Review the simple (haha!) case of number theory over $\mathbb{Z}$:

Continuing discussion of analytical properties of $\zeta(s)$ relevant to *Riemann's Explicit Formula* (von Mangoldt's reformulation):

$$\sum_{p^m<X} \log p \;=\; X \;-\; (b+1) \;-\; \lim_{T\to\infty} \sum_{|\mathrm{Im}\,(\rho)|<T} \frac{X^\rho}{\rho} \;+\; \sum_{n\geq 1} \frac{X^{-2n}}{2n}$$

We are in the course of proving that the *completed* zeta function

$$\xi(s) \;=\; \pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})\,\zeta(s)$$

has an analytic continuation to $s \in \mathbb{C}$, except for simple poles at $s = 0, 1$, and has the *functional equation*

$$\xi(1-s) \;=\; \xi(s)$$

... and (anticipating the Riemann-Hadamard product issues) $s(s-1)\xi(s)$ is entire and **bounded in vertical strips**.

We need the simplest *theta function*

$$\theta(z) \;=\; \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} \qquad\qquad (\text{with } z \in \mathfrak{H})$$

By Riemann's time, Jacobi's functional equation of $\theta(z)$ was well-known, as the simplest example of a larger thing:

$$\theta(z) \;=\; \frac{1}{\sqrt{-iz}} \cdot \theta(-1/z)$$

(Proof below.) The modified version

$$\frac{\theta(iy) - 1}{2} \;=\; \sum_{n=1}^{\infty} e^{-\pi n^2 y}$$

gets used just below.

The connection to $\zeta(s)$ is the *integral presentation*:

Claim: For $\mathrm{Re}(s) > 1$

$$\pi^{-s/2}\,\Gamma(\frac{s}{2}) \cdot \zeta(s) \;=\; \int_0^\infty \frac{\theta(iy) - 1}{2} \cdot y^{s/2} \cdot \frac{dy}{y}$$

Meaning? An integral against t^s with dt/t, a *Mellin transform*, is just a *Fourier transform* in different coordinates.

Starting from the integral, for $\mathrm{Re}(s) > 1$, compute directly

$$\int_0^\infty \frac{\theta(iy) - 1}{2}\, y^{s/2}\, \frac{dy}{y} \;=\; \int_0^\infty \sum_{n\geq 1} e^{-\pi n^2 y}\, y^{s/2}\, \frac{dy}{y}$$

$$= \sum_{n\geq 1} \int_0^\infty e^{-\pi n^2 y}\, y^{s/2}\, \frac{dy}{y} \;=\; \pi^{-s/2} \sum_{n\geq 1} \frac{1}{n^{2s}} \int_0^\infty e^{-y}\, y^{s/2}\, \frac{dy}{y}$$

by replacing y by $y/(\pi n^2)$, and interchanging sum an integeral, giving

$$= \pi^{-s/2}\Gamma(\frac{s}{2}) \cdot \sum_{n\geq 1} \frac{1}{n^s} \;=\; \xi(s) \qquad\qquad (\text{for } \mathrm{Re}(s) > 1)$$

$\dfrac{\theta(iy)-1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 y}$ is of rapid decay as $y \to +\infty$:

$$\frac{\theta(iy)-1}{2} \;=\; \sum_{n\geq 1} e^{-\pi n^2 y} \;\leq\; e^{-\pi y/2} \sum_{n\geq 1} e^{-\pi n^2/2}$$

$$=\; \text{const} \cdot e^{-\pi y/2} \qquad\qquad (\text{for } y \geq 1)$$

Thus, the integral from 1 (not 0) to $+\infty$ is nicely convergent for *all* values of s, and

$$\int_1^\infty \frac{\theta(iy)-1}{2}\, y^{s/2}\, \frac{dy}{y} \;=\; \text{entire in } s$$

The trick (known before Riemann) is to use Jacobi's functional equation for $\theta(z)$ to convert the part of the integral from 0 to 1 into a similar integral from 1 to $+\infty$.

It is not obvious that $\theta(iy)$ has any property that would ensure this. However, in the early 19th century theta functions were intensely studied.

