Continuing the pre/review ...

Continuing: solving equations mod p^n, p-adic numbers, Hensel.

Completions versus *projective limits.*

Mapping-property characterizations... unique up to unique isomorphism.

Another forgotten point: not only are the $p^n\mathbb{Z}_p$ the only ideals in $\mathbb{Z}_p$, but also $\mathbb{Z}_p/p^n\mathbb{Z}_p \approx \mathbb{Z}/p^n\mathbb{Z}$. This is used to compare the *metric completion* version of $\mathbb{Z}_p$ to the *limit* characterization.

Introducing **rational adeles**.

Claim: For positive integers n, $\mathbb{Z}_p/p^n\mathbb{Z}_p \approx \mathbb{Z}/p^n\mathbb{Z}$.

Proof: Inclusion $\mathbb{Z} \to \mathbb{Z}_p$ compose with $\mathbb{Z}_p \to \mathbb{Z}_p/p^n\mathbb{Z}_p$ has kernel

$$\mathbb{Z} \cap p^n\mathbb{Z}_p \;=\; \mathbb{Z} \cap \{x \in \mathbb{Z}_p : |x|_p \le \frac{1}{p^n}\} \;=\; \{x \in \mathbb{Z} : |x|_p \le \frac{1}{p^n}\}$$

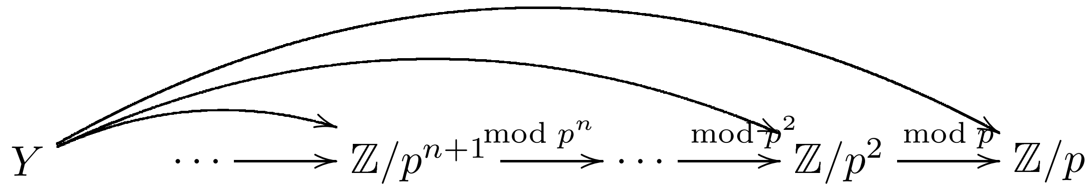
$$=\; \{\text{integers divisible by } p^n\} \;=\; p^n\mathbb{Z}$$

Thus, $\mathbb{Z}/p^n\mathbb{Z}$ *injects* to $\mathbb{Z}_p/p^n\mathbb{Z}_p$. On the other hand, because $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, given $x \in \mathbb{Z}_p$ there is $y \in \mathbb{Z}$ such that $|x - y| \le \frac{1}{p^n}$. That is, $x \in y + p^n\mathbb{Z}_p$. Then

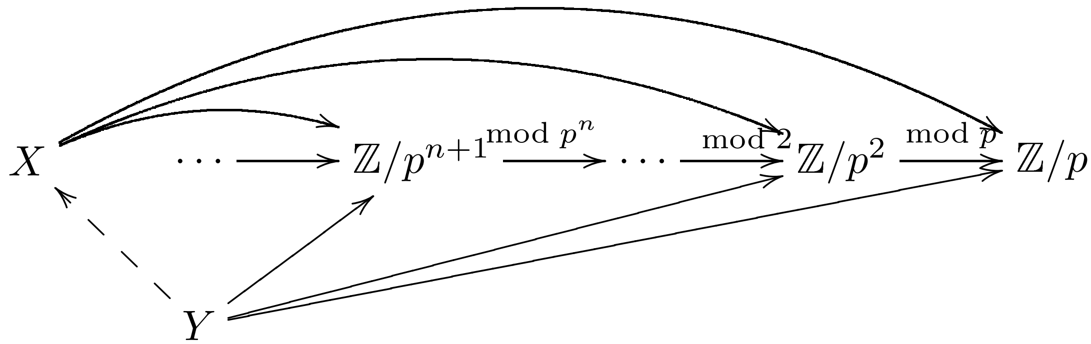
$$x + p^n\mathbb{Z}_p \;=\; y + p^n\mathbb{Z}_p + p^n\mathbb{Z}_p \;=\; y + p^n\mathbb{Z}_p \qquad (\text{with } y \in \mathbb{Z})$$

That is, the map is also *surjective*. ///

Back to projective limits: *map* means *continuous ring hom.* Require that, for every topological ring Y with compatible maps



there is a *unique* map $Y \to X$ giving a commutative diagram



A topological ring $X = \lim \mathbb{Z}/p^n$ meeting these conditions is the *(projective) limit* of the $\mathbb{Z}/p^n$'s, and is provably the same $\mathbb{Z}_p$!!!

