Example of the way that more classical *reciprocity laws* are corollaries of fancier-looking things...

For global field k with *completions* k_v of k, for K a *quadratic* extension of k, put

$$K_v \;=\; K \otimes_k k_v$$

The **local norm residue symbol** $\nu_v : k_v^\times \to \{\pm 1\}$ is

$$\nu_v(\alpha) \;=\; \begin{cases} +1 & (\text{for } \alpha \in N(K_v^\times)) \\ \\ -1 & (\text{for } \alpha \notin N(K_v^\times)) \end{cases}$$

A small *local* Theorem:

$$[k_v^\times : N(K_v^\times)] \;=\; \begin{cases} 2 & (\text{when } K_v \text{ is a field}) \\ \\ 1 & (\text{when } K_v \approx k_v \times k_v) \end{cases}$$

Cor: ν_v is a group homomorphism $k_v^\times \to \{\pm 1\}$. ///

Immediate, opaque, definition of *ideles*:

$$\mathbb{J} \;=\; \mathbb{J}_k \;=\; (\text{ideles of } k)$$

$$=\; \{\{\alpha_v\} \in \prod_v k_v^\times \;:\; \alpha_v \in \mathfrak{o}_v^\times \text{ for all but finitely-many } v\}$$

Let

$$\nu = \prod_v \nu_v \;:\; \mathbb{J} \longrightarrow \{\pm 1\}$$

A (memorable, if obscure) big *global* **Theorem:** ν is a $k^\times$-invariant function on $\mathbb{J}$: it *factors through* $\mathbb{J}/k^\times$.

Granting this, some interesting deductions follow...

Quadratic Hilbert symbols

For $a, b \in k_v$ the (quadratic) **Hilbert symbol** is

$$(a,b)_v \;=\; \begin{cases} \;\;1 & (\text{if } ax^2+by^2=z^2 \text{ has non-trivial solution in } k_v) \\ \\ -1 & (\text{otherwise}) \end{cases}$$

Memorable theorem: For $a, b \in k^\times$

$$\Pi_v \;(a,b)_v \;=\; 1$$

Proof: ... from the fact that the quadratic norm residue symbol is a Hecke character. Especially, for b a non-square in $k^\times$, $(a,b)_v$ is $\nu_v(a)$ for the field extension $k(\sqrt{b})$, and reciprocity for the norm residue symbol gives the result for the Hilbert symbol. ///

Traditional-looking quadratic reciprocity laws follow from that reciprocity for the quadratic Hilbert symbol. Define

$$\left(\frac{x}{v}\right)_2 = \begin{cases} 1 & (\text{for } x \text{ a non-zero square mod } v) \\ 0 & (\text{for } x = 0 \text{ mod } v) \\ -1 & (\text{for } x \text{ a non-square mod } v) \end{cases}$$

Quadratic Reciprocity ('main part'): For π and ϖ two elements of $\mathfrak{o}$ generating distinct odd prime ideals,

$$\left(\frac{\varpi}{\pi}\right)_2 \left(\frac{\pi}{\varpi}\right)_2 = \Pi_v \, (\pi, \varpi)_v$$

where v runs over all *even or infinite* primes, and $(,)_v$ is the (quadratic) Hilbert symbol.

Proof We claim that, since $\pi\mathfrak{o}$ and $\varpi\mathfrak{o}$ are odd primes,

$$(\pi,\varpi)_v = \begin{cases} \left(\frac{\varpi}{\pi}\right)_2 & \text{for } v = \pi\mathfrak{o} \\ \left(\frac{\pi}{\varpi}\right)_2 & \text{for } v = \varpi\mathfrak{o} \\ 1 & \text{for } v \text{ odd and } v \neq \pi\mathfrak{o}, \varpi\mathfrak{o} \end{cases}$$

Let $v = \pi\mathfrak{o}$. Suppose that there is a solution x, y, z in k_v to

$$\pi x^2 + \varpi y^2 = z^2$$

Via the ultrametric property, $\text{ord}_v y$ and $\text{ord}_v z$ are identical, and less than $\text{ord}_v x$, since ϖ is a v-unit and $\text{ord}_v \pi x^2$ is *odd*. Multiply through by π^{2n} so that $\pi^n y$ and $\pi^n z$ are v-units. Then that ϖ must be a square modulo v.

