Introducing Dedekind zeta function of number fields

Proposition: For $\alpha \neq 0$ in the integral closure $\mathfrak{O}$ of $\mathbb{Z}$ in a number field K, the Galois norm and ideal norm are essentially the same:

$$|N_{\mathbb{Q}}^{K}(\alpha)| \;=\; N(\alpha\mathfrak{O})$$

with *ideal* norm $N(\mathfrak{A}) = \#\mathfrak{O}/\mathfrak{A}$ for ideals $\mathfrak{A}$ in $\mathfrak{O}$. [Last time.]

Now, at least for a little while, k is a finite extension of $\mathbb{Q}$.

Dedekind zeta functions:

$$\zeta_k(s) \;=\; \sum_{0\neq\mathfrak{a}\subset\mathfrak{o}} \frac{1}{N\mathfrak{a}^s}$$

Granting convergence, the Dedekind property suggests the Euler product

$$\zeta_k(s) \;=\; \sum_{0\neq\mathfrak{a}\subset\mathfrak{o}} \frac{1}{N\mathfrak{a}^s} \;=\; \prod_{\mathfrak{p}\text{ prime in }\mathfrak{o}} \frac{1}{1-N\mathfrak{p}^{-s}}$$

Understanding splitting/factorization of primes in extensions of $\mathbb{Z}$ or of $\mathbb{F}_q[x]$ gives

Proposition: The Euler product expression for $\zeta_k(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$. [Last time.]

Remark: The proof used the estimate

$$|\zeta_k(s)| \;\leq\; \zeta_{\mathbb{Q}}(\sigma)^{[k:\mathbb{Q}]} \qquad (\sigma = \mathrm{Re}(s) > 1)$$

This is a bad estimate. It suggests that the meromorphically continued $\zeta_k(s)$ has a pole of order $[k : \mathbb{Q}]$ at $s = 1$. In reality, this pole is of order 1, but this is non-trivial to prove. It is related to *finiteness of class number* $h(\mathfrak{o})$ (order of ideal class group), and *Dirichlet's Units Theorem* (the units group $\mathfrak{o}^\times$ is as large as possible).

Thus, the expected sum over principal ideals

$$Z_{[\mathfrak{o}]}(s) \;=\; \sum_{0 \neq \alpha \,\in\, \mathfrak{o}/\mathfrak{o}^\times} \frac{1}{N(\alpha\mathfrak{o})^s} \;=\; \sum_{0 \neq \alpha \,\in\, \mathfrak{o}/\mathfrak{o}^\times} \frac{1}{|N^k_{\mathbb{Q}}(\alpha)|^s}$$

is only a *partial zeta function*, because it is only *part* of $\zeta_k(s)$. For any ideal class $[\mathfrak{b}]$, the corresponding partial zeta function is

$$Z_{[\mathfrak{b}]}(s) \;=\; \sum_{0 \neq \mathfrak{a} \subset \mathfrak{o},\; \mathfrak{a} \in [\mathfrak{b}]} \frac{1}{N\mathfrak{a}^s}$$

and

$$\zeta_k(s) \;=\; \sum_{\text{classes } [\mathfrak{b}]} Z_{[\mathfrak{b}]}(s)$$

The partial zetas can be rewritten as sums over field elements, as follows. Given ideal class $[\mathfrak{b}]$, to say $\mathfrak{a} \in [\mathfrak{b}]$ is to say $\mathfrak{a} = \alpha \cdot \mathfrak{b}$ for some $\alpha \in k^\times$. That $\mathfrak{a} \subset \mathfrak{o}$ is $\alpha\mathfrak{b} \subset \mathfrak{o}$, or $\alpha \in \mathfrak{b}^{-1}$.