Again, the functional equation of θ, proven below, is

$$\theta(z) \;=\; \frac{1}{\sqrt{-iz}} \cdot \theta(-1/z)$$

Book-keeping:

$$\frac{\theta(-1/iy) - 1}{2} = y^{1/2}\frac{\theta(iy) - 1}{2} + \frac{y^{1/2}}{2} - \frac{1}{2}$$

Then

$$\int_0^1 \frac{\theta(iy) - 1}{2}\, y^{s/2}\, \frac{dy}{y} = \int_1^\infty \frac{\theta(-1/iy) - 1}{2}\, y^{-s/2}\, \frac{dy}{y}$$

$$= \int_1^\infty \left(y^{1/2}\frac{\theta(iy) - 1}{2} + \frac{y^{1/2}}{2} - \frac{1}{2}\right) y^{-s/2}\, \frac{dy}{y}$$

$$= \int_1^\infty \frac{\theta(iy) - 1}{2}\, y^{-s/2}\, \frac{dy}{y} + \int_1^\infty \left(\frac{y^{(1-s)/2}}{2} - \frac{y^{-s/2}}{2}\right) \frac{dy}{y}$$

$$= \int_1^\infty \frac{\theta(iy) - 1}{2}\, y^{-s/2}\, \frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s}$$

$$= \;(\text{entire}) + \frac{1}{s-1} - \frac{1}{s}$$

The elementary expressions $1/(s-1)$ and $1/s$ certainly have meromorphic continuations to $\mathbb{C}$, with explicit poles. Thus, together with the first integral from 1 to ∞, we have

$$\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s)$$

$$= \int_1^\infty \frac{\theta(iy)-1}{2}\,(y^{s/2}+y^{(1-s)/2})\,\frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s}$$

$$= (\text{entire}) + \frac{1}{s-1} - \frac{1}{s}$$

The right-hand side is visibly symmetrical under $s \to 1-s$, which gives the functional equation. ///

Comments: Attempting to avoid the gamma factor $\pi^{-s/2}\,\Gamma(\frac{s}{2})$ leads to an unsymmetrical and unenlightening form.

The fact that $\Gamma(s/2)$ has no zeros assures that it masks no poles of $\zeta(s)$. Non-vanishing of $\Gamma(s)$ follows from the identity

$$\Gamma(s)\cdot\Gamma(1-s) \;=\; \frac{\pi}{\sin \pi s}$$

Claim: Jacobi's functional equation for $\theta(z)$

$$\theta(-1/iy) = \sqrt{y} \cdot \theta(iy)$$

Proof: This symmetry itself follows from a more fundamental fact, the **Poisson summation formula**

$$\sum_{n \in \mathbb{Z}} f(n) \ = \ \sum_{n \in \mathbb{Z}} \widehat{f}(n) \qquad\qquad (\widehat{f} \text{ is Fourier transform})$$

$$\text{Fourier transform of } f \ = \ \widehat{f}(\xi) \ = \ \int_{\mathbb{R}} f(x)\, e^{-2\pi i x \xi}\, dx$$

The Poisson summation formula is applied to

$$f(x) \ = \ \varphi(\sqrt{y} \cdot x) \quad \text{with} \quad \varphi(x) \ = \ e^{-\pi x^2}$$

The Gaussian $\varphi(x) = e^{-\pi x^2}$ has the useful property that it is its own Fourier transform.

Prove that the Gaussian is its own Fourier transform by completing the square and a contour integration shift:

$$\widehat{\varphi}(\xi) \;=\; \int_{\mathbb{R}} e^{-\pi x^2}\, e^{-2\pi i x\xi}\, dx \;=\; \int_{\mathbb{R}} e^{-\pi(x+i\xi)^2 - \pi\xi^2}\, dx$$

$$=\; e^{-\pi\xi^2} \int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\, dx$$

By moving the contour of integration, the latter integral is

$$\int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\, dx \;=\; \int_{\mathbb{R}+i\xi} e^{-\pi x^2}\, dx \;=\; \int_{\mathbb{R}} e^{-\pi x^2}\, dx$$

