z}) = G/G^{\mathrm{der}} = G^{\mathrm{ab}}$. Also, $\widehat{H}^0(G_{K/k}, K^\times) = k^\times/N_k^K K^\times$, so:

Corollary/Definition: the **inverse reciprocity map** or **inverse norm residue symbol** $\alpha_{K/k}^{-1} : \alpha \to \alpha \cdot u_{K/k}$ is an *isomorphism* $\mathrm{Gal}(K/k)^{\mathrm{ab}} \to k^\times/N_k^K K^\times$.

Local Existence Theorem: Given an open finite-index subgroup U of $k^\times$, there is a unique abelian K/k with $N_k^K K^\times = U$.

Remark: Because the (inverse) reciprocity map is given by cup product with a canonical element, for $L \supset K \supset k$, some diagrams will obviously commute. Further, for *groundfields* $k \subset k' \subset K$,

$$\begin{array}{ccc} k^\times & \xrightarrow{\alpha_{K/k'}} & \mathrm{Gal}(K/k)^{\mathrm{ab}} \\ {\scriptstyle \mathrm{inc}}\downarrow & & \downarrow{\scriptstyle V} \\ k'^\times & \xrightarrow{\alpha_{K/k'}} & \mathrm{Gal}(K/k')^{\mathrm{ab}} \end{array}$$

where V is the *Verlagerung*, or *transfer* (below...) In fact, this may provide a minor motivation to understand *transfer*.

Global:

Lemma: For all $w|v$ in K_w/k_v, the groups $H^q(G_{K_w/k_v}, K_w^\times)$ are canonically isomorphic to each other, so we identify them. Then

$$\widehat{H}^q(G_{K/k}, \mathbb{J}_K) \approx \coprod_v \widehat{H}^q(G_{K_w/k_v}, K_w^\times)$$

Corollary: $H^1(G_{K/k}, \mathbb{J}_K) = 0$ and

$$H^2(G_{K/k}, \mathbb{J}_K) \approx \coprod_v \frac{1}{n_v}\mathbb{Z}\Big/\mathbb{Z} \qquad\qquad (\text{with } n_v = [K_w : k_v])$$

Proposition: $(\mathbb{J}_K/K^\times)^{\mathrm{Gal}(K/k)} = \mathbb{J}_k/k^\times$

Theorem: With Herbrand quotient $q(G, A) = |H^2(A)|/|H^1(A)|$ of G-module A, for *cyclic* K/k, $q(G_{K/k}, \mathbb{J}_K/K^\times) = n$.

Corollary: For K/k cyclic of degree n, $|\mathbb{J}_k/k^\times N_k^K \mathbb{J}_K| \geq n$.

Remark: This was formerly the *second* inequality, but by 1960's became the *first* inequality.

Corollary: For $N_k^K \mathbb{J}_K \ \subset \ U \ \subset \ \mathbb{J}_k$ and $k^\times U$ dense in $\mathbb{J}_k$, necessarily $K = k$.

Corollary: (For finite abelian K/k), for S any finite set of primes containing ramified primes, $\mathrm{Gal}(K/k)$ is generated by Frobenius elements from primes *not* in S.

Corollary: There are infinitely-many primes outside S which do *not* split completely.

Theorem: *(second/other inequality)* the orders of $\widehat{H}^0(G_{K/k}, \mathbb{J}_K/K^\times)$ and $\widehat{H}^2(G_{K/k}, \mathbb{J}_K/K^\times)$ divide $[K : k]$, and $\widehat{H}^1(G_{K/k}, \mathbb{J}_K/K^\times) = 0$.

Theorem: The global reciprocity law map is the product of the local ones (as earlier), so is a product of cup-product maps in cohomology.

Some explanations... if not proofs... :

Tate cohomology of finite groups: Fitting into the Herbrand quotient situation, a finite cyclic group $G = \langle\sigma\rangle$ with $t = \sum_{g\in G} g$ attaches to every G-module A a *periodic complex*

$$\cdots \xrightarrow{\sigma-1} A \xrightarrow{t} A \xrightarrow{\sigma-1} A \xrightarrow{t} \cdots$$

with (co-)homology $\ker(\sigma-1)|_A/\mathrm{im}\, t|_A$ and $\ker t|_A/\mathrm{im}\,(\sigma-1)|_A$. Of course, $\ker(\sigma-1)|_A = A^G$. It is standard to define *Tate cohomology* for finite cyclic G by

$$\widehat{H}^n(G,A) = \begin{cases} \dfrac{A^G}{\mathrm{im}\, t|_A} & (n \text{ even}) \\[2ex] \dfrac{\ker t|_A}{\mathrm{im}\,(\sigma-1)|A} & (n \text{ odd}) \end{cases}$$

Remark: Tate cohomology is defined for all $n\in\mathbb{Z}$. The hat does not mean *completion* or *dual*: it is merely a distinguishing mark.

More generally: for merely *finite* G and G-module A, for reasons that are not instantly clear, Tate cohomology is defined as follows. Let $t = \sum_{g \in G} g$ be the *trace/norm* element as before, and I_G the *augmentation ideal* in $\mathbb{Z}[G]$ generated by $g-1$ for all $g \in G$. Tate cohomology is

$$\widehat{H}^n(G,A) \;=\; \begin{cases} H^n(G,A) & (\text{for } n \geq 1) \\[2ex] \dfrac{A^G}{\operatorname{im} t|_A} & (n=0) \\[4ex] \dfrac{\ker t|_A}{I_G \cdot A} & (n=-1) \\[2ex] H_{1+|n|}(G,A) & (\text{for } n \leq -2) \end{cases}$$

where (!?!) $H^n(G,A)$ with $n \;\geq\; 0$ is *group cohomology* and $H_n(G,A)$ with $n \geq 0$ is *group homology*, defined as follows.