Also, $N(\alpha\mathfrak{b}) = |N^k_{\mathbb{Q}}(\alpha)| \cdot N\mathfrak{b}$, so the subsum over ideals $[\mathfrak{b}]$ is

$$Z_{[\mathfrak{b}]}(s) \;=\; \sum_{0\neq\alpha \,\in\, \mathfrak{b}^{-1}/\mathfrak{o}^\times} \frac{1}{(|N^k_{\mathbb{Q}}\alpha| \cdot N\mathfrak{b})^s} \;=\; \frac{1}{N\mathfrak{b}^s} \sum_{0\neq\alpha \,\in\, \mathfrak{b}^{-1}/\mathfrak{o}^\times} \frac{1}{|N^k_{\mathbb{Q}}\alpha|^s}$$

The units group $\mathfrak{o}^\times$ is finite for *complex quadratic fields* $k = \mathbb{Q}(\sqrt{-D})$ for $D > 0$ [and *only* in that case and for $k = \mathbb{Q}$ itself, by Dirichlet's Units Theorem, below...]. With $|\mathfrak{o}^\times| < \infty$,

$$Z_{[\mathfrak{b}]}(s) \;=\; \frac{1}{N\mathfrak{b}^s} \frac{1}{|\mathfrak{o}^\times|} \sum_{0\neq\alpha \,\in\, \mathfrak{b}^{-1}} \frac{1}{|N^k_{\mathbb{Q}}\alpha|^s}$$

We will obtain a formula for the class number $h(\mathfrak{o})$ *of* $\mathfrak{o}$ *for complex quadratic fields.* In particular, this proves finiteness in that case.

For $k = \mathbb{Q}(\sqrt{-D})$ for $D > 0$, the ring of algebraic integers $\mathfrak{o}$ is either $\mathbb{Z}[\sqrt{-D}]$ or $\mathbb{Z}[\frac{1+\sqrt{-D}}{2}]$, depending whether $-D = 2, 3 \bmod 4$, or $-D = 1 \bmod 4$.

More to the point, *qualitatively* $\mathfrak{o}$ is a free $\mathbb{Z}$-module of rank 2, and is a *lattice* in $\mathbb{C}$, in the sense that $\mathfrak{o}$ is a *discrete* subgroup of $\mathbb{C}$, and $\mathbb{C}/\mathfrak{o}$ is *compact.*

For *any* complex quadratic field, the *Galois* norm is the *complex* norm squared, because the non-trivial Galois automorphism is the restriction of complex conjugation:

$$N^k_{\mathbb{Q}}(\alpha) \;=\; \alpha \cdot \bar{\alpha} \;=\; |\alpha|^2 \qquad\qquad \text{(for complex quadratic } k\text{)}$$

Thus, in particular, as we know well, in this situation $N^k_{\mathbb{Q}}(\alpha)$ is the square of the distance of α from 0.

Lemma: For a lattice Λ in $\mathbb{C}$, the associated Epstein zeta function

$$Z_\Lambda(s) \;=\; \sum_{0\neq\lambda\in\Lambda} \frac{1}{|\lambda|^{2s}}$$

has a meromorphic continuation to $\mathrm{Re}(s) > 1-\varepsilon$ for small $\varepsilon > 0$, and

$$Z_\Lambda(s) \;=\; \frac{\pi}{\text{co-area}\,\Lambda} \cdot \frac{1}{s-1} \;+\; (\text{holomorphic near } s=1)$$

where *co-area* is intended to be the natural area of the quotient $\mathbb{C}/\Lambda$, or the inverse of the *density* of Λ. Formulaically,

$$(\text{co-area})\ \Lambda \;=\; |\det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}|$$

for $\mathbb{Z}$-basis $\lambda_1 = x_1 + iy_1$, $\lambda_2 = x_2 + iy_2$ of Λ. Equivalently,

$$(\text{co-area})\ \Lambda \;=\; \text{area of fundamental parallelogram for } \Lambda$$

$$=\; \text{area of parallelogram with vertices } 0,\, \lambda_1,\, \lambda_2,\, \lambda_1+\lambda_2$$

Proof: This is a slight sharpening of a higher-dimensional *integral test* applied to this situation. Part of the idea is that for some (or *any*) r_o

$$\sum_{0\neq\lambda\in\Lambda} \frac{1}{|\lambda|^{2s}} \sim \int_{|z|\geq r_o} \frac{(\text{density of }\Lambda)}{|z|^{2s}}\,dx\,dy$$

$$= 2\pi\int_{r_o}^{\infty} \frac{(\text{density of }\Lambda)}{r^{2s}}\,r\,dr = \frac{2\pi(\text{density of }\Lambda)}{2(s-1)}\cdot r_o^{2-2s}$$

$$= \frac{\pi}{\text{co-area }\Lambda}\cdot\frac{1}{s-1}\cdot r_o^{2-2s} = \frac{\pi}{\text{co-area }\Lambda}\cdot\frac{1}{s-1} + (\text{holo near } s=1)$$

This correctly suggests the blow-up at $s = 1$ and the dependence on the co-area of Λ.

