Topologies, completions/limits

An **absolute value** or *norm* $x \to |x|$ on a field k is a non-negative real-valued function on k such that

$$\begin{cases} |x| = 0 \text{ only for } x = 0 & \text{(positivity)} \\ |xy| = |x| \cdot |y| & \text{(multiplicativity)} \\ |x+y| \le |x| + |y| & \text{(triangle inequality)} \end{cases}$$

When $|x+y| \le \max(|x|, |y|)$, the norm is *non-archimedean*, or a *valuation*.

A norm gives k has a metric *topology* by $d(x,y) = |x-y|$. Since $|x| = |x \cdot 1| = |x| \cdot |1|$ we have $|1| = 1$. Also, $|\omega|^n = |\omega^n| = |1|$ for an n^{th} root of unity, so $|\omega| = 1$. Then *reflexivity, symmetry*, and the triangle inequality follow for the metric.

Theorem: Two norms $| * |_1$ and $| * |_2$ on k give the same *non-discrete* topology on a field k if and only if $| * |_1 = | * |_2^t$ for some $0 < t \in \mathbb{R}$. [Last time]

Theorem: Over a complete, non-discrete normed field k,
• A *finite-dimensional* k-vectorspace V has just one Hausdorff topology so that vector addition and scalar multiplication are continuous (a *topological vectorspace* topology). All linear endomorphisms are *continuous*.
• A finite-dimensional k-subspace V of a topological k-vectorspace W is necessarily a *closed* subspace of W.
• A k-linear map $\phi : X \to V$ to a finite-dimensional space V is continuous if and only if the kernel is closed.

Remark: The argument also succeeds over complete non-discrete *division algebras*.

A subset E of V is **balanced** when $xE \subset E$ for every $x \in k$ with $|x| \leq 1$.

Lemma: Let U be a neighborhood of 0 in V. Then U contains a *balanced* neighborhood N of 0. [Last time]

Proposition: For a one-dimensional topological vectorspace V, that is, a free module on one generator e, the map $k \to V$ by $x \to xe$ is a *homeomorphism*. [Last time]

Corollary: Fix $x_o \in k$. A not-identically-zero k-linear k-valued function f on V is *continuous* if and only if the affine hyperplane

$$H = \{v \in V : f(v) = x_o\}$$

is *closed* in V. [Last time]

Proof of theorem: To prove the uniqueness of the topology, prove that for any k-basis $e_1, \ldots, e_n$ for V, the map $k \times \ldots \times k \to V$ by

$$(x_1, \ldots, x_n) \to x_1 e_1 + \ldots + x_n e_n$$

is a homeomorphism. Prove this by induction on the dimension n.

$n = 1$ was treated already. Granting this, since k is complete, the lemma asserting the closed-ness of complete subspaces shows that any one-dimensional subspace is closed.

Take $n > 1$, and let $H = ke_1 + \ldots + ke_{n-1}$. By induction, H is closed in V, so V/H is a topological vector space. Let q be the quotient map. V/H is a one-dimensional topological vectorspace over k, with basis $q(e_n)$. By induction,

$$\varphi : xq(e_n) = q(xe_n) \to x$$

is a homeomorphism to k.

Likewise, ke_n is a closed subspace and we have the quotient map $q' : V \to V/ke_n$. We have a basis $q'(e_1), \ldots, q'(e_{n-1})$ for the image, and by induction

$$\phi' : x_1 q'(e_1) + \ldots + x_{n-1} q'(e_{n-1}) \to (x_1, \ldots, x_{n-1})$$

is a homeomorphism. By induction,

$$v \to (\phi \circ q)(v) \times (\phi' \circ q')(v)$$

is continuous to

$$k^{n-1} \times k \approx k^n$$

On the other hand, by the continuity of scalar multiplication and vector addition, the map

$$k^n \to V \quad \text{by} \quad x_1 \times \ldots \times x_n \to x_1 e_1 + \ldots + x_n e_n$$

is continuous.

The two maps are mutual inverses, proving they are homeomorphisms.

Thus, a n-dimensional subspace is homeomorphic to k^n, so is complete, since (as follows readily) a finite product of complete spaces is complete.

Thus, by the lemma asserting the closed-ness of complete subspaces, an n-dimensional subspace is always closed.