On the other hand, when ϖ is a square modulo v, use Hensel's lemma to infer that ϖ is a square in k_v. Then

$$\varpi y^2 \;=\; z^2$$

certainly has a non-trivial solution.

For v an odd prime distinct from $\pi\mathfrak{o}$ and $\varpi\mathfrak{o}$, π and ϖ are v-units. When ϖ is a square in k_v, $\varpi \;=\; z^2$ has a solution, so the Hilbert symbol is 1. For unit ϖ not a square in k_v, the quadratic field extension $k_v(\sqrt{\varpi})$ has the property that the norm map is *surjective* to units in k_v. Thus, there are $y, z \in k_v$ so that

$$\pi \;=\; N(z + y\sqrt{\varpi}) \;=\; z^2 - \varpi y^2$$

Thus, all but even-prime and infinite-prime quadratic Hilbert symbols are quadratic symbols. ///

Simplest examples Let's recover quadratic reciprocity for two (positive) odd prime numbers p, q:

$$\left(\frac{q}{p}\right)_2 \left(\frac{p}{q}\right)_2 \;=\; (-1)^{(p-1)(q-1)/4}$$

We have

$$\left(\frac{q}{p}\right)_2 \left(\frac{p}{q}\right)_2 \;=\; (p,q)_2 (p,q)_\infty$$

Since both p, q are positive, the equation

$$px^2 + qy^2 \;=\; z^2$$

has non-trivial *real* solutions x, y, z. That is, the 'real' Hilbert symbol $(p, q)_\infty$ for the archimedean completion of $\mathbb{Q}$ has the value 1. Therefore, only the 2-adic Hilbert symbol contributes to the right-hand side of Gauss' formula:

$$\left(\frac{q}{p}\right)_2 \left(\frac{p}{q}\right)_2 \;=\; (p,q)_2$$

Hensel's lemma shows that the solvability of the equation above (for p, q both 2-adic units) depends only upon their residue classes mod 8. The usual formula is but one way of interpolating the 2-adic Hilbert symbol by elementary-looking formulas. ///

For contrast, let us derive the analogue for $\mathbb{F}_q[T]$ with q odd: for distinct *monic* irreducible polynomials π, ϖ in $\mathbb{F}_q[T]$,

$$\left(\frac{\varpi}{\pi}\right)_2 \left(\frac{\pi}{\varpi}\right)_2 \;=\; \left(\frac{-1}{\mathbb{F}_q}\right)_2^{(\deg \pi)(\deg \varpi)}$$

Proof: From the general assertion above,

$$\left(\frac{\varpi}{\pi}\right)_2 \left(\frac{\pi}{\varpi}\right)_2 \;=\; (\pi, \varpi)_\infty$$

where ∞ is the prime (valuation)

$$P \;\longrightarrow\; q^{\deg P}$$

This norm has local ring consisting of rational functions in t writable as power series in the local parameter $t_\infty = t^{-1}$. Then

$$\pi \;=\; t_\infty^{-\deg \pi}(1 + t_\infty(\ldots))$$

where $(1 + t_\infty(\ldots))$ is a power series in t_∞. A similar assertion holds for ϖ. Thus, if either degree is *even*, then one of π, ϖ is a local square, so the Hilbert symbol is $+1$.

When $t_\infty^{-\deg \pi}(1 + t_\infty(\ldots))$ is a non-square, $\deg \pi$ is odd. Nevertheless, *any* expression of the form

$$1 + t_\infty(\ldots)$$

is a local square (by Hensel). Thus, without loss of generality, we are contemplating the equation

$$t_\infty(x^2 + y^2) \;=\; z^2$$

The t_∞-order of the right-hand side is even.

If there is no $\sqrt{-1}$ in $\mathbb{F}_q$, then the left-hand side is t_∞-times a norm from the unramified extension

$$\mathbb{F}_q(\sqrt{-1})(T) \;=\; \mathbb{F}_q(T)(\sqrt{-1})$$

so has odd order. This is impossible. On the other hand if there is a $\sqrt{-1}$ in $\mathbb{F}_q$ then the equation has non-trivial solutions.

Thus, if neither π nor ϖ is a local square (i.e., both are of odd degree), then the Hilbert symbol is 1 if and only if there is a $\sqrt{-1}$ in $\mathbb{F}_q$. The formula given above is an elementary interpolation of this assertion (as for the case $k = \mathbb{Q}$). ///