Note: each finite ring $\mathbb{Z}/p^n$ has a unique Hausdorff topology!!!

Prove *existence* of projective limits by a *construction.* Here, as is typical, $\lim_n X_n$ is a *subset* of the (topological) product $\prod_n X_n$. Specifically, with

$$\ldots \longrightarrow X_{n+1} \xrightarrow{\varphi_{n+1}} \ldots \xrightarrow{\varphi_3} X_2 \xrightarrow{\varphi_2} X_1$$

a projective limit $X = \lim_n X_n$ can be constructed as

$$X \;=\; \{\{x_n\} \;:\; x_n \in X_n \text{ such that } \varphi_n(x_n) = x_{n-1} \text{ for all } n\}$$

That is, X consists exactly of *compatible sequences*

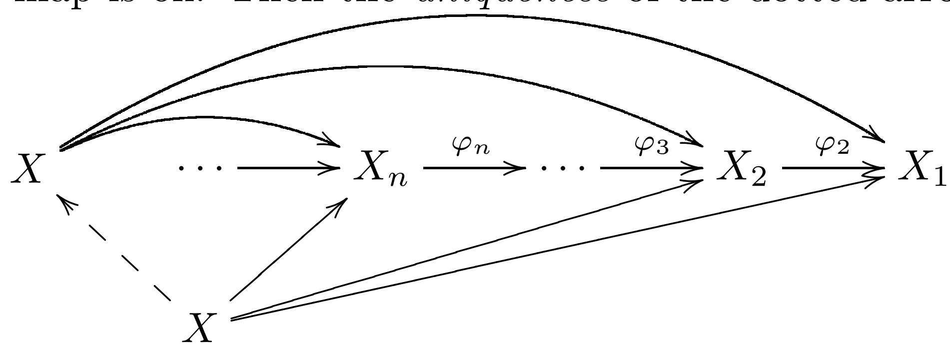
$$\ldots \longrightarrow x_{n+1} \xrightarrow{\varphi_{n+1}} \ldots \xrightarrow{\varphi_3} x_2 \xrightarrow{\varphi_2} x_1$$

as produced by Hensel. For continuous φ_n and *compact* X_n's, *Tychonoff's theorem* says the product is *compact.* The limit is a *closed* subset of a compact Hausdorff space, so is *compact.* This proves compactness of $\mathbb{Z}_p$!!!

Uniqueness (up to unique isomorphism) of projective limits

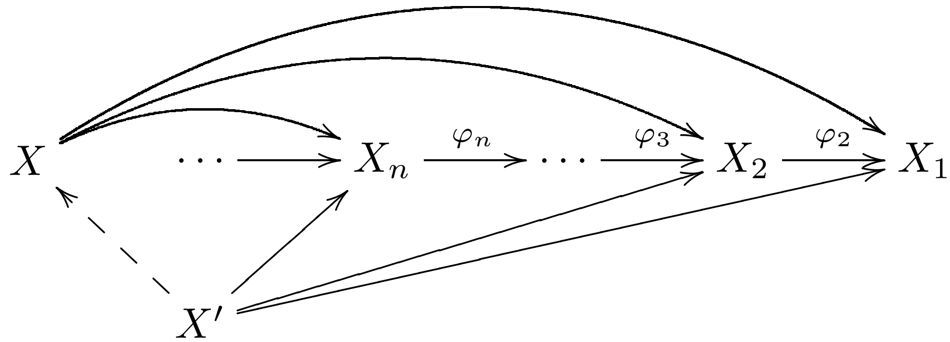
The diagrammatic characterization can be used to assure that there's *no ambiguity* in what $\mathbb{Z}_p$ is, as long as it functions as a projective limit:

First, claim the only map of $X = \lim_n X_n$ to *itself*, compatible with the maps of it to the X_n, is the *identity*. Certainly the identity map is ok. Then the *uniqueness* of the dotted arrow