A small amount of care clarifies this, as a very easy example of a line of reasoning brought to classical perfection by Minkowski *circa* 1900.

Let $\nu(r) = \#\{0 \neq \lambda \in \Lambda : |\lambda| \leq r\}$ be the number of lattice points inside a circle of radius r.

Claim:

$$\nu(r) \;=\; \frac{\pi r^2}{\text{co-area } \Lambda} + O(r)$$

where $O(r)$ denotes a function bounded by some constant multiple of r as $r \to \infty$.

Proof: Let F be any fundamental parallelogram for Λ with one vertex at 0. Let d be the diameter of F. Let B_r be the ball in $\mathbb{C}$ of radius r centered at 0.

For $|\lambda| \leq r$, $\lambda + F \subset B_{r+d}$, so the number of lattice points inside B_r is bounded by the number of (disjoint!) copies of F inside B_{r+d}. Comparing areas,

$$\nu(r) \;\leq\; \frac{\text{area } B_{r+d}}{\text{area } F} \;=\; \frac{\pi(r+d)^2}{\text{area } F} \;=\; \frac{\pi(r+d)^2}{\text{co-area } \Lambda}$$

On the other hand, for $\lambda + F \subset B_r$, certainly $\lambda \in B_r$. The smaller B_{r-d} is entirely covered by $\lambda + F$'s fitting inside B_r, so

$$\nu(r) \;\geq\; \frac{\text{area } B_{r-d}}{\text{area } F} \;=\; \frac{\pi(r-d)^2}{\text{co-area } \Lambda}$$

Together,

$$\frac{\pi(r-d)^2}{\text{co-area } \Lambda} \;\leq\; \nu(r) \;\leq\; \frac{\pi(r+d)^2}{\text{co-area } \Lambda}$$

which proves the claim that $\nu(r) = \pi r^2/\text{co-area}(\Lambda) + O(r)$. ///

Using Riemann-Stieljes integrals and integration by parts,

$$\sum_{0 \neq \lambda \in \Lambda} \frac{1}{|\lambda|^{2s}} \;=\; \int_{r_o}^{\infty} \frac{1}{r^{2s}}\, d\nu(r) \;=\; 2s \int_{r_o}^{\infty} \nu(r)\, \frac{dr}{r^{2s+1}}$$

And

$$2s \int_{r_o}^{\infty} \nu(r)\, \frac{dr}{r^{2s+1}} \;=\; 2s \int_{r_o}^{\infty} \frac{\pi r^2}{\text{co-area } \Lambda}\, \frac{dr}{r^{2s+1}} + 2s \int_{r_o}^{\infty} O(r)\, \frac{dr}{r^{2s+1}}$$

The second summand is holomorphic for $\mathrm{Re}(2s) > 1$, and the first is

$$2s \frac{\pi}{\text{co-area } \Lambda} \cdot \int_{r_o}^{\infty} \frac{dr}{r^{2s-1}} \;=\; \frac{s\pi}{\text{co-area } \Lambda} \cdot \frac{1}{s-1}$$

The residue at $s = 1$ is $\pi/\text{co-area}(\Lambda)$. ///

That is, again, the Epstein zeta function $Z_\Lambda(s)$ attached to a lattice Λ is meromorphic on $\mathrm{Re}(s) > \frac{1}{2}$, with simple pole at $s = 1$ with residue $\pi/\text{co-area}(\Lambda)$.

Corollary: For complex quadratic k, assuming $h(\mathfrak{o}) < \infty$,

$$\zeta_k(s) \;=\; \sum_{[\mathfrak{b}]} \sum_{\mathfrak{a} \sim \mathfrak{b}} \frac{1}{N\mathfrak{a}^s} \;\sim\; \frac{\pi \cdot h(\mathfrak{o})}{|\mathfrak{o}^\times| \cdot \text{co-area}(\mathfrak{o})} + (\text{holo at } s = 1)$$

