Review the simple (haha!) case of number theory over $\mathbb{Z}$:

Continuing discussion of analytical properties of $\zeta(s)$ relevant to \textit{Riemann’s Explicit Formula} (von Mangoldt’s reformulation):

\[
\sum_{p^m < X} \log p = X - (b + 1) - \lim_{T \to \infty} \sum_{|\text{Im}(\rho)| < T} \frac{X^\rho}{\rho} + \sum_{n \geq 1} \frac{X^{-2n}}{2n}
\]

We are in the course of proving that the \textit{completed} zeta function

\[
\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)
\]

has an analytic continuation to $s \in \mathbb{C}$, except for simple poles at $s = 0, 1$, and has the \textit{functional equation}

\[
\xi(1 - s) = \xi(s)
\]

... and (anticipating the Riemann-Hadamard product issues) $s(s - 1)\xi(s)$ is entire and \textbf{bounded in vertical strips}. 
We need the simplest *theta function*

\[ \theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi in^2 z} \quad \text{(with } z \in \mathfrak{H}) \]

By Riemann’s time, Jacobi’s functional equation of \( \theta(z) \) was well-known, as the simplest example of a larger thing:

\[ \theta(z) = \frac{1}{\sqrt{-iz}} \cdot \theta(-1/z) \]

(Proof below.) The modified version

\[ \frac{\theta(iy) - 1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 y} \]

gets used just below.
The connection to \(\zeta(s)\) is the \textit{integral presentation}:

**Claim:** For \(\text{Re}(s) > 1\)

\[
\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s) = \int_0^\infty \frac{\theta(iy) - 1}{2} \cdot y^{s/2} \cdot \frac{dy}{y}
\]

Meaning? An integral against \(t^s\) with \(dt/t\), a \textit{Mellin transform}, is just a \textit{Fourier transform} in different coordinates.

Starting from the integral, for \(\text{Re}(s) > 1\), compute directly

\[
\int_0^\infty \frac{\theta(iy) - 1}{2} y^{s/2} \frac{dy}{y} = \int_0^\infty \sum_{n \geq 1} e^{-\pi n^2 y} y^{s/2} \frac{dy}{y}
\]

\[
= \sum_{n \geq 1} \int_0^\infty e^{-\pi n^2 y} y^{s/2} \frac{dy}{y} = \pi^{-s/2} \sum_{n \geq 1} \frac{1}{n^{2s}} \int_0^\infty e^{-y} y^{s/2} \frac{dy}{y}
\]

by replacing \(y\) by \(y/(\pi n^2)\), and interchanging sum and integral, giving

\[
= \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \cdot \sum_{n \geq 1} \frac{1}{n^s} = \xi(s) \quad \text{for \(\text{Re}(s) > 1\)}
\]
\[
\frac{\theta(iy) - 1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 y}
\] is of rapid decay as \( y \to +\infty \):

\[
\frac{\theta(iy) - 1}{2} = \sum_{n \geq 1} e^{-\pi n^2 y} \leq e^{-\pi y/2} \sum_{n \geq 1} e^{-\pi n^2 / 2}
\]

\[
= \text{const} \cdot e^{-\pi y/2} \quad \text{(for } y \geq 1)\]

Thus, the integral from 1 (not 0) to +\( \infty \) is nicely convergent for all values of \( s \), and

\[
\int_{1}^{\infty} \frac{\theta(iy) - 1}{2} y^{s/2} \frac{dy}{y} = \text{entire in } s
\]

The trick (known before Riemann) is to use Jacobi’s functional equation for \( \theta(z) \) to convert the part of the integral from 0 to 1 into a similar integral from 1 to +\( \infty \).