To begin, the 0^{th} cohomology and homology are just *fixed* and *cofixed* vectors: $H^0(G,A) = A^G$ and $H_0(G,A) = A_G$.

The *functor* $A \rightsquigarrow A^G$ is *left-exact* in that, provably,

$$0 \to A \to B \to C \to 0 \text{ exact} \quad \Longrightarrow \quad 0 \to A^G \to B^G \to C^G \text{ exact}$$

Dually, the *functor* $A \rightsquigarrow A_G = A/I_G A$ is *right-exact*:

$$0 \to A \to B \to C \to 0 \text{ exact} \quad \Longrightarrow \quad A_G \to B_G \to C_G \to 0 \text{ exact}$$

These one-sided exactnesses can be proven directly, but are also corollaries of the *adjunction*

$$\mathrm{Hom}_G(A_G, B) \approx \mathrm{Hom}_G(A, B^G)$$

and the general fact that *left adjoints like* $LA = A_G$ *are right exact*, and *right adjoints like* $RB = B^G$ *are left exact.*

For fixed G, the higher cohomology and homology functors $A \rightsquigarrow H^n(A)$ and $A \rightsquigarrow H_n(A)$ are characterized as the *universal* things that from a short exact sequence $0 \to A \to B \to C \to 0$ produce one-sided *long exact sequences completing* $0 \to A^G \to B^G \to C^G$ and $A_G \to B_G \to C_G \to 0$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^G & \longrightarrow & B^G & \longrightarrow & C^G & \longrightarrow & H^1(A) \\ \swarrow & & & & & & & & \\ H^1(B) & \longrightarrow & H^1(C) & \longrightarrow & H^2(A) & \longrightarrow & H^2(B) & \longrightarrow & H^2(C) \\ \swarrow & & & & & & & & \\ H^3(A) & \longrightarrow & H^3(B) & \longrightarrow & H^3(C) & \longrightarrow & H^4(A) & \longrightarrow & \cdots \end{array}$$

in cohomology, and for homology going in the opposite direction:

$$\begin{array}{ccccccccc}
H^1(C) & \longrightarrow & A_G & \longrightarrow & B_G & \longrightarrow & C_G & \longrightarrow & 0 \\
H^2(A) & \longrightarrow & H^2(B) & \longrightarrow & H^2(C) & \longrightarrow & H^1(A) & \longrightarrow & H^1(B) \\
\cdots & \longrightarrow & H^4(C) & \longrightarrow & H^3(A) & \longrightarrow & H^3(B) & \longrightarrow & H^3(C)
\end{array}$$

In both cases, *naturality* is required, meaning that a map of *short exact sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

gives a map of long-exact sequences:

In cohomology,

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & A^G & \longrightarrow & B^G & \longrightarrow & C^G & \longrightarrow & H^1(A) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'^G & \longrightarrow & B'^G & \longrightarrow & C'^G & \longrightarrow & H^1(A') & \longrightarrow & \cdots
\end{array}
$$

and for homology

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^1(C) & \longrightarrow & A_G & \longrightarrow & B_G & \longrightarrow & C_G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^1(C') & \longrightarrow & A'_G & \longrightarrow & B'_G & \longrightarrow & C'_G & \longrightarrow & 0
\end{array}
$$

Any functors like H^n and H_n that fit into such diagrams are δ-**functors**.

Finally, the H^n's and H_n's are characterized as **universal** δ-functors extending $A \rightsquigarrow A^G$ and $A \rightsquigarrow A_G$: for any other collection T^n extending H^n or T_n extending H_n, there are unique $H^n(A) \to T^n(A)$ or $T_n(A) \to H_n(A)$ such that for all $0 \to A \to B \to C \to 0$, there are maps between long exact sequences, in cohomology

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^n & \longrightarrow & H^n(B) & \longrightarrow & H^n(C) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & T^n & \longrightarrow & T^n(B) & \longrightarrow & T^n(C) & \longrightarrow & \cdots \end{array}$$

and in homology

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & T_n & \longrightarrow & T_n(B) & \longrightarrow & T_n(C) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_n & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) & \longrightarrow & \cdots \end{array}$$

Complexes come from ... *derived functors* of things like the G-fixed submodule functor $A \rightsquigarrow A^G$.

For cyclic G, $\ker(\sigma - 1)|_A = A^G$.

Remark: Those *partial failures* of exactness are reasonable:

Consider $G = \{\pm 1\}$, acting on $\mathbb{Z}$ by multiplication, and trivially on $\mathbb{Z}/2$. Application of the G-invariants functor gives $\mathbb{Z}^G = \{0\}$ and $(\mathbb{Z}/2)^G = \mathbb{Z}/2$, so from the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{quot}} \mathbb{Z}/2 \longrightarrow 0$$

we obtain only the exact sequence

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2$$

that is, losing surjectivity to $\mathbb{Z}/2 = (\mathbb{Z}/2)^G$. Applying the G-*coinvariants* functor gives $\mathbb{Z}_G = \mathbb{Z}/\{n - (-n) : n \in \mathbb{Z}\} \approx \mathbb{Z}/2$, losing exactness at the left end, leaving only the exact sequence

$$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\approx} \mathbb{Z}/2 \longrightarrow 0$$