Proof: As observed earlier,

$$\zeta_k(s) \;=\; \sum_{[\mathfrak{b}]} \frac{1}{N\mathfrak{b}^s} \frac{1}{|\mathfrak{o}^\times|} \sum_{0 \neq \alpha \in \mathfrak{b}^{-1}} \frac{1}{N\alpha^s} \;=\; \sum_{[\mathfrak{b}]} \frac{1}{N\mathfrak{b}^s} \cdot Z_{\mathfrak{b}^{-1}}(s)$$

By the lemma, this will have residue

$$\mathrm{Res}_{s=1}\zeta_k(s) \;=\; \sum_{[\mathfrak{b}]} \frac{1}{N\mathfrak{b}} \cdot \frac{\pi}{|\mathfrak{o}^\times| \cdot \text{co-area } \mathfrak{b}^{-1}}$$

The co-area of $\mathfrak{b}^{-1}$ is determined as follows. Observe that for an ideal $\mathfrak{a}$

$$N\mathfrak{a} \;=\; [\mathfrak{o} : \mathfrak{a}] \;=\; \frac{\text{area } \mathbb{C}/\mathfrak{a}}{\text{area } \mathbb{C}/\mathfrak{o}} \;=\; \frac{\text{co-area } \mathfrak{a}}{\text{co-area } \mathfrak{o}}$$

By multiplicativity, the co-area of $\mathfrak{b}^{-1}$ is $N(\mathfrak{b}^{-1}) = (N\mathfrak{b})^{-1}$. That is, the $\mathfrak{b}^{th}$ summand in the residue does not depend on $\mathfrak{b}$, and we have the assertion. ///

Corollary With $\chi(p) = (-D/p)_2$,

$$\frac{\pi \cdot h(\mathfrak{o})}{|\mathfrak{o}^\times| \cdot \text{co-area } \mathfrak{o}} \;=\; L(1, \chi)$$

Proof: Recall (!?!) the *factorization*

$$\zeta_k(s) \;=\; \zeta_{\mathbb{Q}}(s) \cdot L(s, \chi)$$

Since $\zeta(s) = \zeta_{\mathbb{Q}}(s)$ has residue 1 at $s = 1$, the value $L(1, \chi)$ is the residue of $\zeta_k(s)$ at $s = 1$. ///

Remark: For complex quadratic k, all the units are roots of unity, and the number of roots of unity is often denoted w. Thus, rewriting,

$$\frac{\pi \cdot h}{w \cdot \text{coarea}(\mathfrak{o})} \;=\; L(1, \chi)$$

In particular, not only is $L(1, \chi) \neq 0$, it is *positive*.

Further, for complex quadratic k, the special value $L(1, \chi)$ has a finite, closed-form expression. Let N be the conductor of χ.

From the Fourier expansion of the sawtooth function

$$x - \tfrac{1}{2} \;=\; \frac{-1}{2\pi i} \sum_{n \neq 0} \frac{e^{2\pi i n x}}{n}$$

$$\sum_{a \bmod N} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a) \;=\; \frac{-1}{2\pi i} \sum_{n \neq 0} \frac{1}{n} \sum_{a} \chi(a) e^{2\pi i n a/N}$$

$$= \frac{-1}{2\pi i} \sum_{n \neq 0} \frac{\chi(n)}{n} \cdot \sum_{a} \chi(a) e^{2\pi i a/N}$$

by replacing a by $an^{-1} \bmod N$. Since $\chi(-1) = -1$ (!!!)

$$\sum_{a \bmod N} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a) \;=\; \frac{-1}{\pi i} \cdot L(1, \chi) \cdot \sum_{a} \chi(a) e^{2\pi i a/N}$$

Thus,

$$L(1,\chi) \;=\; \frac{-\pi i}{\sum_a \chi(a) e^{2\pi i a/N}} \sum_{a \bmod N} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a)$$

Thus,

$$\frac{\pi \cdot h(\mathfrak{o})}{w \cdot \mathrm{coarea}(\mathfrak{o})} \;=\; \frac{-\pi i}{\sum_a \chi(a) e^{2\pi i a/N}} \sum_{a \bmod N} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a)$$

and

$$h(\mathfrak{o}) \;=\; \frac{-iw \cdot \mathrm{coarea}(\mathfrak{o})}{\sum_a \chi(a) e^{2\pi i a/N}} \sum_{a \bmod N} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a)$$

Again, this is for *complex quadratic* fields. ///