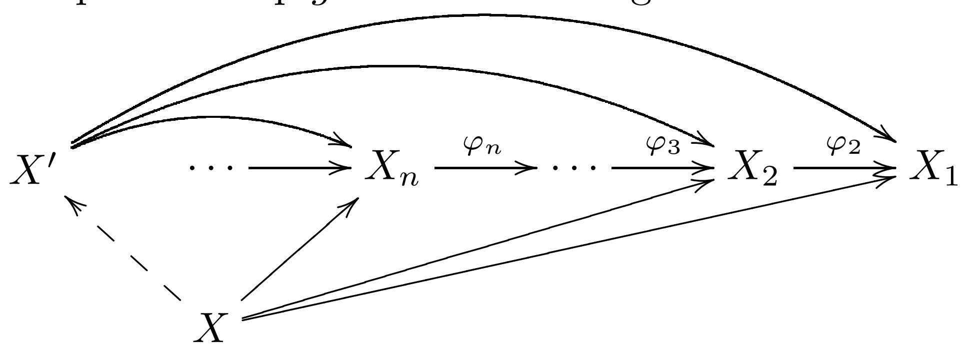


proves that the identity is the *only* compatible map. Next, ...

Suppose X and X' were *two* projective limits. On one hand, there is a unique $f : X' \to X$ giving commutative diagram



On the other hand, reversing the roles of X and X', there is a unique compatible map $g : X \to X'$ fitting into

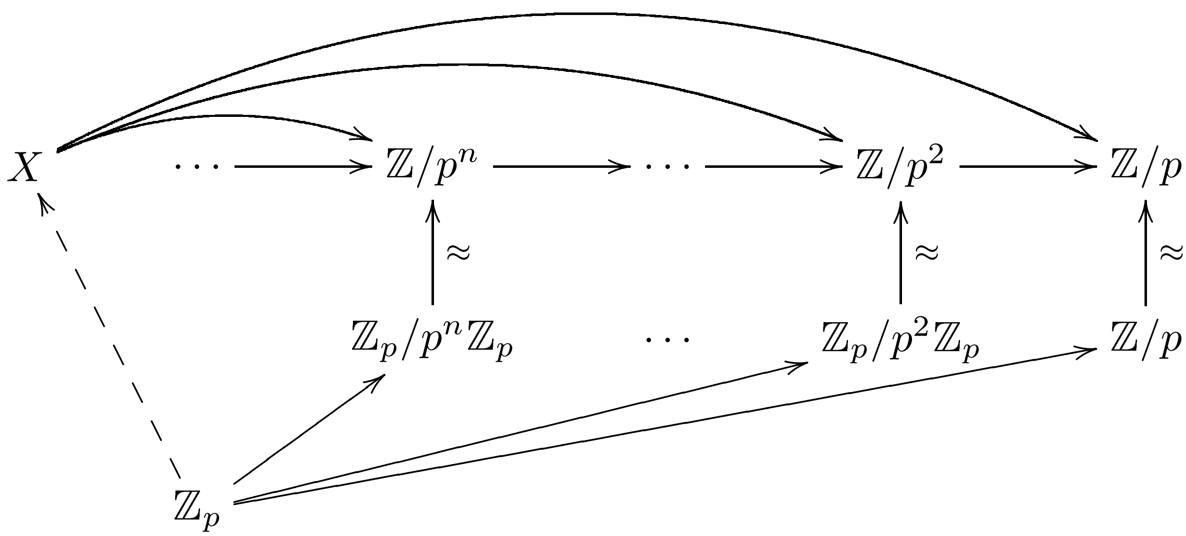


The composites $f \circ g : X \to X$ and $g \circ f : X' \to X'$ are also compatible, so must be the identities on X and X', by the first part. Thus, f, g are mutual inverses. ///

The (projective) limit $X = \lim_n \mathbb{Z}/p^n\mathbb{Z}$ is naturally isomorphic to the metric completion $\mathbb{Z}_p$.

(Further details appear in the proof.)