It is not obvious that \( \theta(iy) \) has any property that would ensure this. However, in the early 19th century theta functions were intensely studied.
Again, the functional equation of $\theta$, proven below, is

$$\theta(z) = \frac{1}{\sqrt{-iz}} \cdot \theta(-1/z)$$

Book-keeping:

$$\frac{\theta(-1/iy) - 1}{2} = \frac{y^{1/2} \theta(iy) - 1}{2} + \frac{y^{1/2}}{2} - \frac{1}{2}$$

Then

$$\int_0^1 \frac{\theta(iy) - 1}{2} \frac{y^{s/2}}{y} dy = \int_1^\infty \frac{\theta(-1/iy) - 1}{2} \frac{y^{-s/2}}{y} dy$$

$$= \int_1^\infty \left( \frac{y^{1/2} \theta(iy) - 1}{2} + \frac{y^{1/2}}{2} - \frac{1}{2} \right) \frac{y^{-s/2}}{y} dy$$

$$= \int_1^\infty \frac{\theta(iy) - 1}{2} \frac{y^{-s/2}}{y} dy + \int_1^\infty \left( \frac{y^{(1-s)/2}}{2} - \frac{y^{-s/2}}{2} \right) \frac{dy}{y}$$

$$= \int_1^\infty \frac{\theta(iy) - 1}{2} \frac{y^{-s/2}}{y} dy + \frac{1}{s-1} - \frac{1}{s}$$

$$= (\text{entire}) + \frac{1}{s-1} - \frac{1}{s}$$
The elementary expressions $1/(s - 1)$ and $1/s$ certainly have meromorphic continuations to $\mathbb{C}$, with explicit poles. Thus, together with the first integral from 1 to $\infty$, we have

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

$$= \int_1^\infty \frac{\theta(iy) - 1}{2} \left( y^{s/2} + y^{(1-s)/2} \right) \frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s}$$

$$= \text{entire} + \frac{1}{s-1} - \frac{1}{s}$$

The right-hand side is visibly symmetrical under $s \to 1 - s$, which gives the functional equation.

///

**Comments:** Attempting to avoid the gamma factor $\pi^{-s/2} \Gamma\left(\frac{s}{2}\right)$ leads to an unsymmetrical and unenlightening form.

The fact that $\Gamma(s/2)$ has no zeros assures that it masks no poles of $\zeta(s)$. Non-vanishing of $\Gamma(s)$ follows from the identity

$$\Gamma(s) \cdot \Gamma(1 - s) = \frac{\pi}{\sin \pi s}$$
Claim: Jacobi’s functional equation for $\theta(z)$

$$\theta(-1/iy) = \sqrt{y} \cdot \theta(iy)$$

Proof: This symmetry itself follows from a more fundamental fact, the Poisson summation formula

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \quad (\hat{f} \text{ is Fourier transform})$$

Fourier transform of $f = \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx$

The Poisson summation formula is applied to

$$f(x) = \varphi(\sqrt{y} \cdot x) \quad \text{with} \quad \varphi(x) = e^{-\pi x^2}$$

The Gaussian $\varphi(x) = e^{-\pi x^2}$ has the useful property that it is its own Fourier transform.
Prove that the Gaussian is its own Fourier transform by completing the square and a contour integration shift:

\[ \hat{\varphi}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x \xi} \, dx = \int_{\mathbb{R}} e^{-\pi (x+i\xi)^2 - \pi \xi^2} \, dx \]

\[ = e^{-\pi \xi^2} \int_{\mathbb{R}} e^{-\pi (x+i\xi)^2} \, dx \]

By moving the contour of integration, the latter integral is

\[ \int_{\mathbb{R}} e^{-\pi (x+i\xi)^2} \, dx = \int_{\mathbb{R}+i\xi} e^{-\pi x^2} \, dx = \int_{\mathbb{R}} e^{-\pi x^2} \, dx \]