Continuity of a linear map $f : X \to k^n$ implies that the kernel $N = \ker f$ is closed. On the other hand, if N is closed, then X/N is a topological vectorspace of dimension at most n. Therefore, the induced map $\bar{f} : X/N \to V$ is unavoidably continuous. But then $f = \bar{f} \circ q$ is continuous, where q is the quotient map.

In particular, any k-linear map $V \to V$ has finite-dimensional kernel, so the kernel is closed, and the map is continuous.

This completes the induction. ///

Corollary: Finite field extensions K of complete, non-discrete k have unique Hausdorff topologies making addition and multiplication continuous.

Proof: K is a finite-dimensional k-vectorspace, and the theorem gives uniqueness of a topological k vector space structure on K so that addition and scalar multiplication by k are continuous. The only ingredient perhaps not overtly supplied by the theorem is the continuity of the multiplication by elements of K. Such multiplications are k-linear endomorphisms of the vector space K, so are continuous, by the theorem. ///

Remark: This discussion still did *not* use *local compactness* of the field k, and is *not* specifically number theoretic.

Constructions/existence: For any Dedekind domain $\mathfrak{o}$, and for a non-zero prime $\mathfrak{p}$ in $\mathfrak{o}$, the $\mathfrak{p}$-adic norm is

$$|x|_{\mathfrak{p}} \;=\; C^{-\mathrm{ord}_{\mathfrak{p}} x} \qquad\qquad (\text{where } x \cdot \mathfrak{o} = \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}} x} \cdot \text{prime-to-}\mathfrak{p})$$

and $C > 1$ is a constant. Since this norm is ultrametric/non-archimedean, the choice of C does not immediately matter, but it *can* matter in interactions of norms for varying $\mathfrak{p}$, as in the **product formula** for number fields and function fields. Recall the product formula for $\mathbb{Q}$:

$$\prod_{v \le \infty} |x|_v \;=\; 1 \qquad\qquad (\text{for } x \in \mathbb{Q}^\times)$$

That is, with $|*|_\infty$ the 'usual' absolute value on $\mathbb{R}$,

$$|x|_\infty \cdot \prod_{p \text{ prime}} |x|_p \;=\; 1 \qquad\qquad (\text{for } x \in \mathbb{Q}^\times)$$

Recall the *Proof:* Both sides are *multiplicative* in x, so it suffices to consider $x = \pm 1$ and $x = q$ prime. For units ± 1, both sides are 1. For $x = q$ prime, $|q|_\infty = q$, while $|q|_q = 1/q$, and $|q|_p = 1$ for $p \neq q$, so again both sides are 1. ///

One normalization to have the product formula hold for *number fields* k: for $\mathfrak{p}$ lying over p, letting $k_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of k and Q_p the usual p-adic completion of $\mathbb{Q}$,

$$|x|_\mathfrak{p} \;=\; |N^{k_\mathfrak{p}}_{\mathbb{Q}_p} x|_p$$

For *archimedean* completion k_v of k, define (or renormalize)

$$|x|_v \;=\; |N^{k_v}_{\mathbb{R}} x|_\infty$$

The latter entails a normalization which (harmlessly) fails to satisfy the triangle inequality:

$$|x|_\mathbb{C} \;=\; |N^{\mathbb{C}}_{\mathbb{R}} x|_\infty \;=\; x \cdot \overline{x} \;=\; \textit{square} \text{ of usual complex abs value}$$

This normalization is used only in a multiplicative context, so failure of the triangle inequality is harmless. The metric topology is given by the *usual* norm.

In other words, for primes $\mathfrak{p}$ in $\mathfrak{o}$, in the formula above take $C = N\mathfrak{p} = |\mathfrak{o}/\mathfrak{p}|$, so

$$|x|_\mathfrak{p} \;=\; N\mathfrak{p}^{-\mathrm{ord}_\mathfrak{p} x}$$

Theorem: *(Product formula for number fields)*

$$\prod_{\text{places } w \text{ of } k} |x|_w \;=\; \prod_{\text{places } v \text{ of } \mathbb{Q}} \; \prod_{w|v} |N^{k_w}_{Q_v}(x)|_v \;=\; 1 \qquad\qquad (\text{for } x \in k^\times)$$

[Proof next]