Proof: The maps $q_n : \mathbb{Z}_p \to \mathbb{Z}_p/p^n\mathbb{Z}_p \approx \mathbb{Z}/p^n$ are a compatible family of (continuous!) maps to the limitands in $\lim_n \mathbb{Z}/p^n$, inducing a unique map of $\mathbb{Z}_p$ to the limit:



For $0 \neq x \in \mathbb{Z}_p$, take n such that $|x|_p > \frac{1}{p^n}$. Then the image of x in $\mathbb{Z}_p/p^n\mathbb{Z}_p$ is non-zero, so $\mathbb{Z}_p$ *injects* to X.

Prove that the map $\mathbb{Z}_p \to X$ is an *isomorphism.*

Let $f_n : Z \to \mathbb{Z}/p^n$ be afamily of maps from another object Z, compatible in the sense that

$$f_{n+1}(z) = f_n(z) \bmod p^n \qquad \text{(for all } z \in Z\text{, for all } n\text{)}$$

For each $z \in Z$, for each n choose $x_n \in \mathbb{Z}$ such that

$$f_n(z) = x_n + p^n\mathbb{Z}$$

Compatibility implies $\{x_n\}$ is *Cauchy.* By completeness, take a limit

$$f(z) = \lim_n x_n \in \mathbb{Z}_p$$

defining a map $f : Z \to \mathbb{Z}_p$ compatible with the f_n's and the projections.

Still need uniqueness of $f : Z \to \mathbb{Z}_p$...

To show that there is a *unique* such f, let $q_n : \mathbb{Z}_p \to \mathbb{Z}_p/p^n\mathbb{Z}_p \approx \mathbb{Z}/p^n$. For two maps f and g $\mathbb{Z} \to \mathbb{Z}_p$ compatible with the projections and f_n's,

$$q_n\big(f(z) - g(z)\big) \;=\; q_n f(z) - q_n g(z)$$

$$=\; f_n(z) - f_n(z) \;=\; 0 \in \mathbb{Z}/p^n$$

That is, $f(z) - g(z) \in p^n\mathbb{Z}_p$ for all n. Taking the intersection over n gives $f(z) = g(z)$. This proves that $\mathbb{Z}_p$ is the projective limit.

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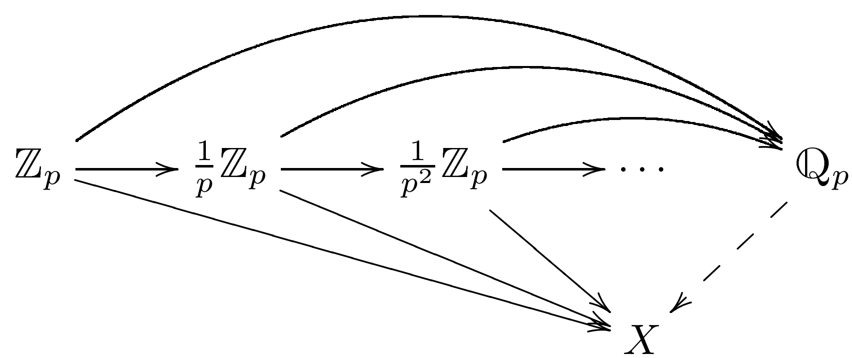
Colimit (inductive limit) $\mathbb{Q}_p$

As topological *rings*, $\mathbb{Q}_p$ is the *field of fractions* of $\mathbb{Z}_p$. Good, but we need more flexibility. Forgetting multiplication for a moment, $\mathbb{Q}_p$ is a *nested union*

$$\mathbb{Q}_p \;=\; \mathbb{Z}_p \cup \frac{1}{p}\mathbb{Z}_p \cup \frac{1}{p^2}\mathbb{Z}_p \cup \ldots$$

That is, it is a *colimit*, where all maps are inclusions,

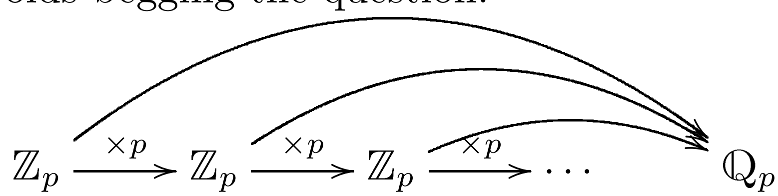


The defining property of the colimit is that all compatible collections of maps from another object X to the limitands give a unique compatible map $X \to \mathbb{Q}_p$. Colimits are unique up to unique isomorphism, as usual.