Thus, the integral is independent of \( \xi \). In fact, the constant is 1. By a straightforward change of variables, Fourier transform behaves well with respect to dilations:

\[ \hat{f}(\xi) = \int_{\mathbb{R}} \varphi(\sqrt{y} \, x) e^{-2\pi i x \xi} \, dx = \frac{1}{\sqrt{y}} \int_{\mathbb{R}} \varphi(x) e^{-2\pi i x \xi/\sqrt{y}} \, dx \]

\[ = \frac{1}{\sqrt{y}} \hat{\varphi}(\xi/\sqrt{y}) = \frac{1}{\sqrt{y}} e^{-\pi \xi^2 / y} \quad \text{(replacing } x \text{ by } x/\sqrt{y}) \]
Applying Poisson summation to $f(x) = e^{-\pi x^2 y}$,

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 y} = \frac{1}{\sqrt{y}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 / y}$$

This gives

$$\theta(iy) = \frac{1}{\sqrt{y}} \theta(-1/iy)$$

**Remark** For $z \in \mathfrak{H}$, also $-1/z \in \mathfrak{H}$, and the series for $\theta(z)$ and $\theta(-1/z)$ are nicely convergent. The identity proven for $\theta$ is $\theta(-1/z) = \sqrt{-iz} \theta(z)$ on the imaginary axis. The *Identity Principle* from complex analysis implies that the same equality holds for all $z \in \mathfrak{H}$. 
Heuristic for **Poisson summation** \( \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \)

The periodicized version of a function \( f \) on \( \mathbb{R} \) is

\[
F(x) = \sum_{n \in \mathbb{Z}} f(x + n)
\]

A periodic function should be (!) represented by its **Fourier series**:

\[
F(x) = \sum_{\ell \in \mathbb{Z}} e^{2\pi i \ell x} \int_0^1 F(x) e^{-2\pi i \ell x} \, dx
\]

Fourier *coefficients* of \( F \) expand to be the Fourier *transform* of \( f \):

\[
\int_0^1 F(x) e^{-2\pi i \ell x} \, dx = \int_0^1 \sum_{n \in \mathbb{Z}} f(x + n) e^{-2\pi i \ell x} \, dx
\]

\[
= \sum_{n \in \mathbb{Z}} \int_n^{n+1} f(x) e^{-2\pi i \ell x} \, dx = \int_{\mathbb{R}} f(x) e^{-2\pi i \ell x} \, dx = \hat{f}(\ell)
\]

Evaluating at 0, we should have

\[
\sum_{n \in \mathbb{Z}} f(n) = F(0) = \sum_{\ell \in \mathbb{Z}} \hat{f}(\ell)
\]
What would it take to legitimize this?

Certainly $f$ must be of sufficient decay so that the integral for its Fourier transform is convergent. and so that summing its translates by $\mathbb{Z}$ is convergent.

We’d want $f$ to be continuous, probably differentiable, so that we can talk about pointwise values of $F$

... and to make plausible the hope that the Fourier series of $F$ converges to $F$ pointwise.

For $f$ and several derivatives rapidly decreasing, the Fourier transform $\hat{f}$ will be of sufficient decay so that its sum over $\mathbb{Z}$ does converge.

A simple sufficient hypothesis for convergence is that $f$ be in the Schwartz space of infinitely-differentiable functions all of whose derivatives are of rapid decay, that is,

$$\text{Schwartz space} = \{ \text{smooth } f : \sup_x (1 + x^2)^{\ell} |f^{(i)}(x)| < \infty \text{ for all } i, \ell \}$$
Representability of a periodic function by its Fourier series is a serious question, with several possible senses. We want pointwise convergence. A special, self-contained argument gives a good-enough result for immediate purposes.

Consider \((\mathbb{Z})\)-periodic functions on \(\mathbb{R}\), that is, complex-valued functions \(f\) on \(\mathbb{R}\) such that \(f(x + n) = f(x)\) for all \(x \in \mathbb{R}, n \in \mathbb{Z}\). For periodic \(f\) sufficiently nice so that integrals

\[
\hat{f}(n) = \int_0^1 f(x) e^{-2\pi inx} \, dx \quad (n^{th} \text{ Fourier coefficient of } f)
\]

make sense, the **Fourier expansion** of \(f\) is

\[
f \sim \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi inx}
\]

We want

\[
f(x_o) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi inx_o}
\]
Consider periodic piecewise-$C^\infty$ functions which are left-continuous and right-continuous at any discontinuities.