Thus, the integral is a independent of ξ. In fact, the constant is 1. By a straightforward change of variables, Fourier transform behaves well with respect to *dilations*:

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{R}} \varphi(\sqrt{y}\, x)\, e^{-2\pi i x\xi}\, dx \;=\; \frac{1}{\sqrt{y}} \int_{\mathbb{R}} \varphi(x)\, e^{-2\pi i x\xi/\sqrt{y}}\, dx$$

$$=\; \frac{1}{\sqrt{y}} \widehat{\varphi}(\xi/\sqrt{y}) \;=\; \frac{1}{\sqrt{y}} e^{-\pi\xi^2/y} \qquad\qquad \text{(replacing } x \text{ by } x/\sqrt{y}\text{)}$$

Applying Poisson summation to $f(x) = e^{-\pi x^2 y}$,

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 y} \;=\; \frac{1}{\sqrt{y}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/y}$$

This gives

$$\theta(iy) \;=\; \frac{1}{\sqrt{y}}\theta(-1/iy)$$

Remark For $z \in \mathfrak{H}$, also $-1/z \in \mathfrak{H}$, and the series for $\theta(z)$ and $\theta(-1/z)$ are nicely convergent. The identity proven for θ is $\theta(-1/z) = \sqrt{-iz}\,\theta(z)$ on the imaginary axis. The *Identity Principle* from complex analysis implies that the same equality holds for all $z \in \mathfrak{H}$.

Heuristic for **Poisson summation** $\sum_{n\in\mathbb{Z}} f(n) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)$

The periodicized version of a function f on $\mathbb{R}$ is

$$F(x) \;=\; \sum_{n\in\mathbb{Z}} f(x+n)$$

A periodic function should be (!) represented by its *Fourier series*:

$$F(x) \;=\; \sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell x} \int_0^1 F(x)\, e^{-2\pi i\ell x}\, dx$$

Fourier *coefficients* of F expand to be the Fourier *transform* of f:

$$\int_0^1 F(x)\, e^{-2\pi i\ell x}\, dx \;=\; \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)\, e^{-2\pi i\ell x}\, dx$$

$$=\; \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x)\, e^{-2\pi i\ell x}\, dx \;=\; \int_{\mathbb{R}} f(x)\, e^{-2\pi i\ell x}\, dx \;=\; \widehat{f}(\ell)$$

Evaluating at 0, we should have

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; F(0) \;=\; \sum_{\ell\in\mathbb{Z}} \widehat{f}(\ell)$$

What would it take to legitimize this?

Certainly f must be of sufficient decay so that the integral for its Fourier transform is convergent. and so that summing its translates by $\mathbb{Z}$ is convergent.

We'd want f to be continuous, probably differentiable, so that we can talk about pointwise values of F

... and to make plausible the hope that the Fourier series of F converges to F pointwise.

For f and several derivatives rapidly decreasing, the Fourier transform $\widehat{f}$ will be of sufficient decay so that its sum over $\mathbb{Z}$ does converge.

A simple sufficient hypothesis for convergence is that f be in the *Schwartz space* of infinitely-differentiable functions all of whose derivatives are of *rapid decay*, that is,

$$\text{Schwartz space} = \{\text{smooth}\, f : \sup_x \,(1+x^2)^\ell |f^{(i)}(x)| < \infty \text{ for all } i, \ell\}$$

Representability of a periodic function by its Fourier series is a serious question, with several possible senses. We want *pointwise convergence.* A special, self-contained argument gives a good-enough result for immediate purposes.

Consider ($\mathbb{Z}$-)periodic functions on $\mathbb{R}$, that is, complex-valued functions f on $\mathbb{R}$ such that $f(x+n) = f(x)$ for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$. For periodic f sufficiently nice so that integrals

$$\widehat{f}(n) \;=\; \int_0^1 f(x)\, e^{-2\pi i n x}\, dx \qquad\qquad (n^{th} \text{ Fourier coefficient of } f)$$

make sense, the **Fourier expansion** of f is

$$f \;\sim\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{2\pi i n x}$$

We want

$$f(x_o) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{2\pi i n x_o}$$

Consider periodic *piecewise*-C^o functions which are left-continuous and right-continuous at any discontinuities.