To *construct* $\mathbb{Q}_p$ as a colimit, we can't divide $\mathbb{Z}_p$ by p^n's, since this begs the question. We avoid that by converting *inclusions* to *multiplications*:

$$
\begin{array}{ccccccc}
\mathbb{Z}_p & \xrightarrow{\text{inc}} & \frac{1}{p}\mathbb{Z}_p & \xrightarrow{\text{inc}} & \frac{1}{p^2}\mathbb{Z}_p & \xrightarrow{\text{inc}} & \cdots \\
\downarrow{\scriptstyle \times 1} & & \downarrow{\scriptstyle \times p} & & \downarrow{\scriptstyle \times p^2} & & \\
\mathbb{Z}_p & \xrightarrow{\times p} & \mathbb{Z}_p & \xrightarrow{\times p} & \mathbb{Z}_p & \xrightarrow{\times p} & \cdots
\end{array}
$$

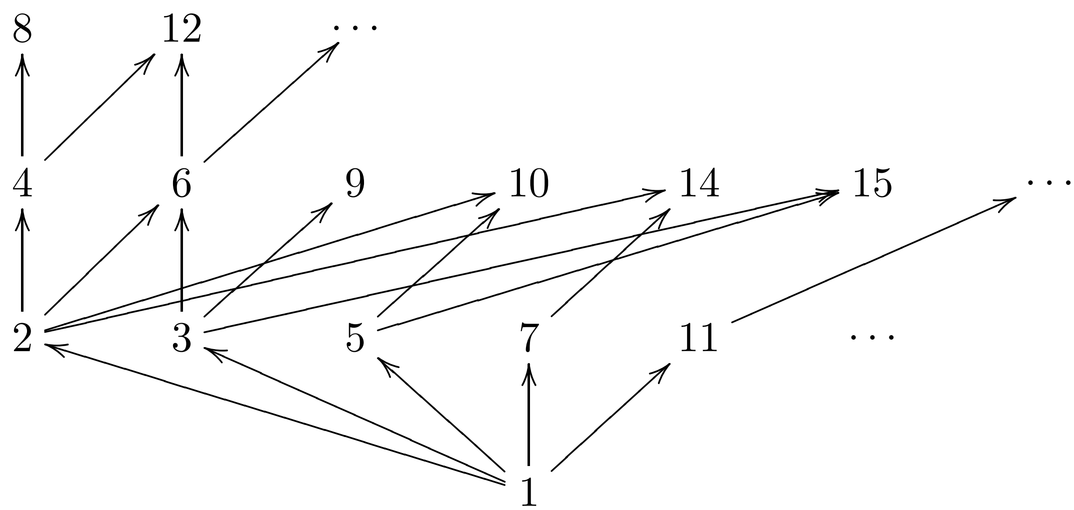
All the squares *commute*, so there is a unique natural isomorphism of the colimits. Thus, we have a (second) colimit description of $\mathbb{Q}_p$ which avoids begging the question:



Toward adeles: $\widehat{\mathbb{Z}}$

An immediate definition is $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, but this doesn't tell how $\widehat{\mathbb{Z}}$ arises in nature.

Better: instead of considering the dinky (directed) posets $\{p^n : n = 1, 2, 3, \ldots\}$ of powers of single primes, consider the (directed) poset of *all* integers, ordered by *divisibility*:



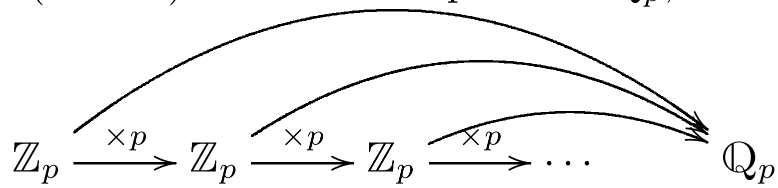
A robust definition:

$$\widehat{\mathbb{Z}} \;=\; \lim_N \mathbb{Z}/N \qquad\qquad \text{(proj lim over } N \text{ ordered by divisibility)}$$