**Theorem:** For periodic piecewise-$C^\infty$ function $f$, left-continuous and right-continuous at discontinuities, for points $x_o$ at which $f$ is $C^0$ and left-differentiable and right-differentiable, the Fourier series of $f$ evaluated at $x_o$ converges to $f(x)$:

$$f(x_o) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x_o}$$

That is, for such functions, at such points, the Fourier series represents the function pointwise.

A notable missing conclusion is uniform pointwise convergence. For more serious applications, pointwise convergence not known to be uniform is often useless.
Proof: Can reduce to $x_0 = 0$ and $f(0) = 0$. Representability of $f(0)$ by the Fourier series is the assertion that

$$0 = f(0) = \lim_{M,N \to +\infty} \sum_{-M \leq n < N} \hat{f}(n) e^{2\pi i n \cdot 0}$$

$$= \lim_{M,N \to +\infty} \sum_{-M \leq n < N} \hat{f}(n)$$

Substituting the defining integral for the Fourier coefficients:

$$\sum_{-M \leq n < N} \hat{f}(n) = \sum_{-M \leq n < N} \int_0^1 f(u) e^{-2\pi i n u} \, du$$

$$= \int_0^1 \sum_{-M \leq n < N} f(u) e^{-2\pi i n u} \, du = \int_0^1 f(u) \cdot \frac{e^{2\pi i M u} - e^{-2\pi i N u}}{1 - e^{-2\pi i u}} \, du$$

We will show that

$$\lim_{\ell \to \pm \infty} \int_0^1 f(u) \cdot \frac{e^{-2\pi i \ell u}}{1 - e^{-2\pi i u}} \, du = 0$$
Since \( f(0) = 0 \), the function

\[
g(x) = \frac{f(x)}{1 - e^{-2\pi i x}}
\]

is piecewise-\( C^0 \), and left-continuous and right-continuous at discontinuities. The only issue is at integers, and by the periodicity it suffices to prove continuity at 0.

\[
\frac{f(x)}{1 - e^{-2\pi i x}} = f(x) \cdot \frac{x}{1 - e^{-2\pi i x}}
\]

The two-sided limit

\[
\lim_{x \to 0} \frac{x}{1 - e^{-2\pi i x}} = \frac{d}{dx} \bigg|_{x=0} \frac{x}{1 - e^{-2\pi i x}}
\]

exists, by differentiability. Similarly, we have left and right limits

\[
\lim_{x \to 0^-} \frac{f(x)}{x} \quad \text{and} \quad \lim_{x \to 0^+} \frac{f(x)}{x}
\]

by the one-sided differentiability of \( f \). So both one-sided limits exist, giving the one-sided continuity of \( g \) at 0. ///
We find ourselves wanting a Riemann-Lebesgue lemma, that the Fourier coefficients of a periodic, piecewise-$C^o$ function $g$, with left and right limits at discontinuities, go to 0.

The essential property approximability by step functions: given $\varepsilon > 0$ there is a step function $s(x)$ such that

$$\int_0^1 |s(x) - g(x)| \, dx < \varepsilon$$

With such $s$,

$$|\hat{s}(n) - \hat{g}(n)| \leq \int_0^1 |s(u) - g(u)| \, du < \varepsilon \quad \text{(for all } \varepsilon > 0)$$

It suffices that Fourier coefficients of step functions go to 0, an easy computation:

$$\int_a^b e^{-2\pi ilx} \, dx = \left[ \frac{e^{-2\pi ilx}}{-2\pi il} \right]_a^b = \frac{e^{-2\pi ilb} - e^{-2\pi ila}}{-2\pi il} \to 0$$
as $l \to \pm\infty$. Thus, the Fourier coefficients of $g$ go to 0, so the Fourier series of $f$ converges to $f(0)$ when $f$ is $C^1$ at 0.