Theorem: For periodic piecewise-C^o function f, left-continuous and right-continuous at discontinuities, for points x_o at which f is C^0 and *left-differentiable* and *right-differentiable*, the Fourier series of f evaluated at x_o *converges* to $f(x)$:

$$f(x_o) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{2\pi i n x_o}$$

That is, for such functions, at such points, the Fourier series *represents* the function *pointwise.*

A notable missing conclusion is *uniform* pointwise convergence. For more serious applications, pointwise convergence not known to be uniform is often useless.

Proof: Can reduce to $x_o = 0$ and $f(0) = 0$. Representability of $f(0)$ by the Fourier series is the assertion that

$$0 = f(0) = \lim_{M,N\to+\infty} \sum_{-M\le n<N} \widehat{f}(n)\, e^{2\pi i n\cdot 0}$$

$$= \lim_{M,N\to+\infty} \sum_{-M\le n<N} \widehat{f}(n)$$

Substituting the defining integral for the Fourier coefficients:

$$\sum_{-M\le n<N} \widehat{f}(n) = \sum_{-M\le n<N} \int_0^1 f(u)\, e^{-2\pi i n u}\, du$$

$$= \int_0^1 \sum_{-M\le n<N} f(u)\, e^{-2\pi i n u}\, du = \int_0^1 f(u)\cdot \frac{e^{2\pi i M u} - e^{-2\pi i N u}}{1 - e^{-2\pi i u}}\, du$$

We will show that

$$\lim_{\ell\to\pm\infty} \int_0^1 \frac{f(u)\cdot e^{-2\pi i \ell u}}{1 - e^{-2\pi i u}}\, du = 0$$

Since $f(0) = 0$, the function

$$g(x) \;=\; \frac{f(x)}{1 - e^{-2\pi i x}}$$

is piecewise-C^o, and left-continuous and right-continuous at discontinuities. The only issue is at integers, and by the periodicity it suffices to prove continuity at 0.

$$\frac{f(x)}{1 - e^{-2\pi i x}} \;=\; \frac{f(x)}{x} \cdot \frac{x}{1 - e^{-2\pi i x}}$$

The two-sided limit

$$\lim_{x \to 0} \frac{x}{1 - e^{-2\pi i x}} \;=\; \frac{d}{dx}\Big|_{x=0} \frac{x}{1 - e^{-2\pi i x}}$$

exists, by differentiability. Similarly, we have left and right limits

$$\lim_{x \to 0^-} \frac{f(x)}{x} \qquad \text{and} \qquad \lim_{x \to 0^+} \frac{f(x)}{x}$$

by the one-sided differentiability of f. So both one-sided limits exist, giving the one-sided continuity of g at 0. ///

We find ourselves wanting a *Riemann-Lebesgue lemma*, that that the Fourier coefficients of a periodic, piecewise-C^o function g, with left and right limits at discontinuities, go to 0.

The essential property approximability by step functions: given $\varepsilon > 0$ there is a *step function* $s(x)$ such that

$$\int_0^1 |s(x) - g(x)|\, dx \; < \; \varepsilon$$

With such s,

$$|\widehat{s}(n) - \widehat{g}(n)| \; \le \; \int_0^1 |s(u) - g(u)|\, du \; < \; \varepsilon \qquad \text{(for all } \varepsilon > 0\text{)}$$

It suffices that Fourier coefficients of *step functions* go to 0, an easy computation:

$$\int_a^b e^{-2\pi i\ell x}\, dx \; = \; [\frac{e^{-2\pi i\ell x}}{-2\pi i\ell}]_a^b \; = \; \frac{e^{-2\pi i\ell b} - e^{-2\pi i\ell a}}{-2\pi i\ell} \; \longrightarrow \; 0$$

as $\ell \to \pm\infty$. Thus, the Fourier coefficients of g go to 0, so the Fourier series of f converges to $f(0)$ when f is C^1 at 0.