Projective limits and products fall into a broader class of "limits", which allows proof of their compatibility with each other... Using Sun-Ze, factoring each N into primes $N = \prod_p p^{e_p(N)}$,

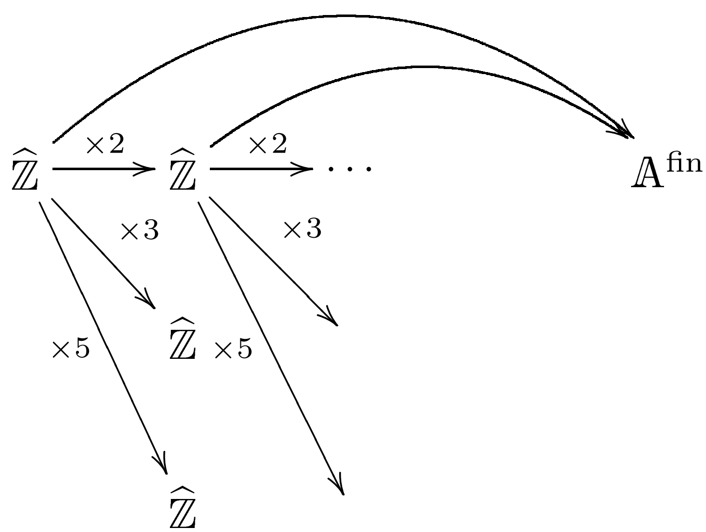
$$\widehat{\mathbb{Z}} \;=\; \lim_N \mathbb{Z}/N \;\approx\; \lim_N \Big(\prod_p \mathbb{Z}/p^{e_p(N)} \Big) \;\approx\; \prod_p \lim_e \mathbb{Z}/p^e \;\approx\; \prod_p \mathbb{Z}_p$$

Recalling the (second) colimit description of $\mathbb{Q}_p$,



we could do the analogous thing with $\widehat{\mathbb{Z}}$ and *all* multiplications. Since the ring $\widehat{\mathbb{Z}}$ has many zero divisors, there's no option to talk about fields-of-fractions! For $0 < n \in \mathbb{Z}$, let $X_n \approx \widehat{\mathbb{Z}}$, and for $m|n$, let $\varphi_{mn} : X_m \to X_n$ by $\varphi_{mn}(x) = \frac{n}{m}x$. With these transition maps $\varphi_{m,n}$ implied,

$$\text{finite rational adeles } \mathbb{A}^{\text{fin}} \; = \; \text{colim}_N X_N$$



The common/immediate description of $\mathbb{A}^{\mathrm{fin}}$:

You will hear $\mathbb{A}^{\mathrm{fin}}$ described [sic] as a *restricted direct product* [sic], meaning

$$\begin{aligned}&\mathbb{A}^{\mathrm{fin}}\\ &= \{\{x_p\} \in \textstyle\prod_p \mathbb{Q}_p \;:\; x_p \in \mathbb{Z}_p \text{ for all but finitely-many primes } p\}\end{aligned}$$

Since restricted direct products [sic] do not occur anywhere else, this is perhaps not an illuminating description [sic]. Its motivation is certainly completely obscure.

But it's tangible.

The **rational adeles** are $\mathbb{A} = \mathbb{R} \times \mathbb{A}^{\text{fin}}$.

This captures not only all the p-adic stuff, but also archimedean (real-number) stuff.

The subgroup $\mathbb{R} \times \widehat{\mathbb{Z}}$ is both open and closed.

One last point: imbed $\mathbb{Q}$ *diagonally* in $\mathbb{A}$, meaning into each $\mathbb{Q}_p$ and into $\mathbb{R}$ in the usual way. For $m|n$, let $\mathbb{R}/n\mathbb{Z} \to \mathbb{R}/m/Z$ by $r + n\mathbb{Z} \to r + m\mathbb{Z}$. Then (we claim)

$$\mathbb{A}/\mathbb{Q} \ = \ \lim_{N} \ \mathbb{R}/N\mathbb{Z} \ = \ \text{compact}$$
